# Queueing Analysis and Nash Equilibria in an Unobservable Taxi-passenger System with Two Types of Passenger

Hung Q. Nguyen[1] and Tuan Phung-Duc[2][a]

[1]*Graduate School of Science and Technology, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8573, Japan*

[2]*Faculty of Engineering, Information and Systems, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8573, Japan*

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Abstract: This paper considers an unobservable double-ended queueing system motivated by the application of a taxi station where passengers and taxis arrive at two sides of the queue, and there are two types of passengers, differentiated by their mean matching times with taxis. We use a three-dimensional Markov chain to model the system and derive several system performance measures (mean queue lengths, waiting times and social welfare). Furthermore, when agents are strategic, we model the system as a multi-population game among three populations of agents and find their joining rates in equilibrium.

## 1 INTRODUCTION

A taxi-passenger system is a typical real-life example of a double-ended queueing system in which agents arrive at both sides of the queue for matching. In airports and railway stations, it is common to see both passengers and taxis form queues to be matched: a queue of passengers who wait for taxis, and a queue of taxis that wait for passengers. This type of system was first studied by Kendall (1951), followed by Dobbie (1961) and Di Crescenzo et al. (2012). Those early studies did not incorporate strategic agent behaviors into the model.

A queueing analysis considering strategic customer behavior was first undertaken by Naor (1969). This research identified a queue length threshold that determines customers' decision between joining or balking, under the most basic setting of an observable M/M/1 queue. Edelson and Hildebrand (1975) complemented by considering the unobservable case. Multiple extensions followed, comprehensively summarized in Hassin and Haviv (2003); Hassin (2016). Hassin and Haviv (2002); Economou (2021) provided a concrete theoretical framework for this type of problem by defining the problem under the scope of game theory. However, only one-population games were considered in those studies.

[a] https://orcid.org/0000-0002-5002-4946

Combining the two aforementioned concepts results in a new topic—the strategic behavior of agents in double-ended queueing systems, with several notable studies: Shi and Lian (2016) considered a system where matching times are zero; hence, the system is described by a single number which is the difference between the number of passengers and the number of taxis in the system. Wang et al. (2017) considered a system with a gated policy. Jiang et al. (2020) incorporated customer loss aversion into the model. Wang and Liu (2019) discussed the impacts of different levels of information. In all of these studies, only passengers are assumed to be rational.

In reality, due to bulky luggage or to communication between passengers and taxi drivers, the matching process usually takes more time than can be dismissed as negligible. Furthermore, there are access points where multiple passenger–taxi pairs can match at the same time. Another consideration is that there may be multiple types of passengers whose matching times are not identical. For example, domestic passengers can match with local taxi drivers more quickly, while it may take a longer time for foreign visitors to communicate with taxis drivers. Moreover, agents from the supply side may also be strategic.

Motivated by these complex circumstances, this paper analyzes a taxi-passenger and multi-server queueing system, and studies agents' strategies for joining the queue based on individual utility. The



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main contributions of this paper are:

- We consider a system with two passenger types, distinguished by their differing service rates.
- We consider general non-zero matching times which follow exponential distributions, and multiple access points.
- From the modeling perspective, we use a three-dimensional Markovian description of the system. When agents are strategic, the system is modeled as a three-population game.

The remainder of this paper is structured as follows: Section 2 describes the queueing system and presents notations. Next, we derive some performance measures in Section 3. In Section 4, we derive Nash equilibria of the system when agents are strategic. In Section 5, we analyze the system through several numerical examples. Finally, Section 6 concludes the paper.

# 2 PRELIMINARIES

We consider an unobservable double-ended queueing system with a taxi stand containing $S$ identical access points. There are three populations of agents arriving at the system: type-1 passengers, type-2 passengers, and taxis. The area (including $S$ taxi access points) can accommodate at most $K$ taxis at the same time ($K \geq S$). In an ideal situation where agents are given enough incentive to join the queue without balking, passengers and taxis arrive at the taxi stands according to Poisson processes with potential arrival rates $\Lambda_p$ and $\Lambda_t$, respectively. An access point receives a type-1 passenger with probability $\varepsilon$, and a type-2 passenger with probability $1-\varepsilon$. The matching times of type-1 and type-2 passengers follow exponential distributions with parameters $\mu_1$ and $\mu_2$, respectively. Without loss of generality, we can assume that $\mu_1 < \mu_2$, meaning that type-1 passengers have a larger mean matching time. When the parking area reaches its maximum capacity, the arrival of any new taxi is blocked, so that taxi leaves immediately. On the other hand, we assume that there is no limit on the buffer of passengers. If a passenger arrives when all access points are busy, or when there are no taxis available for matching, he will wait in the queue under the first-come, first-served (FCFS) service principle.

Let $R_p$ and $R_t$ denote the service values, and $C_p$ and $C_t$ denote the waiting cost per time unit of passengers and taxis, respectively. Let $q_{p_1}, q_{p_2}$ and $q_t$ denote the joining probabilities of type-1 passengers, type-2 passengers, and taxis, respectively. Let $\lambda_p = (q_{p_1}\varepsilon + q_{p_2}(1-\varepsilon))\Lambda_p$, $\lambda_t = q_t\Lambda_t$ and $\alpha = \frac{q_{p_1}\varepsilon}{q_{p_1}\varepsilon + q_{p_2}(1-\varepsilon)}$. Then, $\lambda_p$ and $\lambda_t$ are the actual arrival rates of agents; these will be are used to derive performance measures in the following section.

# 3 PERFORMANCE MEASURES

Let $L(t)$, $I(t)$ and $J(t)$ respectively denote the number of type-1 passengers being matched at the access points, the number of taxis in the system, and the total number of passengers in the system, at time $t$. The process $\{(L(t), I(t), J(t)) \mid t \geq 0\}$ is a continuous-time Markov Chain with the state space $\mathbb{S}$ given by

$$\mathbb{S} = \{(l,i,j) \in \{0,1,...,S\} \times \{0,1,...,K\} \times \{0,1,2,\ldots\}\}.$$

Also, as implied by their definitions, it should be noted that $l \leq i$ and $l \leq j$.

The system can be modeled as a quasi-birth-death process with the infinitesimal generator $Q$ being expressed as follows.

$$Q = \begin{pmatrix} \mathcal{B}^{(0)} & \mathcal{C}^{(0)} & O & O & O & \ldots & \ldots & \ldots & \ldots \\ \mathcal{A}^{(1)} & \mathcal{B}^{(1)} & \mathcal{C}^{(1)} & O & O & \ldots & \vdots & \vdots & \vdots \\ O & \mathcal{A}^{(2)} & \mathcal{B}^{(2)} & \mathcal{C}^{(2)} & O & \ldots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ O & O & \ldots & \mathcal{A}^{(K)} & \mathcal{B}^{(K)} & \mathcal{C}^{(K)} & O & \ldots & \vdots \\ O & O & \ldots & O & \mathcal{A}^{(K)} & \mathcal{B}^{(K)} & \mathcal{C}^{(K)} & O & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \end{pmatrix}, \quad (1)$$

where $O$ denotes a zero matrix of appropriate dimension. If we denote by $\mathbb{M}(m,n)$ the set of all $m \times n$-dimensional matrices, and $\mathcal{M}(i,j)$ $(i,j \in \mathbb{Z}^+)$ the element at the $i^{th}$ row, $j^{th}$ column of a matrix $\mathcal{M}$, then the block matrices in $Q$ can be defined by the system parameters $\lambda_p, \lambda_t, \mu_1, \mu_2$ and $\alpha$, as shown in the Appendix.

Letting $Q^* = \mathcal{A}^{(K)} + \mathcal{B}^{(K)} + \mathcal{C}^{(K)}$, we can then derive the stability condition of the system by simultaneously solving the following equations for $\eta$.

$$\eta Q^* = \mathbf{0},$$

and

$$\eta \mathbf{e} = 1,$$

where $\eta$ is the row vector representing the stationary distribution of the infinitesimal generator $Q^*$, $\mathbf{0}$ is a row vector with all elements equal to 0 and $\mathbf{e}$ is a column vector with all elements equal to 1. The stability condition, then, is

$$\eta \mathcal{C}^{(K)} \mathbf{e} < \eta \mathcal{A}^{(K)} \mathbf{e}. \quad (2)$$

If stability condition (2) is not satisfied, the system becomes an M/$H_2$/$S$/$K$ queue of taxis in which the passengers become "servers". The mean waiting times of two type-1 and type-2 passengers are given by

$$W_{p_1} = W_{p_2} = +\infty. \tag{3}$$

When the stability condition holds, we can derive the steady state probabilities $\pi = (\pi_0, \pi_1, \pi_2, ...)$, where $\pi_j = (\pi_{0,0,j}, \pi_{1,0,j}, ..., \pi_{S,K,j})$ is the vector encoding all probabilities when there are $j$ passengers in the system at the steady state. For $j \geq K$, there exists a constant matrix $\mathcal{R}$ such as

$$\pi_j = \pi_K \mathcal{R}^{j-K},$$

where $\mathcal{R}$ satisfies

$$\mathcal{C}^{(K)} + \mathcal{R}\mathcal{B}^{(K)} + \mathcal{R}^2 \mathcal{A}^{(K)} = O. \tag{4}$$

The solution of the matrix equation (4) is obtained by the Matrix Geometric Method proposed by Neuts (1981). Furthermore, for $1 \leq j \leq K$, we also have

$$\pi_j = \pi_{j-1} \mathcal{R}^{(j)},$$

where $\mathcal{R}^{(K)} = \mathcal{R}$, and $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, ..., \mathcal{R}^{(K-1)}; \pi_0, \pi_1, ..., \pi_K$ are recursively calculated as

$$\mathcal{R}^{(j)} = -\mathcal{C}^{(j)} (\mathcal{B}^{(j)} + \mathcal{R}^{(j+1)} \mathcal{A}^{(j+1)})^{-1}.$$

$\pi_0$ is determined by solving

$$\pi_0 (\mathcal{B}^{(0)} + \mathcal{R}^{(1)} \mathcal{A}^{(1)})^{-1} = \mathbf{0}, \tag{5}$$

and

$$\pi_0 \left( I + \sum_{j=1}^{S-1} \prod_{i=1}^{j} R^{(i)} + \left( \prod_{i=1}^{S} \mathcal{R}^{(i)} \right) (I - \mathcal{R})^{-1} \right) \mathbf{e} = 1, \tag{6}$$

where $I$ denotes an identity matrix of appropriate dimension.

Next, we derive performance measures of the system, including average queue lengths, and average waiting times of passengers and taxis. The average number of passengers in the waiting line is given by

$$\begin{aligned} \mathcal{L}_p &= \sum_{j=0}^{\infty} \sum_{i=0}^{K} \sum_{l=0}^{S} (j - \min\{i, j, S\}) \pi_{l,i,j} \\ &= \sum_{j=0}^{K-1} j \pi_j \mathbf{e}_j - \sum_{j=0}^{K-1} \pi_j \mathbf{g}_j \\ &\quad + \pi_K (I - \mathcal{R})^{-1} [KI + (I - \mathcal{R})^{-1} \mathcal{R}] \mathbf{e}_K \\ &\quad - \pi_K (I - \mathcal{R})^{-1} \mathbf{g}_K, \end{aligned} \tag{7}$$

where

- $\mathbf{e}_j$ is a unit vector of the same dimension as $\pi_j$.
- $\mathbf{g}_j = (g_{0,0,j}, g_{1,0,j}, ..., g_{S,K,j})$, where $g_{l,i,j} = \min\{i, j, S\}$.

Note that the summation in (7) excludes $(l, i, j)$ not existing in the state space. The same rule applies to all later summations and products.

The average number of taxis in the system is given by

$$\begin{aligned} L_t &= \sum_{j=0}^{\infty} \sum_{i=0}^{K} \sum_{l=0}^{S} i \pi_{l,i,j} \\ &= \sum_{j=0}^{K-1} \pi_j \mathbf{f}_j + \pi_K (I - \mathcal{R})^{-1} \mathbf{f}_K. \end{aligned} \tag{8}$$

Here, $\mathbf{f}_j = (f_{0,0,j}, f_{1,0,j}, ..., f_{S,K,j})$, where $f_{l,i,j} = i$.

Corresponding to (7) and (8), the average sojourn time taxis can be calculated via Little's Law as follows

$$W_t = \frac{L_t}{\lambda_t (1 - P_b)}, \tag{9}$$

where $P_b$ is the blocking probability of taxis, calculated by

$$P_b = \sum_{j=0}^{K-1} \pi_j \mathbf{u}_j + \pi_K (I - \mathcal{R})^{-1} \mathbf{u}_K. \tag{10}$$

Here, $\mathbf{u_j}$ is a vector with the same dimension as $\pi_j$, in which the last $\min(j+1, S+1)$ elements equal 1 and the other elements equal 0.

The average number of passengers in the system is given by

$$\begin{aligned} L_p &= \sum_{j=0}^{\infty} j \pi_j \mathbf{e}_j \\ &= \sum_{j=0}^{K-1} j \pi_j \mathbf{e}_j + \pi_K (I - \mathcal{R})^{-1} [KI + (I - \mathcal{R})^{-1} \mathcal{R}] \mathbf{e}_K. \end{aligned} \tag{11}$$

The average sojourn time of all passengers is given by

$$W_p = \frac{L_p}{\lambda_p}. \tag{12}$$

Since a passenger knows his own type before entering the taxi stand, the expected sojourn times of a type-1 and a type-2 passenger are estimated as

$$W_{p_1} = \frac{\mathcal{L}_p}{\lambda_p} + \frac{1}{\mu_1}, \tag{13}$$

and

$$W_{p_2} = \frac{\mathcal{L}_p}{\lambda_p} + \frac{1}{\mu_2}. \tag{14}$$

Finally, social welfare, which equals the total utility of all agents in the system per time unit, is given by

$$SW = \lambda_p R_p + \lambda_t (1 - P_b) R_t - C_p L_p - C_t L_t. \tag{15}$$

For further analysis, we acknowledge the following properties, which are supported by intuition and numerous simulation results.

**Axiom 1.** *The expected sojourn time of an arbitrary agent depends on the strategies of agents in their own population and the other populations. In other words, $W_{p_1}, W_{p_2}$ and $W_t$ are functions of $q_{p_1}, q_{p_2}$ and $q_t$. The monotonic properties of these functions with respect to each variable are given as follows.*
*Under the stability condition given in* (2)*,*
*(1) $W_{p_1}$ is continuously increasing in $q_{p_1}$ and $q_{p_2}$, and decreasing in $q_t$.*
*(2) $W_{p_2}$ is continuously increasing in $q_{p_1}$ and $q_{p_2}$, and decreasing in $q_t$.*
*(3) $W_t$ is continuously decreasing in $q_{p_1}$ and $q_{p_2}$, and increasing in $q_t$.*

# 4 NASH EQUILIBRIA

In this section, we derive all possible Nash equilibria at which agents make a best response to the strategies of other agents. The social profile, which is represented by a triplet $(q_{p_1}, q_{p_2}, q_t)$, is denoted $\mathbf{X}$. Let $\bar{\mathbf{X}} = (\bar{q}_{p_1}, \bar{q}_{p_2}, \bar{q}_t)$ be the social profile in equilibrium.

Denote by $U_{p_1}\left(q_{p_1}|\bar{\mathbf{X}}\right)$ the payoff of an arbitrary type-1 passenger who adopts a strategy $q_{p_1}$ against the social profile $\bar{\mathbf{X}}$, which is given by

$$U_{p_1}\left(q_{p_1}|\bar{\mathbf{X}}\right) = q_{p_1}\left(R_p - C_p W_{p_1}\left(\bar{\mathbf{X}}\right)\right). \tag{16}$$

.

By definition, $\bar{q}_{p_1}$ is a best response against the social profile in equilibrium, which means

$$\bar{q}_{p_1} \in \arg\max_{q_{p_1}} U_{p_1}\left(q_{p_1}|\bar{\mathbf{X}}\right) = \begin{cases} \{0\} & \text{if } R_p - C_p W_{p_1}\left(\bar{\mathbf{X}}\right) < 0, \\ [0,1] & \text{if } R_p - C_p W_{p_1}\left(\bar{\mathbf{X}}\right) = 0, \\ \{1\} & \text{if } R_p - C_p W_{p_1}\left(\bar{\mathbf{X}}\right) > 0. \end{cases} \tag{17}$$

If we similarly define $U_{p_2}\left(q_{p_2}|\bar{\mathbf{X}}\right)$ and $U_t\left(q_t|\bar{\mathbf{X}}\right)$ for the other two populations, we also have

$$\bar{q}_{p_2} \in \arg\max_{q_{p_2}} U_{p_2}\left(q_{p_2}|\bar{\mathbf{X}}\right) = \begin{cases} \{0\} & \text{if } R_p - C_p W_{p_2}\left(\bar{\mathbf{X}}\right) < 0, \\ [0,1] & \text{if } R_p - C_p W_{p_2}\left(\bar{\mathbf{X}}\right) = 0, \\ \{1\} & \text{if } R_p - C_p W_{p_2}\left(\bar{\mathbf{X}}\right) > 0, \end{cases} \tag{18}$$

and

$$\bar{q}_t \in \arg\max_{q_t} U_t\left(q_t|\bar{\mathbf{X}}\right) = \begin{cases} \{0\} & \text{if } R_t - C_t W_t\left(\bar{\mathbf{X}}\right) < 0, \\ [0,1] & \text{if } R_t - C_t W_t\left(\bar{\mathbf{X}}\right) = 0, \\ \{1\} & \text{if } R_t - C_t W_t\left(\bar{\mathbf{X}}\right) > 0. \end{cases} \tag{19}$$

(17), (18) and (19) lead to 27 possible combinations. However, we can reduce the number of cases to consider by noting that $W_{p_1}\left(\bar{\mathbf{X}}\right) > W_{p_2}\left(\bar{\mathbf{X}}\right)$, and considering the following special cases. First, if $\bar{q}_t = 0$, meaning that taxis do not join the system, then it is easily seen that $\bar{q}_{p_1} = \bar{q}_{p_2} = 0$ since the best response of passengers is not to join the system either. On the other hand, if $\bar{q}_{p_2} = 0$, meaning that type-2 passengers have no incentive to join the system, then it is implied that $\bar{q}_{p_1} = 0$ (since type-1 passengers always expect longer sojourn times than type-2 passengers), which leads to $\bar{q}_t = 0$. In other words, $(\bar{q}_{p_1}, \bar{q}_{p_2}, \bar{q}_t) = (0,0,0)$ is an equilibrium and is the only equilibrium where $\bar{q}_{p_2} = 0$ or $\bar{q}_t = 0$. When $\bar{q}_{p_2} > 0$, $\bar{q}_t > 0$ and the stability condition (2) is satisfied, all possible equilibria can be derived as shown in Table 1.

Table 1: Equilibria and corresponding conditions.

| **Equilibria** | **Conditions** |
|---|---|
| $(1,1,1)$ | $R_p - C_p W_{p_1}(1,1,1) > 0$, $R_p - C_p W_{p_2}(1,1,1) > 0$, $R_t - C_t W_t(1,1,1) > 0$. |
| $(\bar{q}_{p_1},1,1)$ | $R_p - C_p W_{p_1}(\bar{q}_{p_1},1,1) = 0$, $R_p - C_p W_{p_2}(\bar{q}_{p_1},1,1) > 0$, $R_t - C_t W_t(\bar{q}_{p_1},1,1) > 0$. |
| $(0,1,1)$ | $R_p - C_p W_{p_1}(0,1,1) < 0$, $R_p - C_p W_{p_2}(0,1,1) > 0$, $R_t - C_t W_t(0,1,1) > 0$. |
| $(0,\bar{q}_{p_2},1)$ | $R_p - C_p W_{p_1}(0,\bar{q}_{p_2},1) < 0$, $R_p - C_p W_{p_2}(0,\bar{q}_{p_2},1) = 0$, $R_t - C_t W_t(0,\bar{q}_{p_2},1) > 0$. |
| $(1,1,\bar{q}_t)$ | $R_p - C_p W_{p_1}(1,1,\bar{q}_t) > 0$, $R_p - C_p W_{p_2}(1,1,\bar{q}_t) > 0$, $R_t - C_t W_t(1,1,\bar{q}_t) = 0$. |
| $(\bar{q}_{p_1},1,\bar{q}_t)$ | $R_p - C_p W_{p_1}(\bar{q}_{p_1},1,\bar{q}_t) = 0$, $R_p - C_p W_{p_2}(\bar{q}_{p_1},1,\bar{q}_t) > 0$, $R_t - C_t W_t(\bar{q}_{p_1},1,\bar{q}_t) = 0$. |
| $(0,1,\bar{q}_t)$ | $R_p - C_p W_{p_1}(0,1,\bar{q}_t) < 0$, $R_p - C_p W_{p_2}(0,1,\bar{q}_t) > 0$, $R_t - C_t W_t(0,1,\bar{q}_t) = 0$. |
| $(0,\bar{q}_{p_2},\bar{q}_t)$ | $R_p - C_p W_{p_1}(0,\bar{q}_{p_2},\bar{q}_t) < 0$, $R_p - C_p W_{p_2}(0,\bar{q}_{p_2},\bar{q}_t) = 0$, $R_t - C_t W_t(0,\bar{q}_{p_2},\bar{q}_t) = 0$. |

Equilibria #1 and #3 can be verified by simply checking their corresponding conditions. The other equilibria are derived by solving their corresponding conditional equations and double-checking other conditions. The solutions to those equations are not explicit but are computationally solvable. We will illustrate results in several numerical examples in the following section.

# 5 NUMERICAL ANALYSIS

In this section, we present the analysis in specific numerical examples. First, we assume that agents are not rational, and numerically examine the variation of some performance measures with respect to system parameters and actual joining rates. In the following examples, we set $\mu_1 = 2$, $\mu_2 = 5$, $\alpha = 0.3$, $R_p = 15$, $R_t = 20$, $C_p = 5$ and $C_t = 4$. Results are illustrated in Figs. 1 to 8.

Figs. 1 to 4 verify the monotonic properties of passengers' and taxis' waiting times with respect to the agents' actual joining rates. The results are intuitive and follow exactly as in Axiom 1.

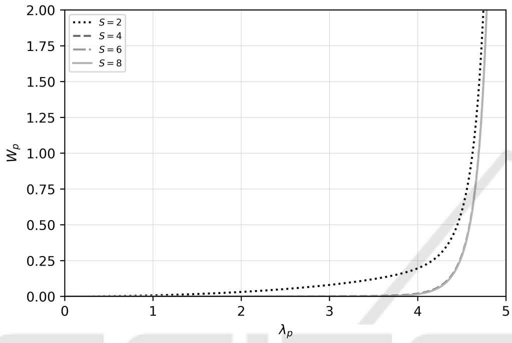


Figure 1: $W_p$ w.r.t $\lambda_p$ ($\lambda_t = 5$, $K = 20$).

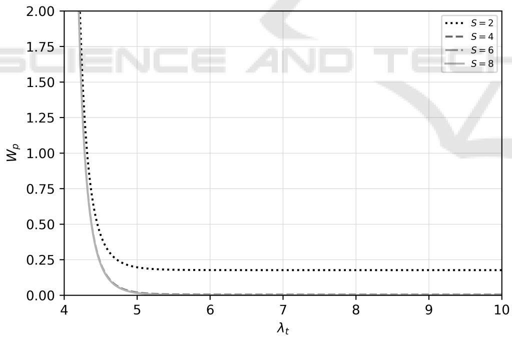


Figure 2: $W_p$ w.r.t $\lambda_t$ ($\lambda_p = 4$, $K = 20$).

Figs. 5 and 6 show that the social welfare function is unimodal with respect to the joining rates of passengers and taxis. There exists a value of passengers' (or taxis') joining rate at which social welfare reaches its maximum. These patterns suggest that the platform manager can control the arrival rate of one side of agents in case the other side is not strategic, to maximize social welfare. More details about applicable control policies can be found in Haviv and Oz (2018). For example, when taxi drivers are not strategic and join the queue with rate $\lambda_t = 5$, policy makers can interfere in the passengers' service value to adjust their arrival rate at around $\lambda_p^* = 4.7$, which yields

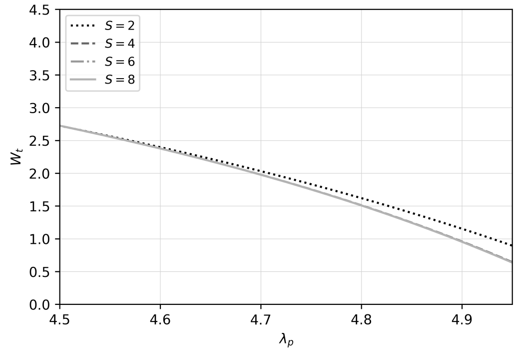


Figure 3: $W_t$ w.r.t $\lambda_p$ ($\lambda_t = 5$, $K = 20$).

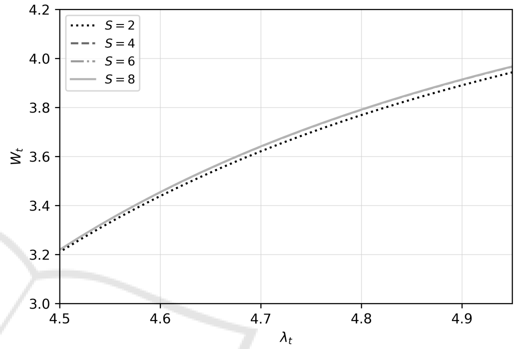


Figure 4: $W_t$ w.r.t $\lambda_t$ ($\lambda_p = 4$, $K = 20$).

the highest social welfare. When $\lambda_p < \lambda_p^*$, passengers need more incentive to join the queue, so a fixed subsidy (such as a discount or coupon) can be granted. On the contrary, when $\lambda_p > \lambda_p^*$, a toll fee can be applied to reduce the joining rate of passengers. The same policies apply in the case where taxi drivers are strategic and passengers are not.

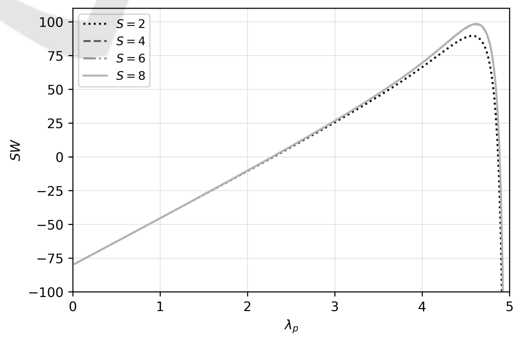


Figure 5: $SW$ w.r.t $\lambda_p$ ($\lambda_t = 5$, $S = 3$, $K = 20$).

Figs. 7 and 8 illustrate how social welfare varies according to the two system parameters $S$ and $K$. In this experiment, social welfare increases quickly at first, then remains almost unchanged as $S$ becomes larger. It can be observed that 5 access points are enough and optimal in this example (considering that

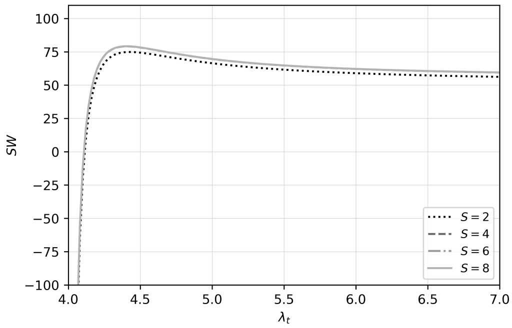


Figure 6: $SW$ w.r.t $\lambda_t$ ($\lambda_p = 4$, $S = 3$, $K = 20$).

a larger parking lot may consume more budget for construction and management). On the contrary, social welfare decreases with increased parking capacity. This phenomenon may stem from the fact that the parking capacity already exceeds a particular "threshold," above which the queue length of taxis gets longer and leads to inefficient waiting times, thus reducing social welfare.

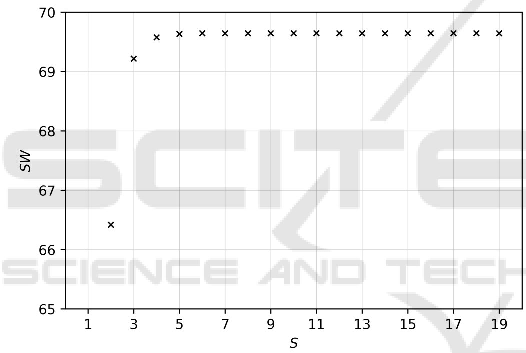


Figure 7: $SW$ w.r.t $S$ ($\lambda_p = 4$, $\lambda_t = 5$, $K = 20$).

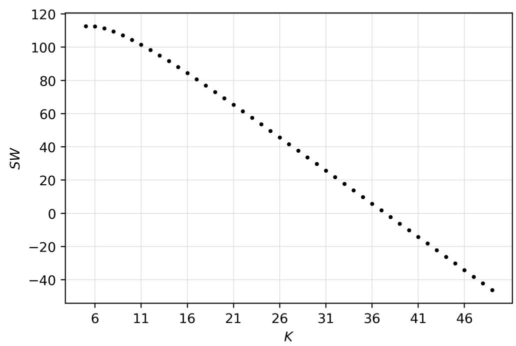


Figure 8: $SW$ w.r.t $K$ ($\lambda_p = 4$, $\lambda_t = 5$, $S = 3$).

In what follows, we derive joining probabilities of agents and calculate social welfare in the case where agents are strategic. For this, we set $\Lambda_p = 4$, $\Lambda_t = 5$, $\mu_1 = 1$, $\mu_2 = 5$, $S = 3$, $K = 8$, $\varepsilon = 0.3$, $C_p = 5$ and $C_t = 4$.

First, it can be seen that the equilibrium $(0,0,0)$ exists in any setting of parameters. In reality, the system may end up at the equilibrium $(0,0,0)$ in extreme situations, for example, when the system is terminated. In the following example, we derive other equilibria from the situations in Table 1.

**Example 1.** Assume $R_p = 15$ and $R_t = 20$. In this case, two equilibria exist: $(1,1,1)$ (and $(0,0,0)$). This means either that potential agents all join, or that none join at all.

**Example 2.** Assume $R_p = 15$ and $R_t = 4$. In this example, we keep the service value of passengers unchanged while reducing the service value of taxis. This make the equilibrium $(1,1,1)$ no longer exist since taxis expect a negative payoff when they join the system at full rate. Instead, there exists an equilibrium at $(0, 0.9099, 0.5499)$, at which type-1 passengers choose not to join the system at all, while both type-2 passengers and taxis join the system at a rate smaller than the corresponding potential rate.

**Example 3.** Assume $R_p = 2.5$ and $R_p = 4$. In this example, we found the equilibrium $(0,1,1)$, meaning that type-1 passengers choose not to join the system at all, while both type-2 passengers and taxis join the system at full potential rates.



## 6 CONCLUSIONS

This paper examined the variations in social welfare and waiting times of agents in a taxi-passenger system with respect to changes in joining rates and system parameters. The derivation of such performance measures provided a basis for further optimization of the system and the identification of equilibrium joining rates when agents are strategic. We derived different patterns of Nash equilibria and showed that multiple equilibria may simultaneously exist in specific numerical examples.

Future work may consider a solution for the social welfare optimization problem on three decision variables corresponding to the joining probabilities of three populations of agents. The results of such investigations provides for the proposal of socially optimal policies.

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## APPENDIX

$$\mathcal{A}^{(K)}, \mathcal{B}^{(K)}, \mathcal{C}^{(K)} \in \mathbb{M}\Big(\frac{(S+1)(S+2)}{2} + (K-S)(S+1), \frac{(S+1)(S+2)}{2} + (K-S)(S+1)\Big)$$

such that

$$\mathcal{C}^{(K)} = diag(\lambda, \lambda, ..., \lambda);$$

$$\mathcal{A}^{(K)}\Big(\frac{(S+1)(S+2)}{2} + i(S+1) + j, \frac{S(S+1)}{2} + i(S+1) + j\Big) = \alpha(j-1)\mu_1 + (1-\alpha)(S-(j-1))\mu_2,$$

$$\mathcal{A}^{(K)}\Big(\frac{(S+1)(S+2)}{2} + i(S+1) + j, \frac{S(S+1)}{2} + i(S+1) + (j+1)\Big) = \alpha(S-(j-1))\mu_2,$$

$$\mathcal{A}^{(K)}\Big(\frac{(S+1)(S+2)}{2} + i(S+1) + (j+1), \frac{S(S+1)}{2} + i(S+1) + j\Big) = (1-\alpha)j\mu_1,$$

$$\mathcal{B}^{(K)}\Big(\frac{S(S+1)}{2} + i(S+1) + j, \frac{(S+1)(S+2)}{2} + i(S+1) + (j+1)\Big) = \lambda_t,$$

for $i = 0, 1, ..., K-S$ and $j = 1, 2, ..., S+1$;

$$\mathcal{A}^{(K)}\Big(\frac{(i+1)(i+2)}{2} + j, \frac{i(i+1)}{2} + j\Big) = (i-(j-1))\mu_2,$$

$$\mathcal{A}^{(K)}\Big(\frac{(i+1)(i+2)}{2} + (j+1), \frac{i(i+1)}{2} + j\Big) = j\mu_1,$$

$$\mathcal{B}^{(K)}\Big(\frac{i(i+1)}{2} + j, \frac{(i+1)(i+2)}{2} + j\Big) = (1-\alpha)\lambda_t,$$

$$\mathcal{B}^{(K)}\Big(\frac{i(i+1)}{2} + j, \frac{(i+1)(i+2)}{2} + (j+1)\Big) = \alpha\lambda_t,$$

for $i = 0, 1, ..., S-1$ and $j = 1, 2, ..., i+1$.

For $n < K$,

$$\mathcal{C}^{(n)} \in \mathbb{M}\Big(\frac{(n+1)(n+2)}{2} + (K-n)(n+1), \frac{(n+1)(n+2)}{2} + (K-n)(n+2)\Big),$$

for $n \geq 1$, and

$$\mathcal{A}^{(n)} \in \mathbb{M}\left(\frac{(n+1)(n+2)}{2} + (K-n)(n+1), \frac{n(n+1)}{2} + (K-(n-1))n\right),$$

$$\mathcal{B}^{(n)} \in \mathbb{M}\left(\frac{(n+1)(n+2)}{2} + (K-n)(n+1), \frac{(n+1)(n+2)}{2} + (K-n)(n+1)\right),$$

such that

$$C^{(n)}(i,i) = \lambda$$

for $i = 1,2,..., \frac{(n+1)(n+2)}{2}$;

$$C^{(n)}\left(\frac{(n+1)(n+2)}{2} + i(n+1) + j, \frac{(n+1)(n+2)}{2} + i(n+2) + j\right) = (1-\alpha)\lambda,$$

$$C^{(n)}\left(\frac{(n+1)(n+2)}{2} + i(n+1) + j, \frac{(n+1)(n+2)}{2} + i(n+2) + (j+1)\right) = \alpha\lambda,$$

$$\mathcal{A}^{(n)}\left(\frac{(n+1)(n+2)}{2} + i(n+1) + j, \frac{n(n+1)}{2} + i(n+1) + j\right) = (n-(j-1))\mu_2,$$

$$\mathcal{A}^{(n)}\left(\frac{(n+1)(n+2)}{2} + i(n+1) + (j+1), \frac{n(n+1)}{2} + i(n+1) + j\right) = j\mu_1,$$

$$\mathcal{B}^{(n)}\left(\frac{n(n+1)}{2} + i(n+1) + j, \frac{(n+1)(n+2)}{2} + i(n+1) + (j+1)\right) = \lambda_t,$$

for $i = 0,1,...,K-n$ and $j = 1,2,...,n+1$;

$$\mathcal{A}^{(n)}\left(\frac{(i+1)(i+2)}{2} + j, \frac{i(i+1)}{2} + j\right) = (i-(j-1))\mu_2,$$

$$\mathcal{A}^{(n)}\left(\frac{(i+1)(i+2)}{2} + (j+1), \frac{i(i+1)}{2} + j\right) = j\mu_1,$$

$$\mathcal{B}^{(n)}\left(\frac{i(i+1)}{2} + j, \frac{(i+1)(i+2)}{2} + j\right) = (1-\alpha)\lambda_t,$$

$$\mathcal{B}^{(n)}\left(\frac{i(i+1)}{2} + j, \frac{(i+1)(i+2)}{2} + (j+1)\right) = \alpha\lambda_t,$$

for $i = 0,1,...,n-1$ and $j = 1,2,...,i+1$; and $n \geq 1$.

Finally,

$$B^{(0)}(i,i) = -\sum_j A^{(0)}(i,j) - \sum_{j \neq i} B^{(0)}(i,j),$$

and

$$B^{(n)}(i,i) = -\sum_j \left(A^{(n)}(i,j) + C^{(n)}(i,j)\right) - \sum_{j \neq i} B^{(n)}(i,j)$$

for $n = 1,...,K$.