ADAPTIVE MIMO MULTI-PERIODIC REPETITIVE CONTROL SYSTEM: LIAPUNOV ANALYSIS

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Abstract: This paper presents a simple feed forward adaptive multi-periodic repetitive control scheme for the ASPR(Almost Strictly Positive Real) or ASNR(Almost Strictly Negative Real, See Appendix for definition) plant to asymptotically track/reject multi-periodic reference/disturbance signals. The lyapunov stability analysis is given. This is an extension work of the lyapunov stability analysis for multi-periodic repetitive control system under a positive real condition. A simulation is included. The extension of the lyapunov stability analysis to ASPR or ASNR plant under certain non-linear perturbations and an exponential stability scheme are discussed as well. Finally an adaptive proportional plus multi-periodic repetitive control scheme is proposed.

1 INTRODUCTION

For a system to track/reject periodic reference/disturbance signal, repetitive control was developed several years ago. This control method, which is based on the internal model principle, has proven to be very effective in practical applications. In most existing repetitive control approaches (Dixon W. E., 2001; Hara S. and M., 1988; Horowitz R., 1991; Jiang Y.A., 1995; Owens D.H. and S.P., 2002; Owens D.H. and S.P., 2003), the asymptotic convergence of the state to the origin and internal stability of the system are guaranteed under some strict assumption on the dynamic system. Hara derived the sufficient conditions for the stability of repetive and modified repetitive control systems by applying the small gain theorem and the stability theorem for time-lag systems. It is shown that the plant $P(s)$ should satisfy $\|f(s)(1 - P(s))\|_\infty < 1$ where $f(s)$ is a low-pass filter introduced to improve the system stability at a cost of losing tracking accuracy at high frequencies. Owens et al gave the lyapunov stability analysis and proved that asymptotic/exponential stability is guaranteed if the linear plant is positive real/strictly positive real or the nonlinear plant is passive/strictly passive. Similar lyapunov stability analysis was done (Dixon W. E., 2001; Horowitz R., 1991; Jiang Y.A., 1995) and some strict assumptions, which are actually passive condition as in (Owens D.H. and S.P., 2003), were made on the nominal system of the plant. In this paper, we will alleviate such restrictive assumptions on the plant to some extent.

In many cases, the reference/disturbance periodic signals may contain different fundamental frequencies and the ratio of these frequencies can be irrational. So the so-called multi-periodic repetitive control was analysed by several authors (G., 1997; Weiss G., 1999; Owens D.H. and S.P., 2002; Owens D.H. and S.P., 2003; Li L.M. and S.P., 2002). Weiss gave a H^∞ stability condition based on input-output transfer function for linear SISO/MIMO single/multi-periodic system. The lyapunov stability analysis is given by Owens and it is studied by Li that a feed forward and feedback compensation can be employed when the real plants are not necessarily positive real. However, the method in (Li L.M. and S.P., 2002) needs some plant parameter information and such information is based on off-line frequency domain system identification of a particular system. Also the plant is restricted to be minimum phase, strictly proper and with relative degree one and positive high-frequency gain, which actually is an ASPR plant.

Adaptive repetitive control design and implementation, which includes internal model principle, have

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been discussed by many authors (G, 1996; Jiang Y.A., 1995; Sun Z., 2000; Tomizuka M., 1989; Tsao T., 1994; Tzou Y., 1999; X.D. and J.P., 1998) both in the discrete-time and continuous-time domain. Most of them (G, 1996; Sun Z., 2000; Tomizuka M., 1989; Tsao T., 1994; Tzou Y., 1999) are indirect adaptive control algorithms. Several estimation algorithms were used to identify the plant models and certainty equivalence principles were applied to design the adaptive control schemes. On the other hand, Jiang gave a direct adaptive control scheme and applied an adaptively adjusted gain in the feedback controller when the upper bound of the plant uncertainty exists, however unknown. Ye designed a global adaptive control of a class of nonlinear systems when the signs of certain system parameters are unknown for learning control system.

In this paper we will use the non-identifier-based direct adaptive control technique (A., 1993) to design adaptive controllers for a class of ASPR or ASNR MIMO LTI systems, which actually are minimum-phase, with relative degree m and unknown high-frequency gain, to track/reject multi-periodic reference/disturbance signals. The Lyapunov stability analysis is applied.

The adaptive MIMO multi-periodic repetitive control system is shown in Figure 1. The R, D, Y, U, E are

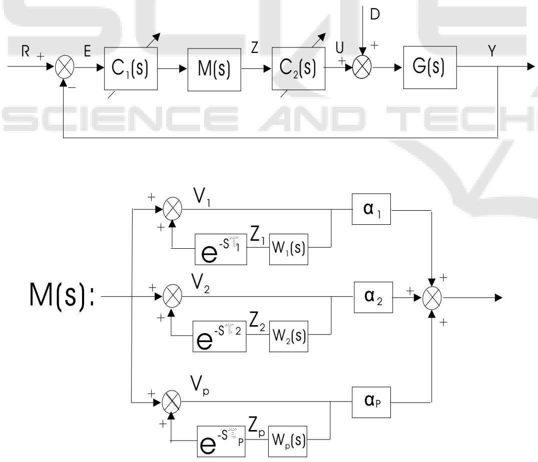

Figure 1: Adaptive MIMO multi-periodic repetitive control system

reference, disturbance, output, control input and error respectively. The plant $\sum_G$ is finite-dimensional, linear time-invariant and described by

$$\begin{array}{l} \dot{x}(t) = Ax(t) + B(u(t) + d(t)) \\ y(t) = Cx(t), x(0) = x_0 \end{array} \tag{1}$$

where $x(t) \in R^n, u(t) \in R^m, y(t) \in R^m$ and the dimensions of constant matrices A, B, C are $n \times n, n \times m, m \times n$ respectively. Both reference $r(t)$ and disturbance $d(t)$ are multi-periodic with components of period $\tau_i, i = 1, ..., p$. These periods are assumed known. The multi-periodic repetitive controller is $M(s) = \sum_{i=1}^{p} \frac{\alpha_i I}{1 - W_i(s)e^{-s\tau_i}}$, we select $\sum_{i=1}^{p} \alpha_i = 1$ without loss of generality. $W_i(s)$ is a low-pass filter. $C_1(s), C_2(s)$ are both feed forward matrix gains given in the following sections designed to guarantee the Lyapunov stability of the whole system including the plant.

The paper is organized as follows. In section 2, we introduce a simple high constant feed forward gain, which realizes the stability of the multi-periodic repetitive control system for an ASPR plant. In section 3, we adopt an adaptive feed forward gain, which alleviate the assumption made in section 2. In section 4, the general problem is solved for the ASPR or ASNR plant and here we introduce a Nussbaum-type feed forward gain. Simulation results are presented in section 5. Section 6 discusses the extension of the lyapunov analysis to the ASPR or ASNR plant under certain non-linear perturbations. Section 7 gives an exponential stabilization control scheme via exponential weighting factor. Section 8 gives an adaptive proportional plus multi-periodic repetitive control scheme. For every control schemes, the Lyapunov stability proof is given. Finally in section 9, conclusions are given.

2 STABILIZATION BY HIGH CONSTANT FEED FORWARD GAIN

In this section, we will show that applying a enough high constant feed forward gain can make the multi-periodic repetitive control system to be lyapunov stable when the plant is ASPR.

Assume the MIMO, LTI plant $\sum_G$ is ASPR, that is there exists an unknown constant matrix $\lambda^* \in R^{m \times m}$ such that the closed-loop system $(A - B\lambda^* C, B, C)$ satisfies the strict-positive-realness conditions, that is

$$\begin{array}{l} P(A - B\lambda^* C) + (A - B\lambda^* C)^T P < -Q \\ PB = C^T \end{array} \tag{2}$$

where the P, Q are positive definite matrix. An ASPR plant $G(s)$ has a strictly minimum-phase $m \times m$ transfer matrix of relative degree m(n poles and $n - m$ zeros). If $G(s)$ has the minimal realization (A, B, C), then $CB > 0$(positive definite).

Theorem 1 *Consider the ASPR system $\sum_G$ described by (1). Suppose that both reference $r(t)$ and disturbance $d(t)$ are identically zero. The feed forward gain $C_1(s) = k\Gamma$ and $C_2(s) = \Gamma$, where k is a positive constant selected to be larger than $\gamma := \|{\lambda^*}^T + \lambda^*\|$ and $\Gamma \in R^{m \times m}$ is a matrix such that $\Gamma + \Gamma^T > 0$. Γ is selected to be $I_{m \times m}$ without* *loss of generality. Then the multi-periodic repetitive system in Figure 1 is globally asymptotically stable in the sense that the state $x(.) \in L_\infty^n[0,\infty)$, control signal $v_i(.) \in L_2^m[0,\infty)$, and output $y(.) \in L_2^m[0,\infty)$.*

Proof: Assume

$$\begin{aligned}\dot{x}_{W_i}(t) &= A_{W_i}x_{W_i}(t) + B_{W_i}v_i(t)\\ z_i(t) &= C_{W_i}x_{W_i}(t)\end{aligned} \tag{3}$$

is a minimal realization of strictly bounded real $W_i(s)$. Then according to Corollary 1 and the inequality (10) in (Owens D.H. and S.P., 2002), we have $(x_{W_i}^T P_{W_i} x_{W_i})' \le \mu^2 v_i^T v_i - z_i^T z_i$, where $0 < \mu < 1$ is a constant. Introduce a positive definitive Lyapunov function V of the form

$$V = x^T Px + \frac{1}{k}\sum_{i=1}^{p}\alpha_i\left(\int_{t-\tau_i}^{t}\|z_i(\theta)\|^2 d\theta + x_{W_i}^T P_{W_i} x_{W_i}\right) \tag{4}$$

The system (1) can be rewritten as follows:

$$\begin{aligned}\dot{x}(t) &= (A - B\lambda^* C)x(t) + Bv(t) + B\lambda^* y(t)\\ y(t) &= Cx(t), z(t) := \textstyle\sum_{i=1}^{p}\alpha_i v_i\end{aligned} \tag{5}$$

By differentiating V along (5), using (2) and $z_i(t-\tau_i) = v_i(t) + k(t)y(t)$, we have

$$\frac{dV}{dt} < -x^T Qx - (k-\gamma)y^T y - \frac{1-\mu^2}{k}\sum_{i=1}^{p}\alpha_i v_i^T v_i \tag{6}$$

Integrating (6) and using (4) and the positivity of V yield

$$\begin{aligned}V(0) > V(t) &+ \int_0^t x^T Qx dt + \int_0^t (k-\gamma)\|y\|^2 dt\\ &+ \int_0^t \frac{1-\mu^2}{k}\sum_{i=1}^{p}\alpha_i v_i^T v_i dt\end{aligned} \tag{7}$$

from which $x(.) \in L_\infty^n[0,\infty)$,$v_i(.) \in L_2^m[0,\infty)$ and $y(.) \in L_2^m[0,\infty)$, which proves the result. □

3 STABILIZATION BY ADAPTIVE FEED FORWARD GAIN

In section 2, we assume γ is known. It is a restrictive assumption which will be excluded in this section by applying an adaptive feed forward gain $k(t)$.

Theorem 2 *Consider the ASPR system $\sum_G$ described by (1). Suppose that both reference $r(t)$ and disturbance $d(t)$ are identically zero. The feedforward gain $C_1(s) = k(s)\Gamma$ and $C_2(s) = \Gamma$, where $k(t)$ is an adaptive scale gain with adaptive law $\dot{k}(t) = e^T(t)e(t), k(0) > 0$, $\Gamma = I_{m\times m}$. Then the adaptive multi-periodic repetitive system in Figure 1 is globally asymptotically stable in the sense that $x(.) \in L_\infty^n[0,\infty)$, $v_i(.) \in L_2^m[0,\infty)$, $y(.) \in L_2^m[0,\infty)$, $k(.) \in L_\infty[0,\infty)$ and $\lim_{t\to\infty} k(t) = k_\infty < \infty$.*

Proof: The proof is an extension of that in section 2. By differentiating (4) we have

$$\begin{aligned}\frac{dV}{dt} &< -x^T Qx - (k-\gamma)y^T y - \frac{1-\mu^2}{k}\sum_{i=1}^{p}\alpha_i v_i^T v_i\\ &- \frac{1}{k^2}\frac{dk}{dt}\sum_{i=1}^{p}\alpha_i\left(\int_{t-\tau_i}^{t}\|z_i(\theta)\|^2 d\theta + x_{W_i}^T P_{W_i} x_{W_i}\right)\end{aligned} \tag{8}$$

Integrating (8) and using the adapting law $\dot{k}(t) = e(t)^T e(t) = y(t)^T y(t)$ yields

$$\begin{aligned}V(t') - V(0) &< -\int_0^{t'} x^T Qx dt - \int_{k(0)}^{k(t')}(\tau-\gamma)d\tau -\\ &\int_0^{t'}\frac{1}{k^2}\frac{dk}{dt}\sum_{i=1}^{p}\alpha_i\left(\int_{t-\tau_i}^{t}\|z_i(\theta)\|^2 d\theta + x_{W_i}^T P_{W_i} x_{W_i}\right)dt\\ &- \int_0^{t'}\frac{1-\mu^2}{k}\sum_{i=1}^{p}\alpha_i v_i^T v_i dt\end{aligned} \tag{9}$$

We will establish $k(t) \in L_\infty[0,t')$ by contradiction. Suppose $k(t) \notin L_\infty[0,t')$, the term $-\int_{k(0)}^{k(t')}(\tau-\gamma)d\tau = -[\frac{k(t')^2}{2} - \gamma k(t') - \frac{k(0)^2}{2} + \gamma k(0)]$ will be negative infinity. The other items of the right part of (9) are definitely negative due to $\frac{dk(t)}{dt} \ge 0$ and $0 < \mu < 1$, hence contradicting the non-negativity of the left hand side of (9). Therefore, we have $k(t) \in L_\infty[0,t')$.

When $t' = \infty$, we have $k(t) \in L_\infty[0,\infty)$. Due to the monotonic increase of $k(t)$, we have $\lim_{t\to\infty} k(t) = k_\infty < \infty$. Also we have $x(.) \in L_\infty^n[0,\infty)$, $v_i(.) \in L_2^m[0,\infty)$ and $y(.) \in L_2^m[0,\infty)$ as before, which proves the result. □

4 STABILIZATION VIA NUSSBAUM-TYPE SWITCHING

In some case, CB, which is called control direction in (X.D. and J.P., 1998), is non-zero, however not definitely positive and we don't know the sign of CB. Such plant $\sum_G$ can be called ASPR or ASNR, that is there exists an unknown positive definite matrix λ^* such that the closed-loop system $(A - \sigma B\lambda^* C, \sigma B, C)$ satisfies the strict-positive-realness conditions, where $\sigma := sign(CB)$ is assumed unknown.

Now we introduce a Nussbaum-type adaptive controller as follows:

$$u(t) = N(\lambda(t))\Gamma z(t) \tag{10}$$

$\Gamma = I_{m\times m}$, $N(.) : R \to R$ is any continuous function of Nussbaum type (R.D., 1983), that is, $N(.)$ has the properties $\sup_{k>k_0}\frac{1}{k-k_0}\int_{k_0}^{k}N(\tau)d\tau = +\infty$ and $\inf_{k>k_0}\frac{1}{k-k_0}\int_{k_0}^{k}N(\tau)d\tau = -\infty$. For example, $N(.) : \tau \to \tau^2\cos\tau$ suffices.

Theorem 3 *Consider the ASPR or ASNR system $\sum_G$ described by (1). The feedforward gain $C_1(s) =$*

$k(s)\Gamma$ and $C_2(s) = N(s)\Gamma$, where $k(t)$ and $\lambda(t)$ are both adaptive scalar gains with adaptive law $\dot{k}(t) = e(t)^T e(t), k(0) > 0$ and $\dot{\lambda}(t) = e(t)^T z(t), \lambda(0) \geq 0$. Then the adaptive multi-periodic repetitive system in Figure 1 is globally asymptotically stable in the sense that $x(.) \in L_\infty^n[0,\infty)$, $e(.) \in L_2^m[0,\infty)$, $\lambda(.) \in L_\infty[0,\infty)$, $k(.) \in L_\infty[0,\infty)$ and $\lim_{t\to\infty} k(t) = k_\infty < \infty$.

Proof: We set the low-pass filter $W_i(s)$ to be 1. The system can be rewritten as follows:

$$\begin{array}{l} \dot{x}(t) = (A - \sigma B\lambda^* C)x(t) + B[N(\lambda)z(t) + d(t)] \\ +\sigma B\lambda^* y(t) \\ y(t) = Cx(t), \quad z(t) = \sum_{i=1}^p \alpha_i z_i(t) \end{array} \tag{11}$$

Also due to the minimum phase property of $\sum_G$, there exists an invariant set, made up of periodic trajectoris vanishing with $r(t)$ and $d(t)$, which is contained in the ker of the output. That is, if the control input u_∞ is carefully selected under some state x_∞, the output of the system output y_∞ will be r. So we have

$$\begin{array}{l} \dot{x}_\infty(t) = Ax_\infty(t) + Bu_\infty(t) \\ r(t) = Cx_\infty(t) \end{array} \tag{12}$$

Then we define $e(t) := r(t) - y(t), e_x(t) := x_\infty(t) - x(t)$, we have

$$\begin{array}{l} \dot{e}_x(t) = (A - \sigma B\lambda^* C)e_x(t) + \sigma B\lambda^* e(t) \\ -BN(\lambda)z(t) - Bd(t) + Bu_\infty(t) \\ e(t) = Ce_x(t) \end{array} \tag{13}$$

Similar to $d(t) = \sum_{i=1}^p \alpha_i d_i(t)$, we have $u_\infty(t) = \sum_{i=1}^p \alpha_i u_{\infty i}(t)$. Introducing a positive definite lyapunov function V:

$$\begin{array}{l} V = e_x^T P e_x + \frac{1}{k}\sum_{i=1}^p \alpha_i \int_{t-\tau_i}^t \|\widetilde{z}_i(\theta)\|^2 d\theta \\ \widetilde{z}_i(\theta) := z_i(\theta) - \sigma u_{\infty i}(\theta) + \sigma d_i(\theta) \end{array} \tag{14}$$

By differentiating V, we have

$$\begin{array}{l} \frac{dV}{dt} < -e_x^T Q e_x - (2\sigma N(\lambda) - 2)z^T e - (k-\gamma)e^T e \\ \quad - \frac{1}{k^2}\frac{dk}{dt}\sum_{i=1}^p \alpha_i \int_{t-\tau_i}^t \|\widetilde{z}_i(\theta)\|^2 d\theta \end{array} \tag{15}$$

Integrating (15) and using law $\dot{k}(t) = e(t)^T e(t)$, $\dot{\lambda}(t) = e(t)^T z(t)$ yield

$$\begin{array}{l} V(t') - V(0) < -\int_0^{t'} e_x^T Q e_x dt - \int_{k(0)}^{k(t')}(\tau - \gamma)d\tau \\ -\int_{\lambda(0)}^{\lambda(t')}(2\sigma N(\tau) - 2)d\tau \\ -\int_0^{t'} \frac{1}{k^2}\frac{dk}{dt}\sum_{i=1}^p \alpha_i \int_{t-\tau_i}^t \|\widetilde{z}_i(\theta)\|^2 d\theta dt \end{array} \tag{16}$$

Suppose $\lambda(t) \notin L_\infty[0,t')$, $k(t) \notin L_\infty[0,t')$, the term $-\int_{\lambda(0)}^{\lambda(t')}(2\sigma N(\tau) - 2)d\tau$ will take arbitrary large negative or positive value when $\lambda(t') = \infty$ according to Theorem A.1 in Appendix. For example, if we select $N(\lambda) = \lambda^2\cos\lambda$ and $\lambda(0) = 0$ without loss of generality, then we have $-\int_0^{\lambda(t')}(2\sigma N(\tau) - 2)d\tau = -2\sigma[\lambda(t')^2\sin\lambda(t') + 2\lambda(t')\cos\lambda(t') - 2\sin\lambda(t')] + 2\lambda(t')$ and it will take arbitrary large negative or positive value when $\lambda(t') = \infty$. So when it takes arbitrary large negative, the right hand side of (16) will be negative, hence contradicting the non-negativity of the left hand side of (16). Therefore, we have $\lambda(t) \in L_\infty[0,t')$, $k(t) \in L_\infty[0,t')$.

When $t' = \infty$, we have $\lambda(t) \in L_\infty[0,\infty)$, $k(t) \in L_\infty[0,\infty)$. As in section 3, we have $\lim_{t\to\infty} k(t) = k_\infty < \infty$, $x(t) \in L_\infty^n[0,\infty)$ and $e(t) \in L_2^m[0,\infty)$, which proves the result. □

It should be pointed out that we can't prove that $\lim_{t\to\infty}\lambda(t) = \lambda_\infty < \infty$ although the simulation seems to show λ converges. Also for above analysis we don't assume that reference $r(t)$ and disturbance $d(t)$ are identically zero. In that case, $W_i(s)$ can only be set as 1 because otherwise $\widetilde{z}_i(t) = z_i(t) - \sigma u_{\infty i}(t) + \sigma d_i(t)$ doesn't satisfy the same evolution equation as $z_i(t)$. It's easy to understand because zero-tracking/full-rejection will be lost when $W_i(s)$ isn't equal to 1.

5 SIMULATION

For sake of simplicity, a SISO system is examined to illustrate the control system performance. The ASPR or ASNR plant under control is described as (1) where $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 1 & 0.5 \end{pmatrix}$, $\mathbf{x(0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $G(s) = \frac{\pm(s+1)}{(s-1)(s+2)}$. The reference is $r = r_1 + r_2$, where $r_1 = \sin\omega_1 t + 1.5\sin 5\omega_1 t$, $r_2 = \sin\omega_2 t$ and $\omega_1 = 0.2 \times 2\pi$rad/sec, $\omega_2 = 0.3 \times 2\pi$rad/sec. The disturbance is a square wave at a period of $7Hz$ and with peak value ± 2. A square wave is chosen to indicate the scheme can cope with signals with infinite frequency content. The weightings are chosen to be 0.4, 0.4, 0.2(for the disturbance rejection repetitive sub-controller). We select $k(0) = 1, \lambda(0) = 0$, $W_i(s) = 1$ and $N(\lambda) = \lambda^2\cos(\lambda)$. The simulation result is given in Figure 2 and 3. Figure 2 is for $G(s) = \frac{(s+1)}{(s-1)(s+2)}$ and Figure 3 is for $G(s) = \frac{-(s+1)}{(s-1)(s+2)}$. The simulation result shows that the control scheme is capable for the ASPR or ASNR plant to asymptotically track/reject a multi-periodic reference/disturbance signal.

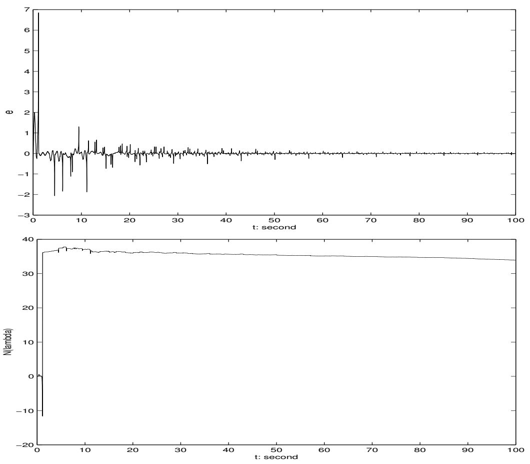

Figure 2: Error $e(t)$ and Nussbaum-type gain $N(\lambda)$ for ASPR plant

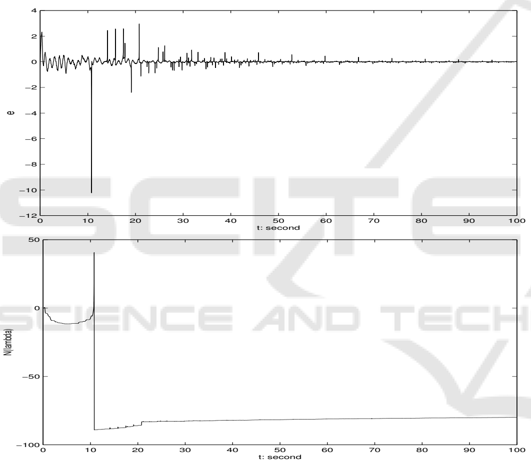

Figure 3: Error $e(t)$ and Nussbaum-type gain $N(\lambda)$ for ASNR plant

6 EFFECT OF NON-LINEAR PERTURBATION

The above lyapunov stability can be extended to the system under certain non-linear perturbations. The plant is described by

$$\begin{aligned} &\dot{x}(t) = Ax(t) + B(u(t) + d(t)) + g_1(t, x(t)) \\ &+g_2(t, y(t)) + d^{'}(t) \\ &y(t) = Cx(t) \end{aligned} \tag{17}$$

The nominal system is ASPR or ASNR as in section 4 and the non-linear perturbations satisfy

$$\begin{aligned} &g_1(.,.) : R \times R^n \to R^n, \|g_1(t,x)\| \le \hat{g}_1 \|x\| \\ &g_2(.,.) : R \times R^m \to R^n, \|g_2(t,y)\| \le \hat{g}_2 \|y\| \\ &d^{'}(.) \in L_2^n[0,\infty) \end{aligned} \tag{18}$$

Here $g_1(.,.), g_2(.,.), d^{'}(.)$ are assumed to be Carathedory function, which, for some unknown $\hat{g}_1, \hat{g}_2 \ge 0$, are linearly bounded for almost all $t \in R$ and for all $x \in R^n, u, y \in R^m$.

Theorem 4 *Consider the system* $\sum_G$ *described by (17) and (18). Suppose that both reference* $r(t)$ *and disturbance* $d(t)$ *are identically zero. Then the adaptive multi-periodic non-linear repetitive system in Figure 1 where the feedforward gain* $C_1(s) = k(s)\Gamma$ *and* $C_2(s) = N(s)\Gamma$ *with* $\dot{k}(t) = e(t)^T e(t), k(0) > 0$ *,* $\dot{\lambda}(t) = e(t)^T z(t), \lambda(0) \ge 0$ *is globally asymptotically stable in the sense that* $x(.) \in L_\infty^n[0,\infty)$*,* $y(.) \in L_2^m[0,\infty)$*,* $\lambda(.) \in L_\infty[0,\infty)$*,* $k(.) \in L_\infty[0,\infty)$ *and* $\lim_{t\to\infty} k(t) = k_\infty < \infty$.

Proof: We set the low-pass filter $W_i(s)$ to be 1 for sake of simplicity. The proof is similar to that in section 4 and here we only outline below. Introducing a positive definite lyapunov function V:

$$V = x^T P x + \frac{1}{k}\sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta \tag{19}$$

Differentiating V along (17) yield

$$\begin{aligned} &(x^T P x)^{'} \\ &< -x^T Q x + 2\sigma N(\lambda) z^T y + \gamma y^T y + 2\hat{g}_1 \|P\| \|x\|^2 \\ &+2\hat{g}_2 \|P\| \|x\| \|y\| + 2\|P\| \left\|d^{'}\right\| \|x\| \\ &\le -x^T \hat{Q} x + 2\sigma N(\lambda) z^T y + (\gamma + \hat{g}_2 \|P\| a_1^2) y^T y \\ &+ \|P\| a_2^2 \left\|d^{'}\right\|^2 \\ &\hat{Q} := Q - (2\hat{g}_1 + \hat{g}_2 a_1^{-2} + a_2^{-2}) \|P\| I, a_1, a_2 > 0 \\ &\tfrac{1}{k}(\textstyle\sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta)^{'} \\ &= -2z^T y - k y^T y - \tfrac{1}{k^2}\tfrac{dk}{dt} \textstyle\sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta \\ &\tfrac{dV}{dt} < -x^T \hat{Q} x - (2\sigma N(\lambda) - 2) z^T(-y) \\ &-(k - \gamma - \hat{g}_2 \|P\| a_1^2) y^T y \\ &- \tfrac{1}{k^2}\tfrac{dk}{dt} \textstyle\sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta + \|P\| a_2^2 \left\|d^{'}\right\|^2 \end{aligned} \tag{20}$$

When the linear bounds $\hat{g}_1, \hat{g}_2 > 0$ are sufficiently small in terms of the system entries (A, B, C) and $a_1, a_2 > 0$ are chosen to be sufficiently large so that $\hat{Q}$ is also positive definite. Integrating (20) yields

$$\begin{aligned} &V(t^{'}) - V(0) \\ &< -\textstyle\int_0^{t^{'}} x^T \hat{Q} x dt - \int_{k(0)}^{k(t^{'})} (\tau - \gamma - \hat{g}_2 \|P\| a_1^2) d\tau \\ &- \textstyle\int_{\lambda(0)}^{\lambda(t^{'})} (2\sigma N(\tau) - 2) d\tau + \int_0^{t^{'}} \|P\| a_2^2 \left\|d^{'}\right\|^2 dt \\ &- \textstyle\int_0^{t^{'}} \tfrac{1}{k^2}\tfrac{dk}{dt} \sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta dt \end{aligned} \tag{21}$$

The item $\int_0^{t'} \|P\| a_2^2 \left\|d'\right\|^2 dt$ is bounded as $d'(.) \in L_2[0,\infty)$. Therefore, it can be shown that $\lambda(.) \in L_\infty[0,\infty)$, $k(.) \in L_\infty[0,\infty)$,$y(.) \in L_2^m[0,\infty)$,$x(.) \in L_\infty^n[0,\infty)$ and $\lim_{t\to\infty} k(t) = k_\infty < \infty$ as before, which proves the result. □

7 EXPONENTIAL STABILIZATION VIA EXPONENTIAL WEIGHTING FACTOR

It has been proved that asymptotical stability of adaptive multi-periodic repetitive control system can be guaranteed if the plant $\sum_G$ is ASPR or ASNR. While when it strictly satisfies a ASPR or ASNR condition, now we show that the system is exponentially stable when modifying the adaptive scheme. According to definition A.1 and A.3 in Appendix, each almost strictly positive real system is almost ϵ -strictly positive real for some sufficiently small but unknown $\epsilon^* > 0$, that is, $(A+\epsilon^* I, \sigma B, C)$ is Almost Strictly Positive Real. Our aim is to find $\epsilon^* > 0$ adaptively by using an exponential weighting factor tuned by $k(t)$. We introduce a function $\epsilon(k(t))$(for example$\frac{0.1}{k(t)+1}$) with following properties: i) $\epsilon(k(t)) > 0$ for all $k(t) > 0$. ii) It is non-increasing for all $k(t) > 0$. iii) $\lim_{t\to\infty} \epsilon(k(t)) = \epsilon_\infty > 0$.

Theorem 5 *Consider the system $\sum_G$ described by (1). Suppose that both reference $r(t)$ and disturbance $d(t)$ are identically zero. Then the adaptive multi-periodic repetitive system in Figure 1 where the feedforward gain $C_1(s) = k(s)\Gamma$ and $C_2(s) = N(s)\Gamma$ with $\dot{k}(t) = e_\epsilon(t)^T e_\epsilon(t), k(0) > 0$, $\dot{\lambda}(t) = e_\epsilon(t)^T z_\epsilon(t), \lambda(0) \geq 0$ by denoting $x_\epsilon(t) := e^{\epsilon(k(t))t}x(t)$ is globally exponentially stable in the sense that $x(.) \in L_\infty^n[0,\infty)$, $y(.) \in L_2^m[0,\infty)$, $\lambda(.) \in L_\infty[0,\infty)$, $k(.) \in L_\infty[0,\infty)$, $\lim_{t\to\infty} k(t) = k_\infty < \infty$, $\lim_{t\to\infty} \epsilon(k(t)) = \epsilon_\infty > 0$ and also $\|x(t)\| \leq M_1 e^{-\epsilon t}$ for all $t \geq 0$ and some $M_1 > 0, \epsilon > 0$.*

Proof: With the notation $x_\epsilon(t) := e^{\epsilon(k(t))t}x(t)$, the plant can be written as

$$\begin{aligned}
&\dot{x}_\epsilon(t) = [A+\epsilon^* I - \sigma B\lambda^* C]x_\epsilon(t) + N(\lambda)Bz_\epsilon(t) \\
&+[\epsilon(k(t)) - \epsilon^* + \tfrac{d\epsilon(k(t))}{dt}t + \sigma B\lambda^* C]x_\epsilon(t) \\
&y_\epsilon(t) = Cx_\epsilon(t), \quad z_\epsilon(t) = \textstyle\sum_{i=1}^p \alpha_i v_{i_\epsilon}(t)
\end{aligned} \tag{22}$$

Also we have

$$\begin{aligned}
&z_{i\epsilon}^T(t-\tau_i)z_{i\epsilon}(t-\tau_i) \\
&= e^{2\epsilon(k(t-\tau_i))(t-\tau_i)} z_i^T(t-\tau_i)z_i(t-\tau_i) \\
&\geq e^{2\epsilon(k(t))(t-\tau_i)} z_i^T(t-\tau_i)z_i(t-\tau_i) \\
&= e^{-2\epsilon(k(t))\tau_i}(v_{i\epsilon}+ky_\epsilon)^T(v_{i\epsilon}+ky_\epsilon) \\
&\geq e^{-2\epsilon(k(0))\tau}(v_{i\epsilon}+ky_\epsilon)^T(v_{i\epsilon}+ky_\epsilon) \\
&\qquad \tau := max(\tau_i)
\end{aligned} \tag{23}$$

Introducing a positive definite lyapunov function V:

$$\begin{aligned}
&V = x_\epsilon^T P x_\epsilon + \tfrac{1}{k}\textstyle\sum_{i=1}^p \alpha_i M(t) \\
&M(t) := \int_{t-\tau_i}^t \|z_{i\epsilon}(\theta)\|^2 d\theta + x_{W_i\epsilon}^T P_{W_i} x_{W_i\epsilon}
\end{aligned} \tag{24}$$

Differentiating V along (22) and using (23) yields

$$\begin{aligned}
&\tfrac{dV}{dt} < -x_\epsilon^T Q x_\epsilon - 2\epsilon^* x_\epsilon^T P x_\epsilon + 2\epsilon(k)x_\epsilon^T P x_\epsilon \\
&+2\tfrac{d\epsilon(k)}{dt} t x_\epsilon^T P x_\epsilon + (\gamma - ke^{-2\epsilon(k(0)\tau)})y_\epsilon^T y_\epsilon \\
&+(2e^{-2\epsilon(k(0))\tau} - 2\sigma N(\lambda))(-y_\epsilon)^T v_\epsilon \\
&-\tfrac{1}{k}(e^{-2\epsilon(k(0))\tau} - \mu^2)\textstyle\sum_{i=1}^p \alpha_i v_{i\epsilon}^T v_{i\epsilon} \\
&-\tfrac{1}{k^2}\tfrac{dk}{dt}\textstyle\sum_{i=1}^p \alpha_i M(t)
\end{aligned} \tag{25}$$

Integrating (25) and using the adaptive law yields

$$\begin{aligned}
&V(t') - V(0) \\
&< -\int_0^{t'} x_\epsilon^T Q x_\epsilon dt - 2\int_0^{t'} (\epsilon^* - \epsilon(k))x_\epsilon^T P x_\epsilon dt \\
&+2\int_0^{t'} \tfrac{d\epsilon(k)}{dt} t x_\epsilon^T P x_\epsilon dt + \int_{k(0)}^{k(t')} (\gamma - se^{-2\epsilon(k(0))\tau})ds \\
&+\int_{\lambda(0)}^{\lambda(t')} (2e^{-2\epsilon(k(0))\tau} - 2\sigma N(s))ds \\
&-\int_{0}^{t'} \tfrac{1}{k^2}\tfrac{dk}{dt}\textstyle\sum_{i=1}^p \alpha_i M(t)dt \\
&+\int_0^{t'} \tfrac{1}{k}\textstyle\sum_{i=1}^p \alpha_i(\mu^2 - e^{-2\epsilon(k(0))\tau})z_{i\epsilon}^T z_{i\epsilon} dt
\end{aligned} \tag{26}$$

Suppose $\lambda(.) \notin L_\infty[0,\infty)$, $k(.) \notin L_\infty[0,\infty)$. Assume $\epsilon(k(0)) > \epsilon^*$, $-2\int_0^{t'} (\epsilon^* - \epsilon(k))x_\epsilon^T P x_\epsilon dt$ is definitely negative. When $\epsilon(k(0)) \leq \epsilon^*$, $-2\int_0^{t'} (\epsilon^* - \epsilon(k))x_\epsilon^T P x_\epsilon dt = -2\int_0^{t_1} (\epsilon^* - \epsilon(k))x_\epsilon^T P x_\epsilon dt - 2\int_{t_1}^{t'} (\epsilon^* - \epsilon(k))x_\epsilon^T P x_\epsilon dt$, where $\epsilon(k(t_1)) = \epsilon^*$. According to Theorem A.2 in Appendix and without loss of generality, we can assume $x_\epsilon = (y_\epsilon^T, \eta_\epsilon^T)^T$, so then $P = \begin{pmatrix} (\sigma CB)^{-1} & 0 \\ 0 & P_4 \end{pmatrix}$. Due to $y_\epsilon(.) \in L_\infty[0,t_1)$ and A_4 is asymptotically stable, we have $\eta_\epsilon(.) \in L_\infty[0,t_1)$, then $-2\int_0^{t_1} (\epsilon^* - \epsilon(k))x_\epsilon^T P x_\epsilon dt$ is a positive finite. So $-2\int_0^{t'} (\epsilon^* - \epsilon(k))x_\epsilon^T P x_\epsilon dt$ is negative infinity. $+\int_{k(0)}^{k(t')} (\gamma - se^{-2\epsilon(k(0))\tau})ds$ is negative infinity and $+\int_{\lambda(0)}^{\lambda(t')} (2e^{-2\epsilon(k(0))\tau} - 2\sigma N(s))ds$ is arbitrarily negative or positive infinity as before. When we select $k(0)$ so that $|e^{-\epsilon(k(0))\tau}| < \mu$, $+\int_0^{t'} \frac{1}{k}\sum_{i=1}^p \alpha_i(\mu^2 - e^{-2\epsilon(k(0))\tau})z_{i\epsilon}^T z_{i\epsilon} dt$ is negative. The other items are definitely negative due to

$\frac{dk}{dt} \geq 0$ and $\frac{d\epsilon}{dt} \leq 0$. So when $+\int_{\lambda(0)}^{\lambda(t')}(2e^{-2\epsilon(k(0))\tau} - 2\sigma N(s))ds$ takes arbitrarily negative, the right hand side of (26) will be negative, hence contradicting the non-negativity of the left hand side. Then from $\int_0^{+\infty} x^T Qx dt \leq \int_0^{+\infty} x_\epsilon^T Q x_\epsilon dt \leq +\infty$, we have $x(.) \in L_\infty^n[0,\infty)$. Similar as before, we have $y(.) \in L_2^m[0,\infty)$, $\lambda(.) \in L_\infty[0,\infty)$, $k(.) \in L_\infty[0,\infty)$ and $\lim_{t\to\infty} k(t) = k_\infty < \infty$. Then we have $\lim_{t\to\infty} \epsilon(k(t)) = \epsilon_\infty > 0$. As $x_\epsilon(t)$ is uniformly bounded such that $\|x_\epsilon(t)\| \leq M_1$, we have $\|x(t)\| \leq M_1 e^{-\epsilon t}$ for some $M_1 > 0, \epsilon > 0$, which indicates exponential stability of the state. □

However, perfect zero-tracking/full-rejecting for periodic reference/disturbance signals will be lost if the low-pass filter is not selected to be 1. So the state can only exponentially decrease to a bound as $\|x(t)\| \leq M_1 e^{-\epsilon t} + M_2$ for all $t \geq 0$ and some $M_1 > 0, M_2 > 0, \epsilon > 0$. Now we need to revise the adaptive scheme of $k(t)$ as

$$\dot{k}(t) = \begin{cases} \|e(t)\|(\|e(t)\| - \delta) & \text{if } \|e(t)\| \geq \delta \\ 0 & \text{if } \|e(t)\| < \delta \end{cases}$$

to prevent the divergence of adaptive gain $k(t)$.

8 ADAPTIVE PROPORTIONAL PLUS MULTI-PERIODIC REPETITIVE CONTROL SYSTEM

Theorem 6 *Consider the ASPR or ASNR system* $\sum_G$ *described by (1). Suppose that both reference* $r(t)$ *and disturbance* $d(t)$ *are identically zero. Then the adaptive multi-periodic repetitive system in Figure 4 with* k_1 *being a positive constant,* $\dot{k}_2(t) = e(t)^T e(t), k_2(0) > 0$ *and* $\dot{\lambda}(t) = e(t)^T z(t) + k_1 k_2(t) e(t)^T e(t), \lambda(0) \geq 0$ *is globally asymptotically stable in the sense that* $x(.) \in L_\infty^n[0,\infty)$, $y(.) \in L_2^m[0,\infty)$, $\lambda(.) \in L_\infty[0,\infty)$, $k_2(.) \in L_\infty[0,\infty)$ *and* $\lim_{t\to\infty} k_2(t) = k_{2\infty} < \infty$.

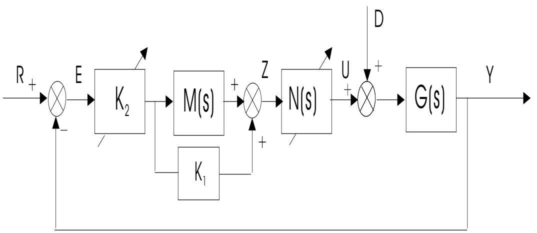

Figure 4: Adaptive MIMO proportional plus multi-periodic repetitive control system

Proof: We set the low-pass filter $W_i(s)$ to be 1 for sake of simplicity. Introducing a positive definite lyapunov function V:

$$V = x^T P x + \frac{1}{k_2}\sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta \quad (27)$$

By differentiating V we have

$$\begin{array}{l} \frac{dV}{dt} < -x^T Qx - (k_2 + 2k_1k_2 - \gamma)y^T y \\ -(2\sigma N(\lambda) - 2)(k_1 k_2 y^T y - z^T y) \\ -\frac{1}{k_2^2}\frac{dk_2}{dt}\sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta \end{array} \quad (28)$$

Integrating (28) yields

$$\begin{array}{l} V(t') - V(0) \\ < -\int_0^{t'} x^T Q x dt - \int_{k_2(0)}^{k_2(t')} (\tau + 2k_1\tau - \gamma) d\tau \\ -\int_{\lambda(0)}^{\lambda(t')} (2\sigma N(\tau) - 2) d\tau \\ -\int_0^{t} \frac{1}{k_2^2}\frac{dk_2}{dt}\sum_{i=1}^{p} \alpha_i \int_{t-\tau_i}^{t} \|z_i(\theta)\|^2 d\theta dt \end{array} \quad (29)$$

Similar to that in section 4, we can conclude $x(.) \in L_\infty^n[0,\infty)$, $y(.) \in L_2^m[0,\infty)$, $\lambda(.) \in L_\infty[0,\infty)$, $k_2(.) \in L_\infty[0,\infty)$ and $\lim_{t\to\infty} k_2(t) = k_{2\infty} < \infty$, which proves the result. □

Also the simulation results show that a higher proportional gain k_1 is helpful for the performance.

9 CONCLUSION

A kind of adaptive MIMO multi-periodic repetitive control system is studied. The stability is analysed in the sense of lyapunov stability. The adapting gains are proved to be bounded and the error decays asymptotically to zero. The similar lyapunov stability analysis is also extended to ASPR or ASNR plant under certain non-linear perturbations. It is also shown that exponential stability can be guaranteed by modifying the adaptive schemes. Finally, an proportional plus adaptive multi-periodic repetitive control system is proposed and its stability is proven in the sense of lyapunov stability as well.

APPENDIX

Theorem A. 1 *(X.D. and J.P., 1998). Let* $V(t)$ *and* $k(t)$ *be smooth functions defined on* $[0,+\infty)$ *with* $V(t) \geq 0, \forall t \in [0,+\infty)$, $N(t)$ *a Nussbaum-type function, and b a nonzero constant. If the following inequality holds:* $V(t) \leq \int_0^{k(t)} [bN(\omega)+1] d\omega + c, \forall t \in [0,+\infty)$ *where c is an arbitrary constant, then* $V(t)$, $k(t)$ *and* $\int_0^{k(t)} [bN(\omega) + 1] d\omega$ *must be bounded on* $[0,+\infty)$.

Theorem A. 2 *(A., 1993). Consider the system (1) with $det(CB) \neq 0$ and let $V \in R^{n\times(n-m)}$ denote a basis matrix of $kerC$. It follows that $S := [B(CB)^{-1}, V]$ has the inverse $S^{-1} = [C^T, N^T]^T$, where $N := (V^TV)^{-1}V^T[I_n - B(CB)^{-1}C]$. Hence the state space transformation $(y^T, \eta^T)^T = S^{-1}x = ((Cx)^T, (Nx)^T)^T$ converts (1) into*

$$\begin{aligned}\dot{y}(t) &= A_1y(t) + A_2\eta(t) + CB(u(t) + d(t)) \\ \dot{\eta}(t) &= A_3y(t) + A_4\eta(t)\end{aligned} \tag{30}$$

Here $A_1 \in R^{m\times m}, A_2 \in R^{m\times(n-m)}, A_3 \in R^{(n-m)\times m}, A_4 \in R^{(n-m)\times(n-m)}$, so that

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = S^{-1}AS$$

If (A, B, C) is minimum phase, then A_4 in (30) is asymptotically stable.

Definition A. 1 *Almost Strictly Positive Real: A system*

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), x(0) = x_0\end{aligned} \tag{31}$$

where $(A, B, C, D) \in R^{n\times n} \times R^{n\times m} \times R^{m\times n} \times R^{m\times m}$, is called Strictly Positive Real, if it satisfies equation (32) for $\mu > 0$ and we say it is Almost Strictly Positive Real, if there exists a $K \in R^{m\times m}$, so that the feedback $u(t) = -Ky(t) + r(t)$ yields a Strictly Positive Real system.

$$\begin{aligned}PA + A^TP &= -QQ^T - 2\mu P \\ PB &= C^T - QW \\ W^TW &= D + D^T\end{aligned} \tag{32}$$

Definition A. 2 *Almost Strictly Negative Real: The system $G(s)$ defined by (31) is called Almost Strictly Negative Real, if $-G(s)$ is a Almost Strictly Positive Real system.*

Definition A. 3 *Almost ϵ-Strictly Positive/Negative Real: Let $\epsilon > 0$, the system (31) is called ϵ-Strictly Positive Real, if it satisfies equation (32) for $\mu > \epsilon$ and we say it is Almost ϵ-Strictly Positive Real, if there exists a $K \in R^{m\times m}$, so that the feedback $u(t) = -Ky(t) + r(t)$ yields a ϵ-Strictly Positive Real system. It is called Almost ϵ-Strictly Negative Real, if $-G(s)$ is a Almost ϵ-Strictly Positive Real system.*

REFERENCES

A., I. (1993). *Non-Identifier-Based High-Gain Adaptive Control, Lecture Notes in Control and Information Sciences 189*. Springer-Verlag, London, 1st edition.

Dixon W. E., Zergeroglu E., D. D. C. B. (Sep, 2001). Repetitive learning control: A lyapunov-based approach. In *Proceedings of the 2001 International Conference on Control Applications, Mexico City, Mexico*.

G, H. (1996). Adaptive suppression of vibrations: A repetitive control approach.

G., W. (1997). Repetitive control systems: old and new ideas.

Hara S., Yamamoto Y., O. T. and M., N. (1988). Repetitive control system: A new type servo system for periodic exogenous signals.

Horowitz R., Messner W., M. J. B. (1991). Exponential convergence of a learning controller for robert manipulators.

Jiang Y.A., Clements D.J., H. T. (Dec, 1995). Adaptive repetitive control of nonlinear systems. In *Proceedings of the 34th Conference on Decidion and Control, New Orleans, LA*.

Li L.M., O. D. and S.P., B. (2002). Multi-periodic repetitive control system part ii: Robustness and practical issues.

Owens D.H., L. L. and S.P., B. (2002). Multi-periodic repetitive control system part i: A liapunov analysis for mimo systems.

Owens D.H., L. L. and S.P., B. (2003). Multi-periodic nonlinear repetitive control: Feedback stability analysis.

R.D., N. (1983). Some remarks on a conjecture in parameter adaptive control.

Sun Z., T. T. (2000). Adaptive control with asymptotic tracking performance and its application to an electro-hydraulic servo system.

Tomizuka M., Tsao T., C. K. (1989). Analysis and synthesis of discrete-time repetitive controllers.

Tsao T., T. M. (1994). Robust adaptive and repetitive digital tracking control and application to a hydraulic servo for noncircular marching.

Tzou Y., Jung S., Y. H. (1999). Adaptive repetitive control of pwm inverters for very low thd ac-voltage regulation with unknown loads.

Weiss G., H. M. (1999). Repetitive control of mimo systems using h^∞ design.

X.D., Y. and J.P., J. (1998). Adaptive nonlinear design without a priori knowledge of control directions.