NEW DERIVATION OF THE FILTER AND FIXED-INTERVAL SMOOTHER WITH CORRELATED UNCERTAIN OBSERVATIONS

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Abstract: A least-squares linear fixed-interval smoothing algorithm is derived to estimate signals from uncertain observations perturbed by additive white noise. It is assumed that the Bernoulli variables describing the uncertainty are only correlated at consecutive time instants. The marginal distribution of each of these variables, specified by the probability that the signal exists at each observation, as well as their correlation function, are known. The algorithm is obtained without requiring the state-space model generating the signal, but just the covariance functions of the signal and the additive noise in the observation equation.

1 INTRODUCTION

The problem of estimating a discrete-time signal from noisy observations in which the signal can be randomly missing is considered. To describe this situation, the observation equation is formulated multiplying the signal at any sample time by a binary random variable taking the values one and zero. So, the observation equation involves both an additive and a multiplicative noise which models the uncertainty about the signal being present or missing at each observation. It is assumed that, for each particular observation, the probability of containing the signal is known for the observer.

In many practical situations, the variables modelling the uncertainty in the observations can be assumed to be independent and, then, the distribution of the multiplicative noise is fully determined by the probability that each particular observation contains the signal. A different situation, in which the variables modelling the uncertainty are correlated at consecutive instants, is considered by Jackson and Murthy (1976) who, using a state-space approach, derived a least-squares linear filtering algorithm which allows to obtain the signal estimator at any time from those in the two preceding instants.

In the last years, the estimation problem in the aforementioned situations has been investigated under a more general approach which does not require the state-space model, but only the autocovariance function of the signal. Assuming that this function can be expressed in a semi-degenerate kernel form, algorithms with a simpler structure than the corresponding ones when the state-space model is known have been obtained for different estimation problems (see Nakamori et al. (2003a) for the linear filter and fixed-point smoother when the uncertainty is modelled by independent random variables). The situation considered by Jackson and Murthy (1976) has been also treated in Nakamori et al. (2003b) under a covariance approach and filtering and fixed-point smoothing algorithms have been derived for this uncertain observation model. The aim in this paper is to propose a fixed-interval smoothing algorithm based on covariance information for this last model.

The fixed-interval smoothing problem appears when all the measurements of the signal inside a time-interval are available before proceeding to the estimation. Fixed-interval smoothing techniques have been applied to stochastic signal processing problems

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(Ferrari-Trecate and De Nicolao, 2001, Young and Pedregal, 1999) as well as to the estimation of time-variable parameters (Young et al., 2001).

In this paper we treat the least-squares linear estimation problem and the fixed-interval smoothing algorithm is derived under an innovation approach. This approach provides an expression for the smoother as the sum of the filter and another term, uncorrelated with it, which can be obtained from a backward-time algorithm.

The filtering and fixed-interval smoothing algorithms are applied to a simulated observation model where the signal cannot be missing in two consecutive observations, situation which can be covered by the correlation form considered in the theoretical study.

2 ESTIMATION PROBLEM

We consider the least-squares (LS) linear estimation problem of a discrete-time signal from noisy uncertain observations described by

$$y(k) = \theta(k)z(k) + v(k) \qquad (1)$$

where the involved processes satisfy:

(I) The signal process $\{z(k);\ k \geq 0\}$ has zero mean and its autocovariance function is expressed in a semi-degenerate kernel form, that is,

$$K_z(k,s) = E[z(k)z^T(s)] = \begin{cases} A(k)B^T(s), & 0 \leq s \leq k \\ B(k)A^T(s), & 0 \leq k \leq s \end{cases}$$

where A and B are known $n \times M'$ matrix functions.

(II) The noise process $\{v(k);\ k \geq 0\}$ is a zero-mean white sequence with known autocovariance function, $E[v(k)v^T(s)] = R_v(k)\delta_K(k-s)$.

(III) The multiplicative noise $\{\theta(k);\ k \geq 0\}$ is a sequence of Bernoulli random variables with $P[\theta(k) = 1] = \overline{\theta}(k)$ and autocovariance function

$$K_\theta(k,s) = \begin{cases} 0, & |k-s| \geq 2 \\ E[\theta(k)\theta(s)] - \overline{\theta}(k)\overline{\theta}(s), & |k-s| < 2 \end{cases}$$

(IV) The processes $\{z(k);\ k \geq 0\}$, $\{v(k);\ k \geq 0\}$ and $\{\theta(k);\ k \geq 0\}$ are mutually independent.

The purpose is to obtain a fixed-interval smoothing algorithm; concretely, assuming that the observations up to a certain time L are available, our aim is to find recursive formulas which allow to obtain the estimators of the signal, $z(k)$, at any time $k \leq L$. For this purpose, we will use an innovation approach. If $\widehat{y}(k,k-1)$ denotes the LS linear estimator of $y(k)$ based on $\{y(1),\ldots,y(k-1)\}$, $\nu(k) = y(k) - \widehat{y}(k,k-1)$ represents the *innovation* contained in the observation $y(k)$, that is, the new information provided by $y(k)$ after its estimation from the previous observations. It is known that the LS linear estimator of $z(k)$ based on the observations $\{y(1),\ldots,y(L)\}$, which is denoted by $\widehat{z}(k,L)$, is equal to the LS linear estimator based on the innovations $\{\nu(1),\ldots,\nu(L)\}$. The advantage of considering the innovation approach to address the LS estimation problem comes from the fact that the innovations constitute a white process; then, by denoting $\Pi(i) = E[\nu(i)\nu^T(i)]$, the Orthogonal Projection Lemma (OPL) leads to

$$\widehat{z}(k,L) = \sum_{i=1}^{L} E[z(k)\nu^T(i)]\Pi^{-1}(i)\nu(i). \qquad (2)$$

In view of (2), the first step to obtain the estimators is to establish an explicit formula for the innovations, which is presented in Theorem 1. Afterwards, in the next section, we present recursive formulas for the fixed-interval smoother, $\widehat{z}(k,L),\ k < L$, including that of the filter, $\widehat{z}(k,k)$. These formulas have been derived by decomposing (2) as the filter and a correction term uncorrelated with it, and obtaining recursive expressions for both terms from the OPL.

2.1 Innovation process

When the variables $\{\theta(k);\ k \geq 0\}$ modelling the uncertainty are independent all the information prior to time k which is required to estimate $y(k)$ is provided by the one-stage predictor of the signal, $\widehat{z}(k,k-1)$. However, for the problem at hand, the correlation between $\theta(k-1)$ and $\theta(k)$, which must be considered to estimate $y(k)$, is not contained in $\widehat{z}(k,k-1)$. Concretely, as it is indicated in Theorem 1, in this case the innovation is obtained by a linear combination of the new observation, the predictor of the signal and the previous innovation.

Theorem 1. Under hypotheses (I)-(IV), the innovation process associated with the observations given in (1) satisfies

$$\begin{aligned} \nu(k) &= y(k) - \overline{\theta}(k)A(k)O(k-1) - K_\theta(k,k-1) \\ &\quad \times A(k)B^T(k-1)\Pi^{-1}(k-1)\nu(k-1), \quad k \geq 2 \\ \nu(1) &= y(1) \end{aligned}$$

where the vectors $O(k)$ are calculated from

$$\begin{aligned} O(k) &= O(k-1) + J(k)\Pi^{-1}(k)\nu(k), \quad k \geq 1 \\ O(0) &= 0 \end{aligned}$$

being

$$\begin{aligned} J(k) &= \overline{\theta}(k)\left[B^T(k) - r(k-1)A^T(k)\right] - K_\theta(k,k-1) \\ &\quad \times J(k-1)\Pi^{-1}(k-1)B(k-1)A^T(k), \quad k \geq 2 \\ J(1) &= \overline{\theta}(1)B^T(1) \end{aligned}$$

and $\Pi(k)$ the covariance matrix of the innovation, which is given by

$$\begin{aligned}\Pi(k) = {} & \overline{\theta}(k)A(k)\left[B^T(k) - \overline{\theta}(k)r(k-1)A^T(k)\right] \\ & -K_\theta^2(k,k-1)A(k)B^T(k-1) \\ & \times\Pi^{-1}(k-1)B(k-1)A^T(k) \\ & -\overline{\theta}(k)K_\theta(k,k-1)A(k)\Big[J(k-1) \\ & \times\Pi^{-1}(k-1)B(k-1) + B^T(k-1) \\ & \times\Pi^{-1}(k-1)J^T(k-1)\Big]A^T(k) + R_v(k).\end{aligned}$$

The covariance $r(k)$ of the vector $O(k)$ verifies

$$\begin{aligned}& r(k) = r(k-1) + J(k)\Pi^{-1}(k)J^T(k), \quad k \geq 1 \\ & r(0) = 0.\end{aligned}$$

3 FIXED-INTERVAL SMOOTHER

Theorem 2. Assuming hypotheses (I)-(IV), the estimators of the signal $z(k)$ from the observations $y(1), \cdots, y(L)$, with $k \leq L$, are given by

$$\begin{aligned}\widehat{z}(k,L) &= \widehat{z}(k,k) + [B(k) - A(k)r(k)]q_1(k,L) \\ &\quad +A(k)J(k)q_2(k,L), \quad k < L \\ \widehat{z}(k,k) &= A(k)O(k), \quad k \leq L\end{aligned}$$

where $q_1(k,L)$ and $q_2(k,L)$ can be recursively calculated, from $q_1(L,L) = 0$ and $q_2(L,L) = 0$, by

$$\begin{aligned}q_1(k,L) = {} & \left[I_{M'} - \Delta_1(k+1)\Pi^{-1}(k+1)J^T(k+1)\right] \\ & \times q_1(k+1,L) + \Delta_1(k+1)q_2(k+1,L) \\ & +\Delta_1(k+1)\Pi^{-1}(k+1)\nu(k+1), \quad k < L\end{aligned}$$

$$\begin{aligned}q_2(k,L) = {} & -\Delta_2(k+1)\Pi^{-1}(k+1)J^T(k+1) \\ & \times q_1(k+1,L) + \Delta_2(k+1)q_2(k+1,L) \\ & +\Delta_2(k+1)\Pi^{-1}(k+1)\nu(k+1), \quad k < L\end{aligned}$$

with

$$\Delta_1(k+1) = \overline{\theta}(k+1)A^T(k+1),$$

$$\Delta_2(k+1) = -K_\theta(k+1,k)\Pi^{-1}(k)B(k)A^T(k+1).$$

3.1 Error covariance matrices

The LS method uses the covariance matrices, $P(k,L)$, of the estimation errors to measure the goodness of the estimators. It is easy to verify that

$$P(k,L) = K_z(k,k) - E[\widehat{z}(k,L)\widehat{z}^{\,T}(k,L)].$$

Hence, using the expression given in Theorem 2 for $\widehat{z}(k,L)$ and the uncorrelation property between each $q_s(k,L)$, $s = 1,2$, and $\widehat{z}(k,k)$ we have

$$\begin{aligned}P(k,L) = {} & P(k,k) - [B(k) - A(k)r(k)]Q_1(k,L) \\ & \times[B(k) - A(k)r(k)]^T \\ & -A(k)J(k)Q_2(k,L)J^T(k)A^T(k) \\ & -[B(k) - A(k)r(k)]Q_{12}(k,L)J^T(k)A^T(k) \\ & -A(k)J(k)Q_{12}^T(k,L)[B(k) - A(k)r(k)]^T\end{aligned}$$

where $P(k,k)$, the filtering error covariance matrix, is given by

$$P(k,k) = A(k)\left[B^T(k) - r(k)A^T(k)\right], \quad k \leq L.$$

The matrices $Q_s(k,L)$, for $s = 1,2$, and $Q_{12}(k,L)$ are obtained by

$$\begin{aligned}Q_1(k,L) = {} & F(k+1)Q_1(k+1,L)F^T(k+1) \\ & +\Delta_1(k+1)\left[Q_2(k+1,L) + \Pi^{-1}(k+1)\right] \\ & \times\Delta_1^T(k+1) + F(k+1)Q_{12}(k+1,L)\Delta_1^T(k+1) \\ & +\Delta_1(k+1)Q_{12}^T(k+1,L)F^T(k+1)\end{aligned}$$

$$\begin{aligned}Q_2(k,L) = {} & \Delta_2(k+1)\Pi^{-1}(k+1)\left[J^T(k+1)\right. \\ & \times Q_1(k+1,L)J(k+1) + \Pi(k+1)\left.\right]\Pi^{-1}(k+1) \\ & \times\Delta_2^T(k+1) + \Delta_2(k+1)Q_2(k+1,L)\Delta_2^T(k+1) \\ & -\Delta_2(k+1)\Pi^{-1}(k+1)J^T(k+1)Q_{12}(k+1,L) \\ & \times\Delta_2^T(k+1) - \Delta_2(k+1)Q_{12}^T(k+1,L)J(k+1) \\ & \times\Pi^{-1}(k+1)\Delta_2^T(k+1)\end{aligned}$$

$$\begin{aligned}Q_{12}(k,L) = {} & -F(k+1)Q_1(k+1,L)J(k+1) \\ & \times\Pi^{-1}(k+1)\Delta_2^T(k+1) + \Delta_1(k+1)Q_2(k+1,L) \\ & \times\Delta_2^T(k+1) + F(k+1)Q_{12}(k+1,L)\Delta_2^T(k,k+1) \\ & -\Delta_1(k+1)Q_{12}^T(k+1,L)J(k+1)\Pi^{-1}(k+1) \\ & \times\Delta_2^T(k+1) + \Delta_1(k+1)\Pi^{-1}(k+1)\Delta_2^T(k+1)\end{aligned}$$

for $k < L$, with initial conditions $Q_s(L,L) = 0$, for $s = 1,2$, and $Q_{12}(L,L) = 0$, being

$$F(k+1) = I_{M'} - \Delta_1(k+1)\Pi^{-1}(k+1)J^T(k+1).$$

4 COMPUTER RESULTS

We consider a sequence of independent Bernoulli random variables, $\{\gamma(k);\ k \geq 0\}$, taking the value one with probability p and we define

$$\theta(k) = 1 - \gamma(k-1) + \gamma(k-1)\gamma(k), \quad k \geq 1.$$

So, the variables $\theta(k)$ are also Bernoulli random variables and, since $\theta(k)$ and $\theta(s)$ are independent for $|k-s| \geq 2$, they are uncorrelated and hypothesis (III) is satisfied. The common mean of these variables is $\overline{\theta} = 1 - p + p^2$ and its covariance function is given by

$$K_\theta(k,s) = \begin{cases} 0, & |k-s| \geq 2 \\ -(1-\overline{\theta})^2, & |k-s| < 2 \end{cases}$$

Let $z(k)$ be the signal to be estimated and $y(k)$ the observation of this signal, defined as in (1) by

$$y(k) = \theta(k)z(k) + v(k)$$

where $v(k)$ represents the measurement noise.

Since $\theta(k) = 0$ corresponds to $\gamma(k-1) = 1$ and $\gamma(k) = 0$, this fact implies that $\theta(k+1) = 1$ and, hence, the possibility of the signal being missing in two successive observations is avoided. So, the considered observation model covers those signal transmission models with stand-by sensors, in which any failure in the transmission is immediately detected and the old sensor is then replaced.

We have assumed that the autocovariance function of the signal is

$$K_z(k,s) = 1.025641 \times 0.95^{k-s}, \quad 0 \leq s \leq k.$$

The noise $\{v(k);\ k \geq 0\}$ has been assumed to be a sequence of independent random variables with

$$E[v(k)] = 0, \ \ R_v(k) = 0.7037037.$$

To show the effectiveness of the algorithm proposed in this paper, we compare the results obtained from 100 observations of the signal, using different values of the parameter p.

First, the performance of the filter and fixed-interval smoother, measured by the error variances, has been calculated for $p = 0.1,\ 0.3$ and 0.5. The results are displayed in Figure 1 which shows that the estimators have a better performance as p is smaller, due to the fact that the mean value, $\overline{\theta}$, decreases with p. Moreover, this figure shows not only that, for each value of p, the error variances are smaller using the fixed-interval smoother instead of the filter, but also that the improvement with the smoother is highly significant since, even the worst results with the smoother ($p = 0.5$) are better than the best ones with the filter ($p = 0.1$). A simulated signal and

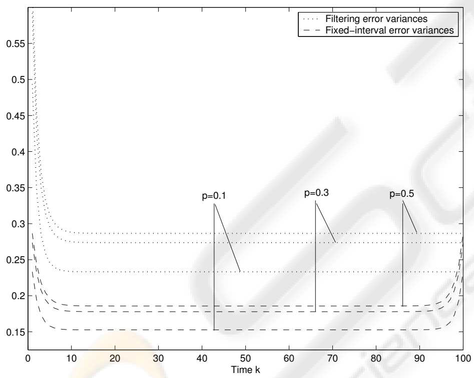

Figure 1: Filtering and fixed-interval smoothing error variances for $p = 0.1,\ 0.3,\ 0.5$.

their filtered and smoothed estimates from 100 observations simulated with $p = 0.1$ are displayed in Figure 2. The result, as expected, is that the smoothing estimates are nearer to the signal and, hence, the behaviour of the fixed-interval smoother is better than that of the filter.

5 CONCLUSION

In this paper, the LS linear fixed-interval smoother is derived from uncertain observations of a signal, when

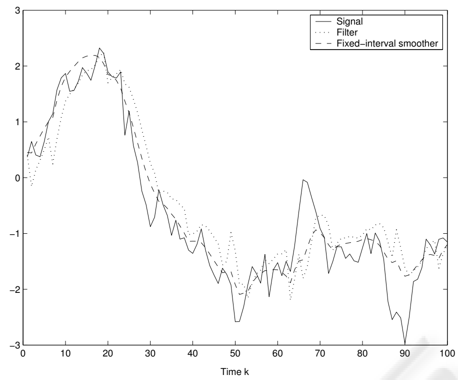

Figure 2: Signal, filtering and fixed-interval smoothing estimates for $p = 0.1$.

the Bernoulli random variables characterizing the uncertainty in the observations are correlated at consecutive time instants, for the case of white observation additive noise. It is not required the knowledge of the state-space model, but only the covariance matrices of the processes involved in the observation equation. The recursive algorithms are derived by an innovation approach.

The results are applied to a particular model which includes signal transmission models with stand-by sensors for the immediate replacement of a failed unit.

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