SIMULATION OF SYSTEMS WITH VARIOUS TIME DELAYS USING PADE'S APPROXIMATION

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Abstract: In this paper, Pade's rational functions have been simulated for approximating several characteristic values of time delay regarding the plant time constant. Several representative plants were tested in order to show in which cases Pade's function approximates time-delay block well. Only if the ratio of time delay versus time constant of the plant is rather great, or the plant contains emphasized numerator dynamics; approximation capabilities get poorer. The convergence rate of n-order Pade's function has been also analyzed by using Taylor series and phase-frequency characteristics.

1 INTRODUCTION

Many of industrial processes and process control systems, along with their structural presentations, contain one or more time delay components (Dugard, Verriest, 1998; Chen, *et al*., 2003). These make an inherent part of the mathematical models used to describe the systems' dynamics of management and biological systems as well. Padé approximations are widely used to approximate a dead-time in continuous control systems (Vajta, 2000). It provides a finite-dimensional rational approximation of a dead-time. The accuracy of applied time delay blocks is particularly important in computer simulation of complex dynamic systems, described by high-order equations, then in computation of convolution integrals etc (Beek, *et al*., 1999). The principal problem in their realization is that their transfer function appears in transcendent form, what is not quite appropriate for simulation (Hebibovic, 1991; Vajta, 2000). In order to avoid the problem, it has long been the practice to approximate the time delay transfer function with a rational function (Hebibovic, 1998).

In this paper MATLAB/Simulink features were exploited and comparison of the first four Pade's functions has been done regarding several typical plants and typical ratios of time-delay versus plant time constant. Convergence that can be seen well from simulations is supported by theoretical analysis using Taylor series.

2 SYSTEM DESCRIPTION

Let's consider a time function u(t) as an input of the time delay block. The output of this system is the same function, but with time delay τ, which can be described by Eq.1.

$$x(t) = u(t-\tau) \tag{1}$$

Time delay block can be described in Laplace form which can be derived from Eq.1.

$$G(s) = \frac{X(s)}{U(s)} = \frac{e^{-\tau s}U(s)}{U(s)} = e^{-\tau s} \tag{2}$$

Pade's approximation of time delay block is very favorable in practice because of good convergence rate of this approximation. It is also very interesting theoretical case when Pade's approximation order reaches infinity.

Pade's function is a rational function determined by Eqs 3-5 (Hebibovic, 1998).

$$W_{\mu\nu}(-\tau s) = \frac{N_{\mu\nu}(-\tau s)}{D_{\mu\nu}(-\tau s)} \tag{3}$$

$$N_{\mu\nu}(-\tau s) = \sum_{i=1}^{\nu} \frac{\nu!}{(\nu-i)!} \frac{(\mu+\nu-i)!}{i!(\mu+\nu)!} (-\tau s)^{i} \tag{4}$$

$$D_{\mu\nu}(-\tau s) = \sum_{j=1}^{\nu} \frac{\mu!}{(\mu-j)!} \frac{(\mu+\nu-j)!}{j!(\mu+\nu)!} (\tau s)^{j} \tag{5}$$

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For practical use, Pade's functon takes form given by Eq.6 ($\mu = \nu = n$)

$$W_{nn}(-\tau s) = \frac{b_{n0} + b_{n1}s + b_{n2}s^2 + \ldots + b_{nn}s^n}{a_{n0} + a_{n1}s + a_{n2}s^2 + \ldots + a_{nn}s^n} \tag{6}$$

It's obvious that coefficients a_{ni} and b_{ni} are functions of time delay τ.

$$a_{ni} = \frac{n!(2n-i)!}{i!(2n)!(n-i)!}\tau^i,$$
$$b_{ni} = (-1)^i a_{ni}, \quad i = 0,1,2,\ldots,n \tag{7}$$

Number n determines the order of Pade's function. It can be seen that Pade's coefficients can be easily arranged into a square matrix. Eqs 8-10 represent some interresting relations between adjacent members of a matrix. These relations can simplify software evaluation of Pade's coefficients (Hebibovic, 1991).

$$\frac{a_{n,i+1}}{a_{n,i}} = \frac{n-i}{(i+1)(2n-i)}\tau, \quad m = 0,1,2,\ldots,n-1 \tag{8}$$

$$\frac{a_{n+1,i}}{a_{n,i}} = \frac{(2n-i+1)(2n-i+2)}{2(2n+1)(n-i+1)}, \quad m = 0,1,\ldots,n \tag{9}$$

$$\frac{a_{n+1,i+1}}{a_{n,i}} = \frac{(2n-i+1)}{2(2n+1)(i+1)}\tau, \quad m = 0,1,\ldots,n \tag{10}$$

3 MAGNITUDE AND PHASE OF n-ORDER PADE'S FUNCTION

Amplitude-frequency characteristic of time delay block can be perfectly approximated by amplitude-frequency characteristic of Pade's function. Beside that, phase-frequency characteristic of Pade's approximation converges to phase-frequency characteristic of time delay block, as order of approximation reaches infinity (Titov, Uspenskij 1969; Doganovskij, Ivanov 1966.)

If "s" from Eq.6 gets replaced with "jω", magnitude and phase of observed function can be easily determined for every number n that represents order of approximation. All Pade's functions, represented with Eq.6 have following form (Hebibovic, 1998).

$$W_{nn}(-j\omega\tau) = \frac{R_{nn} - jI_{nn}}{R_{nn} + jI_{nn}}, \quad n = 1,2,3,\ldots \tag{11}$$

It is obvious that magnitude of n-order Pade-s function is equal to 1 (Eq.12)

$$A_{nn}(\omega\tau) = |W_{nn}(-j\omega\tau)| = 1, \quad n = 1,2,3,\ldots \tag{12}$$

Unfortunately, this is not the case for phase-frequency characteristic (Eqs.13-14)

$$\arg(e^{-j\omega\tau}) = -\omega\tau \tag{13}$$

$$\varphi_{nn}(\omega\tau) = \arg(W_{nn}(-j\omega\tau)) \neq -\omega\tau \tag{14}$$

By increasing order of Pade-s function, equations for phase-frequency characteristic calculation become more complex, and characteristic itself converges to phase-frequency characteristic of pure time delay block (Fig.1, $wT \equiv \omega\tau$).

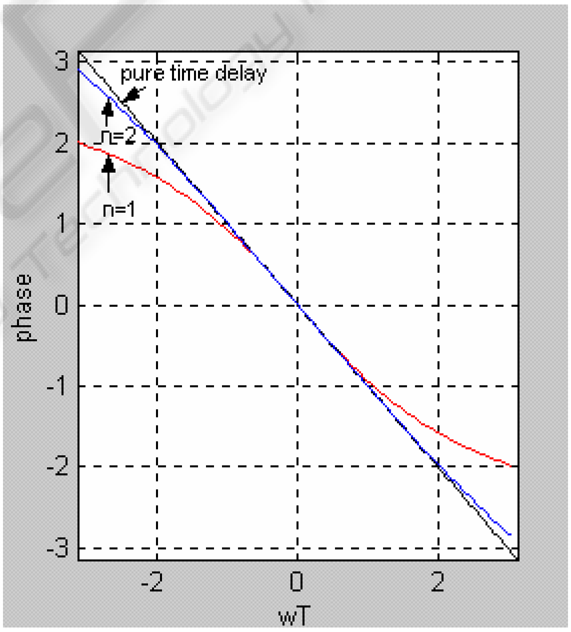

Figure 1: Phase-frequency characteristics of time delay block and Pade's functions

The convergence rate of Pade's function to transfer function of time delay block can also be seen from corresponding Taylor series (Eqs 15-17).

$$e^{-\tau s}=1-\tau s+\frac{\tau^2s^2}{2}-\frac{\tau^3s^3}{6}+\frac{\tau^4s^4}{24}-\frac{\tau^5s^5}{120}+\frac{\tau^6s^6}{720}-\frac{\tau^7s^7}{5040}+\frac{\tau^8s^8}{40320}-\frac{\tau^9s^9}{362880}+\frac{\tau^{10}s^{10}}{3628800}+O_1(e^{-\tau s}) \quad (15)$$

$$W_{22}(-\tau s)=1-\tau s+\frac{\tau^2s^2}{2}-\frac{\tau^3s^3}{6}+\frac{\tau^4s^4}{24}-\frac{\tau^5s^5}{144}+\frac{\tau^7s^7}{1728}-\frac{\tau^8s^8}{3456}+\frac{\tau^9s^9}{10368}-\frac{\tau^{10}s^{10}}{41472}+O_2(W_{22}(-\tau s)) \quad (16)$$

$$W_{55}(-\tau s)=1-\tau s+\frac{\tau^2s^2}{2}-\frac{\tau^3s^3}{6}+\frac{\tau^4s^4}{24}-\frac{\tau^5s^5}{120}+\frac{\tau^6s^6}{720}-\frac{\tau^7s^7}{5040}+\frac{\tau^8s^8}{40320}-\frac{\tau^9s^9}{362880}+\frac{\tau^{10}s^{10}}{3628800}+O_3(W_{55}(-\tau s)) \quad (17)$$

Let k be the number of elements in Taylor's sum of $W_{nn}(-\tau s)$, which are identical to corresponding elements of Taylor's sum of $e^{-\tau s}$. Validity of Eq.18 that binds number k with Pade's function order n can easlily be shown (Hebibovic, 1998).

$$k=2n+1 \quad (18)$$

Hence, if n reaches infinity ($n\to\infty$), than all corresponding elements of two Taylor's sums are identical. Therefore, the next equation can be written:

$$e^{-\tau s}=\lim_{n\to\infty}W_{nn}(-\tau s) \quad (20)$$

In this case, coefficients of Pade's polynoms can be calculated by the following theorem (Hebibovic, 1998).

Theorem: For the order n of Pade's function given by the Eq.6 it can be written:

$$a_{\infty m}=\lim_{n\to\infty}a_{nm}=\frac{2^{-m}}{m!}\tau^m,\quad m=0,1,2,\ldots, \quad (21)$$

Consequences of this theorem are that by increasing the order of Pade's function to infinity, perfect approximation of time delay block by magnitude and phase is obtained. However, this approximation can be considered only in domain of theory. In most practical problems, Pade's function with order up to four can satisfy.

4 SIMULATION RESULTS

It is interesting to compare the step response of block that contains pure time delay to step reponse of block whose time delay sub-block is approximated with n-order Pade's function. Simulations of plants with various τ/T quotients have been run, where T represents the time constant of the process.
The simulation results are illustrated on Figures 2-9 that presented remarkable feature of Pade's approximation. Figures 2-4 show the results obtained with the first-order static plant including pure time delay. It is obvious that approximation is better when τ/T is smaller.
The same procedure has been done with an astatic first-order plant with several time-delays (Figures 5-7). The quality of approximation also increases as τ/T decreases.
Figures 8 and 9 illustrate time responses of somewhat more complex plant that includes time-delay. Time response of this transfer function corresponds to many physiological processes, for example blood glucose component that is depended on stress (Hebibovic, *et al.*, 2003). Empirically, it can be concluded that the quality of approximation increases as expression T_1T/T_d also increases.

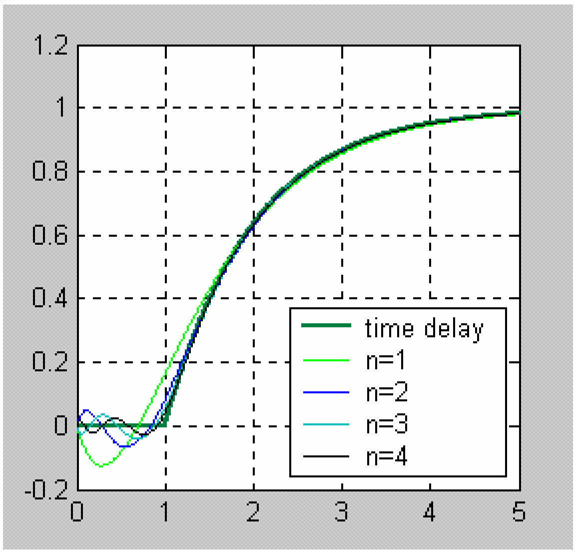

Figure 2: Plant $G(s)=e^{-\tau s}\dfrac{K}{Ts+1}$, ($\tau/T=1$)

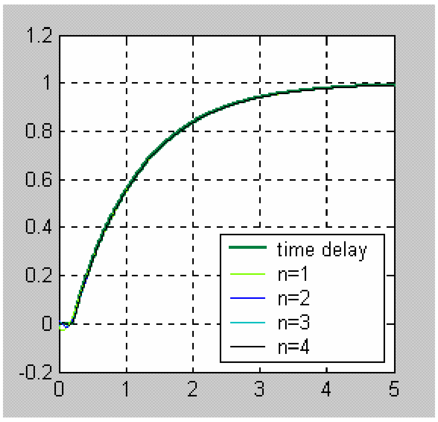

Figure 3: Plant $G(s) = e^{-\tau s}\dfrac{K}{Ts+1}$, ($\tau/T$=0.2)

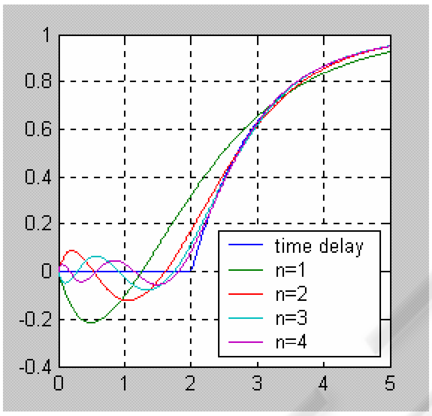

Figure 4: Plant $G(s) = e^{-\tau s}\dfrac{K}{Ts+1}$, ($\tau/T$=2)

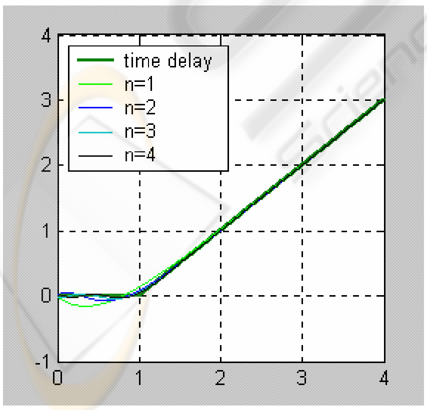

Figure 5: Plant $G(s) = e^{-\tau s}\dfrac{1}{Ts}$, ($\tau/T$=1)

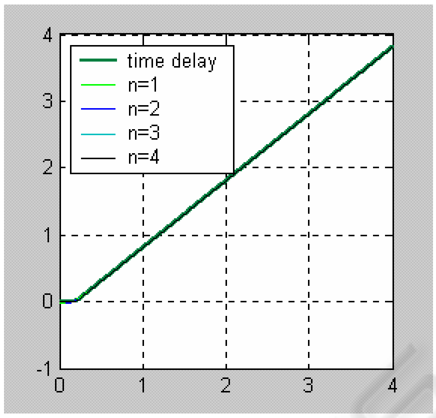

Figure 6: Plant $G(s) = e^{-\tau s}\dfrac{1}{Ts}$, ($\tau/T$=0.2)

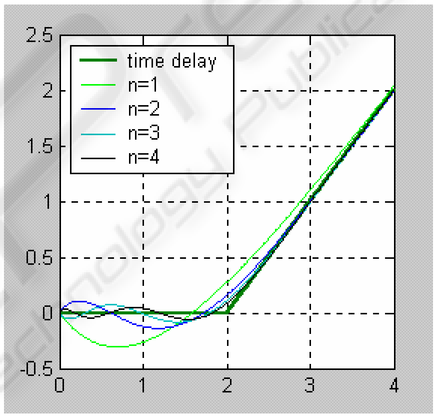

Figure 7: Plant $G(s) = e^{-\tau s}\dfrac{1}{Ts}$, ($\tau/T$=2)

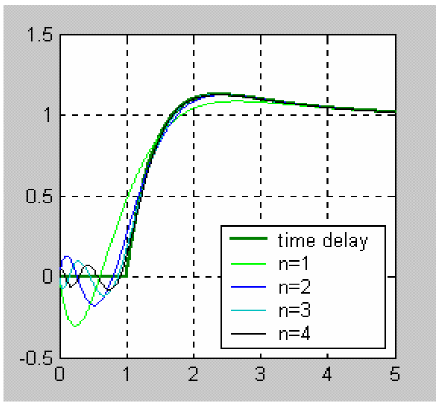

Figure 8: Plant $G(s) = e^{-\tau s}\dfrac{K}{T_1 s+1}\left(1+\dfrac{T_d s}{Ts+1}\right)$,

($T_1 T/T_d \approx \tau$)

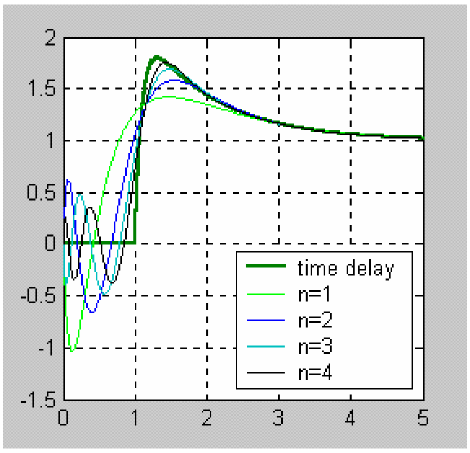

Figure 9: Plant $G(s) = e^{-\tau s} \dfrac{K}{T_1 s + 1}\left(1 + \dfrac{T_d s}{Ts + 1}\right)$,

$(T_1 T / T_d << \tau)$

5 CONCLUSIONS

Simulations have shown remarkable features of Pade's approximation. It can be seen that the time response of the plant that contains fourth order Pade's function fits very well the time response of the plant with pure time delay. This is common for typical static and astatic industrial processes with somewhat smaller time delays. However, the quality of approximation has certain limits. Simulations show that Pade's approximation doesn't give satisfactory results for systems with greater time delays and /or emphasized derivative time constants. Future work will explore some different models of rational functions for approximation of time-delay, especially within the systems where Pade's function didn't show good performance.

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