AN ACCURATE AND EFFICIENT PARAMETER DECOUPLING FOR TRANSFER FUNCTION IDENTIFICATION

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Abstract: We present an improved parameter decoupling algorithm in estimating parameters that characterize the numerator and denominator of transfer function polynomials using the Adaptive Weighted Least Squares arising (AWLS) and Weighted Least Squares (WLS) from Fourier moment functionals of the Shinbrot type. This algorithm gives more accurate estimates and uses less computation than Pearson's algorithm. Also, simulation example shows that this algorithm can be applied for the frequency analysis of lightly damped systems for which establishing steady state or stationary operation may require unreasonably long settling times.

1 INTRODUCTION

A decoupling algorithm for optimal identification of rational transfer function parameters of discrete-time linear systems by least-squares (LS) fitting of observed input/output (I/O) data sequences (Shaw, 1994) was provided. The numerator was estimated by minimizing the optimization criterion, and using the estimated numerator, the optimal denominator was estimated by linear LS in one step. A decoupled parameter estimation (DPE) algorithm for estimating sinusoidal parameters from both 1-D and 2-D data sequences corrupted by autoregressive (AR) noise was presented (Li and Stoica, 1996). In the first step of the DPE algorithm, a nonlinear LS criterion was minimized by a relaxation algorithm to obtain the sinusoidal parameters. These estimates were used in the second step of the DPE algorithm, which estimates the AR noise parameters by a linear LS approach. A parameter decoupling method for transfer function during quasi-harmonic operation was proposed (Pearson, 1998) without any simulation example. This presupposes a non-steady state mode of operation over a single or integral number of periods during which a sinusoidal input is used as a probing signal. This deliberate use of a sinusoid during an otherwise transient state of system operation is motivated by the desire to simplify the identification process via a parameter decoupling that occurs in a particular frequency domain model. We explored Pearson's algorithm with several simulation examples and improved its estimation performance by a more accurate and more effective method.

In contrast to (Pearson, 1998), the use of a high frequency sinusoid is proposed in the modified *alpha*-stage to decouple the denominator parameters (herein called *alpha* parameters). This makes it possible to use lower indexed harmonic Fourier series coefficients of the output than input harmonics for the estimation of denominator parameters which is advantageous because lower harmonics contain more important information on the system. This simple idea causes a huge difference in the estimation performance of denominator parameters and affects to the estimation of numerator parameters through the weighting matrix in the *beta*-stage which use *alpha* parameters.

Moreover, we propose to modify the *beta*-stage by using a non-harmonic input for the probing signal. By using non-harmonic input, one step decoupling of numerator parameters (called *beta* parameters) is possible, which decreases the computation burden and increases estimation performance compared to Pearson's *beta*-stage.

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Following a presentation of the models, the decoupling procedures for the new algorithm is delineated and the least squares identifiers and the weighting matrixes for both stages in the modified algorithm are formulated. Finally, the simulation example is illustrated for the performance comparison.

2 FREQUENCY DOMAIN MODEL

Consider a time-invariant, bounded-input bounded-output stable linear differential system with scalar input $u(t)$ and scalar output $y(t)$ modeled on a finite time interval [0,*T*] by the *n*th order differential equation: $(p = d/dt)$

$$\sum_{j=0}^{n} a_j p^j y(t) = \sum_{j=0}^{n_b} b_j p^j u(t) + \sum_{j=0}^{n} a_j p^j v(t) \tag{1}$$

equivalently with operator polynomials $(A(p), B(p))$ in $p = d/dt$ and with a_0 normalized to unity:

$$A(p) = 1 + \sum_{j=1}^{n} a_j p^j, \qquad B(p) = \sum_{j=0}^{n_b} b_j p^j \tag{2}$$

$$A(p)y(t) = B(p)u(t) + A(p)v(t), \ A(0) = 1 \tag{3}$$

$(u(t), y(t))$ denote an input/output data pair, and $v(t)$ denotes an additive-output white Gaussian noise disturbance as defined by

$$E\{v(t)\} = 0, \ E\{v(t)v(t+\tau)\} = \sigma^2 \delta(\tau) \tag{4}$$

where $\delta(\tau)$ is the Dirac delta function. Assuming orders (n, n_b) of the polynomial pair $(A(s), B(s))$ are specified with $n_b \le n$, the problem is to estimate the parameters $(a_1, a_2, ..., a_n)$ and $(b_0, b_1, b_2, ..., b_{nb})$, given noise-corrupted data truncated to a time interval of length *T*. A “resolving frequency” ω_0 is defined in relation to [0, *T*] by $\omega_0 = 2\pi / T$.

To introduce the Modulating Function Technique (MFT), define a set of the *n*th order complex Fourier type modulating function (Pearson, 1998):

$$\phi_m(t) = \frac{1}{\sqrt{T}} e^{-im\omega_0 t} (e^{-i\omega_0 t} - 1)^n \tag{5}$$

$$m = 0, 1, \cdots, M, \quad 0 \le t \le T$$

where ω_0 is the resolving frequency, T is the time interval of the data block, and *M* is an integer for controlling the highest frequency and number of algebraic equations. Each $\phi_m(t)$ satisfies the end point conditions:

$$p^k \phi_m(t)|_{t=0} = 0, \ p^k \phi_m(t)|_{t=T} = 0, \ k = 0, 1, \cdots, (n-1) \tag{6}$$

Using the binomial expansion, $\phi_m(t)$ can be written as:

$$\phi_m(t) = \frac{1}{\sqrt{T}} \sum_{k=0}^{n} c_k e^{-i(m+k)\omega_0 t}, \qquad c_k = (-1)^{n-k} \binom{n}{k} \tag{7}$$

Then define a Shinbrot-type moment functional (Pearson, 1998) of order *n* given $x(t)$ on [0,*T*]:

$$f_m(x) = \int_0^T \phi_m(t) x(t) dt = \sum_{k=0}^{n} c_k X[m+k] \tag{8}$$

where $X[k]$ is the Fourier coefficient of $x(t)$ at frequency $k\omega_0$ as shown in equation (11).

If $P(p)$ is any polynomial of degree *n* (or less) in the differential operator $p = d/dt$ and if $x(t)$ is any *n*-times differentiable function on [0,*T*] or *n*-times mean-square differentiable in the case of stochastic signals, then as stated (Pearson, 1998):

$$f_m(P(p)x) = \sum_{k=0}^{n} c_k P[m+k] X[m+k] \tag{9}$$

Equation (3) will be converted to the frequency domain via the Shinbrot-type moment functionals of order *n* in equation (9). The result is

$$c'\phi_m = c'\Gamma_m + c'E_m, \quad m \in Z = \{0, 1, \cdots\} \tag{10}$$

where prime denotes the transpose of vector/matrix and the following definitions apply:

$$c = (c_0, c_1, ..., c_n)', \quad c_k = (-1)^{n-k} \binom{n}{k}$$

$$\phi_m = \begin{bmatrix} \phi[m] \\ \phi[m+1] \\ \vdots \\ \phi[m+n] \end{bmatrix}, \quad \Gamma_m = \begin{bmatrix} \Gamma[m] \\ \Gamma[m+1] \\ \vdots \\ \Gamma[m+n] \end{bmatrix}, \quad E_m = \begin{bmatrix} E[m] \\ E[m+1] \\ \vdots \\ E[m+n] \end{bmatrix}$$

$$\phi[k] = [A[k]Y[k]], \ \Gamma[k] = [B[k]U[k]], \ E[k] = A[k]V[k]$$

and $(U[k], Y[k], V[k])$ denote the *k*th harmonic Fourier series coefficient triplet defined by

$$(U[k], Y[k], V[k]) = \frac{1}{\sqrt{T}} \int_0^T (u(t), y(t), v(t)) e^{-ik\omega_0 t} dt \tag{11}$$

In addition to the pair (n, n_b), it is assumed that a bandwith ω_B is specified within which the user will extract frequency components of the data to be used in estimating the parameters. This means that the following constraint applies if $k_{\max}$ is the highest harmonic to be sought from the data using (11): $k_{\max}\omega_0 \le \omega_B$

Assuming $v(t)$ is a bandlimited white noise process with passband > ω_{BW}, the equation error $v(t)$ is transformed to

$$\varepsilon(m) = c'E_m, \quad m \in Z \tag{12}$$

which is still zero-mean Gaussian.

The parameter decoupling and improvement of estimation performance in parameter space for the model (10) is the main focus for the remainder of this paper.

3 DECOUPLING THE ESTIMATE

Given the system bandwidth, a set of M_{BW} integers Z_{BW} is defined by $Z_{BW} = \{1, 2, \cdots, M_{BW}\}$ such that the frequencies $\omega_k = k\omega_0$, $k = 1, 2, \cdots, M_{BW}$ represent selected 'knots' at which to estimate the transfer function $H(i\omega_k) = B(i\omega_k)/A(i\omega_k)$, $i = \sqrt{-1}$. The choice of M_{BW} is assumed to satisfy the equality:

$$(M_{BW} + n)\omega_0 = \omega_{BW} \tag{13}$$

This need is based on the condition that the highest frequency extracted from the data does not exceed the bandwidth. The question of selecting an appropriate ω_0 and *T* is discussed later. Let an $m_\alpha \in Z_{BW}$ be selected along with a complex number C_α such that the input signal

$$u_\alpha(t) = \frac{1}{\sqrt{T}}\left(C_\alpha e^{im_\alpha\omega_0 t} + C_\alpha{}^* e^{-im_\alpha\omega_0 t}\right), \quad t_\alpha \le t \le t_\alpha + T \tag{14}$$

represents the sinusoidal signal with amplitude of $2|C_\alpha|/\sqrt{T}$ and frequency $m_\alpha\omega_0$ that is applied to the system over a [0,*T*] time interval. However, excitation of all modes on this interval is a necessary condition to avoid degeneracy in estimating the α parameters. Corresponding to this choice, the *j*th Fourier series coefficient from (11) for $u_\alpha(t)$ is:

$$U_\alpha(j) = C_\alpha\delta[j - m_\alpha] + C_\alpha{}^*\delta[j + m_\alpha] \tag{15}$$

where $\delta[j]$ denotes the discrete unit pulse.
Substituting (15) into (10) gives

$$c'\phi_m = c_{m_\alpha - m}\Gamma_{m_\alpha} + c'E_m, \tag{16}$$

$$m \in \begin{cases} \{0, 1, \cdots, m_\alpha\} & for \quad m_\alpha < n \\ \{m_\alpha - n, m_\alpha - n + 1, \cdots, m_\alpha\} & for \quad m_\alpha \ge n \end{cases}$$

where $\Gamma_{m_\alpha} = [B[m_\alpha]C_\alpha]$, ϕ_m and E_m have the same definition as in (10) and their components are defined by:

$$\phi[k] = [A[k]Y_\alpha[k]], \quad \Gamma[k] = [B[k]U_\alpha[k]], \quad E[k] = A[k]V[k]$$

where $Y_\alpha[k]$ represents the *k*th harmonic Fourier series coefficient of the observed response $y_\alpha(t)$ on [0,*T*] to the sinusoid (14) as computed from (11).

In the modified algorithm, the least squares formulations will focus on estimating the $n + n_b + 1$ parameters $\{a_1, a_2, \cdots, a_n, b_0, b_1, \cdots, b_{n_b}\}$ in the transfer function

$$H(i\omega_k) = \frac{B(i\omega_k)}{A(i\omega_k)} = \frac{Q_\beta[m_k]\theta_\beta}{1 + \Lambda[m_k]\alpha}, \qquad m_k \in Z_{BW} \tag{17}$$

$$\Lambda(s) = (s, s^2, \cdots, s^n), \quad Q_\beta(s) = (1, s, s^2, \cdots, s^{n_b})$$

and the real-valued α and θ_β parameters are defined by

$$\alpha = (a_1, a_2, \cdots, a_n)', \quad \theta_\beta = (b_0, b_1, \cdots, b_{n_b})'$$

$$\Lambda[m_k] = \Lambda(im_k\omega_0), \quad A(ik\omega_0) = 1 + \Lambda[k]\alpha, \quad Q_\beta[m_k] = Q_\beta(im_k\omega_0)$$

3.1 Modified *Alpha* Stage

In Pearson's *alpha*-stage algorithm, the harmonics of $(m_\alpha + 1)$ through $(m_\alpha + M_\alpha + n)$ were used for regressor and regressand, and the lowest harmonic which can be used is 2 when $m_\alpha = 1$. Because the lower harmonics of the output, especially fundamental, contain the more useful information of the system, we propose to apply a high frequency sinusoid and use lower indices of the Fourier series coefficients than the value of m_α for the estimation of the α parameters. To take advantages of low index Fourier series coefficients, let us set

$$m_\alpha = M_\alpha + n + 1 \tag{18}$$

where M_α is user a chosen frequency index in the modified *alpha*-stage, and its recommended range is shown in (19). i.e., apply a sinusoid with frequency $(M_\alpha + n + 1)\omega_0$ which is right above the bandwidth and amplitude $2|C_\alpha|/\sqrt{T}$ as a probing signal. With this probing input, all low harmonics from DC to $(M_\alpha + n)$ of the output data (which covers the system bandwidth) can be used for the estimation of the denominator by defining a new Z_α, a set of frequency index *m* values that makes the α parameters decouple from the θ_β parameters in the polynomial $B(i\omega_k)$, i.e., define

$$Z_\alpha = \{m : 0 \le m \le M_\alpha \quad with \quad M_\alpha \sim 2n \; to \; 4n\} \tag{19}$$

The Z_α in the modified *alpha*-stage includes DC as well as the fundamental. This is the major difference between Pearson's *alpha*-stage and the modified *alpha*-stage.

The one restriction which is sufficient to facilitate the decoupling over the positive integers, i.e., to ensure that $c'\Gamma_m = 0$ in (10), for the input (14) with m_α in (18), is

$$0 < m < m_\alpha \tag{20}$$

which will provide a total of $M_\alpha + 1$ frequency domain equations including the DC component.

With $m \in Z_\alpha$, the right side of (10) reduces to $c'E_m$ and utilizing the relation $A(ik\omega_0) = 1 + \Lambda[k]\alpha$ it can be rearranged as a linear regression on α.

$$c'Y_m = -c'Q_m\alpha + c'E_m, \qquad m \in Z_\alpha \tag{21}$$

$$Y_m = \begin{bmatrix} Y[m] \\ Y[m+1] \\ \vdots \\ Y[m+n] \end{bmatrix},\ Q_m = \begin{bmatrix} \Lambda[m]Y[m] \\ \Lambda[m+1]Y[m+1] \\ \vdots \\ \Lambda[m+n]Y[m+n] \end{bmatrix}, \text{and}\ \ E_m = \begin{bmatrix} A[m]V[m] \\ A[m+1]E[m+1] \\ \vdots \\ A[m+n]E[m+n] \end{bmatrix}$$

To change the complex-valued regression model into a real-valued column vector linear regression model, an equivalent real-valued regression is defined as follows:

$$Y_c = -Q_c\alpha + \varepsilon_\mathrm{A} \tag{22}$$

where the following notation applied for the combined real and imaginary quantities:

$$Y_c = \begin{bmatrix} \mathrm{Re}\, c'Y_0 \\ \mathrm{Re}\, c'Y_1 \\ \vdots \\ \mathrm{Re}\, c'Y_{M_\alpha} \\ \mathrm{Im}\, c'Y_0 \\ \mathrm{Im}\, c'Y_1 \\ \vdots \\ \mathrm{Im}\, c'Y_{M_\alpha} \end{bmatrix},\quad Q_c = \begin{bmatrix} \mathrm{Re}\, c'Q_0 \\ \mathrm{Re}\, c'Q_1 \\ \vdots \\ \mathrm{Re}\, c'Q_{M_\alpha} \\ \mathrm{Im}\, c'Q_0 \\ \mathrm{Im}\, c'Q_1 \\ \vdots \\ \mathrm{Im}\, c'Q_{M_\alpha} \end{bmatrix},\quad \varepsilon_\mathrm{A} = \begin{bmatrix} \mathrm{Re}\, c'E_0 \\ \mathrm{Re}\, c'E_1 \\ \vdots \\ \mathrm{Re}\, c'E_{M_\alpha} \\ \mathrm{Im}\, c'E_0 \\ \mathrm{Im}\, c'E_1 \\ \vdots \\ \mathrm{Im}\, c'E_{M_\alpha} \end{bmatrix}$$

$Y_c \in \Re^{2(M_\alpha+1)}$, $Q_c \in \Re^{(2M_\alpha+2)\times n}$ and $\varepsilon_\mathrm{A} \in \Re^{2(M_\alpha+1)}$

Note that the row dimension of Y_c, Q_c and ε_A is $2(M_\alpha+1)$. Based on this regression model and assuming linearly independent regressors and zero-mean Gaussian residuals ε_A with a nonsingular covariance matrix $W_\alpha = E\{\varepsilon_\mathrm{A}\varepsilon_\mathrm{A}'\}$, a weighted least squares estimate is defined by

$$\hat{\alpha} = -\left(Q_c' W_\alpha^{-1} Q_c\right)^{-1} Q_c' W_\alpha^{-1} Y_c \tag{23}$$

where W_α is a symmetric positive definite weighting matrix.

Moreover, $\hat{\alpha}$ can be estimated by the iterative method (Shen, 1993), which can be expressed by the following equation:

$$\hat{\alpha}_k = -\left(Q_c' W_{\alpha,k-1}^{-1} Q_c\right)^{-1} Q_c' W_{\alpha,k-1}^{-1} Y_c, \quad k = 1, 2, \cdots \tag{24}$$

where $W_{\alpha,k-1}$ denotes the covariance matrix of the residual vector, which will be shown in equation (41), as a function of unknown parameter θ_α and evaluated at the previous iterate $\theta_{\alpha,k-1}$. Thus $W_{\alpha,k-1} = W_\alpha(\theta_{\alpha,k-1})$. But the initial weighting matrix $W_{\alpha,0}$ is taken as the identity matrix.

3.2 Weighting Matrix in the Modified Alpha Stage

The composite residual vector ε_A can be expressed as follows

$$\varepsilon_\mathrm{A} = \begin{bmatrix} \mathrm{Re}\,\varepsilon_\alpha \\ \mathrm{Im}\,\varepsilon_\alpha \end{bmatrix} = \begin{bmatrix} \varepsilon_\alpha^R \\ \varepsilon_\alpha^I \end{bmatrix} \tag{25}$$

and the following definitions apply:

$$\varepsilon_\alpha = \begin{bmatrix} \varepsilon_\alpha[0] \\ \varepsilon_\alpha[1] \\ \vdots \\ \varepsilon_\alpha[M_\alpha] \end{bmatrix} \quad E_m = \begin{bmatrix} A[m]V[m] \\ A[m+1]V[m+1] \\ \vdots \\ A[m+n]V[m+n] \end{bmatrix} \quad \varepsilon_\alpha[m] = c'E_m$$

Equivalent vector-matrix representation for ε_α is

$$\varepsilon_\alpha = C_\alpha P_\alpha V_\alpha \tag{26}$$

where matrix C_α, diagonal matrix P_α, and vector V_α are given by

$$C_\alpha = \begin{bmatrix} c_0 & c_1 & \cdots & c_n & 0 & 0 & \cdots & 0 \\ 0 & c_0 & c_1 & \cdots & c_n & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & & \ddots & & 0 \\ 0 & 0 & \cdots & c_0 & c_1 & \cdots & & c_n \end{bmatrix}$$

$$P_\alpha = diag\,(A[0], A[1], ..., A[M_\alpha + n])$$

$$V_\alpha = (V[0], V[1], ..., V[M_\alpha + n])'$$

C_α is a $(M_\alpha+1)\times(M_\alpha+n+1)$ real matrix defined by sequentially moving the row vector c' over one entry to the right with 0's elsewhere, as shown above, P_α is complex with $A[m] = A(im\omega_0)$ and V_α is a complex valued column vector. The covariance matrix for the $(2M_\alpha+1)$ dimensional residual vector ε_A will have the block diagonal structure:

$$W_\alpha = E\{\varepsilon_\mathrm{A}\varepsilon_\mathrm{A}'\} = \begin{bmatrix} W_\alpha^R(\theta_a) & \Theta \\ \Theta & W_\alpha^I(\theta_a) \end{bmatrix} \tag{27}$$

where

$$W_\alpha^R(\theta_\alpha) = E\{\varepsilon_\alpha^R(\varepsilon_\alpha^R)'\} = \frac{\sigma^2}{2}\{C_\alpha P_\alpha P_\alpha^H C_\alpha' + c_0^2 A[0]^2 e_1 e_1'\} \tag{28}$$

$$W_\alpha^I(\theta_\alpha) = E\{\varepsilon_\alpha^I(\varepsilon_\alpha^I)'\} = \frac{\sigma^2}{2}\{C_\alpha P_\alpha P_\alpha^H C_\alpha' - c_0^2 A[0]^2 e_1 e_1'\} \tag{29}$$

in which $PP^H = diag(A[0]^2, |A[1]|^2, \cdots, |A[M_\alpha+n]|^2)$, $|A[m]|^2 = |A(im\omega_0)|^2$, $A[0]=1$, Θ is a zero matrix and superscript H denotes conjugate transpose. Unit column vector e_1 is given by $e_1 = (1\ \Theta_{1\times n_\alpha})'$ in which $\Theta_{1\times n_\alpha}$ is a zero vector with a dimension of $1\times n_\alpha$.

3.3 Modified Beta Stage

Pearson's *beta*-stage algorithm needs another computation step to extract the b_i parameters from the number $ceil\{(n_b+1)/2\}$ of algebraic equations where the function *ceil(A)* rounds the elements of *A* to the nearest integer greater than or equal to *A*.. Moreover, his algorithm needs $ceil\{(n_b+1)/2\}$ times of the weighting matrix inversion computation in the *beta*-stage. This inconvenience was eventually caused by the harmonic operation in the *beta*-stage. Since the extracted parameter b_i 's are inversely proportional to $\omega_0{}^i$, as will be shown in (38), and the resolving frequency ω_0 is usually a small number for high resolution, less than 1, the bias and standard deviation of b_i 's are amplified. This problem is inevitable as long as the indirect parameter estimation algorithm, which uses harmonic sinusoids for a probing signal is adopted in the *beta*-stage. To improve the problems of inconvenience and inaccuracy in estimating the numerator parameters with Pearson's *beta*-stage algorithm, the modified *beta*-stage is proposed, which estimates numerator parameters at one shot using non-harmonic operation.

Again, assume that an estimate $\hat{\alpha}$ has been obtained following the completion of the modified *alpha*-stage as described in the previous section, and consider a non-harmonic sinusoidal input, $u_\beta(t)$, like a sweep sine, as a probing signal in the modified *beta*-stage. To ensure the excitation of all modes, a sweep sine with Fourier coefficients, which covers the system bandwidth, should be chosen. The model (10) is changed by:

$$c'\phi_m = c'\gamma_m\theta_\beta + c'E_m \tag{30}$$

where the following definitions apply:

$c = (c_0, c_1, ..., c_n)'$ and $\theta_\beta = (b_0, b_1, \cdots, b_{n_b})'$

$$\phi_m = \begin{bmatrix} \phi[m] \\ \phi[m+1] \\ \vdots \\ \phi[m+n] \end{bmatrix}, \quad \gamma_m = \begin{bmatrix} \gamma[m] \\ \gamma[m+1] \\ \vdots \\ \gamma[m+n] \end{bmatrix}, \quad and \quad E_m = \begin{bmatrix} E[m] \\ E[m+1] \\ \vdots \\ E[m+n] \end{bmatrix}$$

$$\phi[k] = \left[\hat{A}[k]Y[k]\right], \quad \gamma[k] = \left[Q_\beta[k]U_\beta[k]\right], \quad E[k] = \hat{A}[k]V[k]$$

$$Q_\beta(s) = (1, s, s^2, \cdots, s^{n_b}), \quad Q_\beta[k] = Q_\beta(ik\omega_0), \quad m \in Z_\beta$$

where the following definition apply:

$$Z_\beta = \{m : 0 \le m \le M_\beta\} \tag{31}$$

where M_β is explained in (32). In equation (30), $U_\beta[k]$ denotes the *k*th harmonic Fourier coefficient of the input $u_\beta(t)$ on $0 \le t \le T$. Note that $B[k]$ in (17) was separated into two terms, $Q_\beta[k]$ and θ_β, in order to estimate the b_i parameters directly. There are n_b+1 parameters to be estimated. Let $M_\beta \ge n_b+1$ denote a user-selected integer used to specify the number of frequency indices upon which to base the estimate of b_i 's. Choose

$$M_\beta \approx 2(n_b+1) \; to \; 4(n_b+1) \,. \tag{32}$$

To change the complex-valued regression model into a real-valued column vector linear regression model, define combined constituents by

$$\xi = \Phi\theta_\beta + \varepsilon_{\mathrm{B}} \tag{33}$$

where the following notation applies for the combined real and imaginary quantities:

$$\xi = \begin{bmatrix} \mathrm{Re}\, c'\phi_0 \\ \mathrm{Re}\, c'\phi_1 \\ \vdots \\ \mathrm{Re}\, c'\phi_{M_\beta} \\ \mathrm{Im}\, c'\phi_0 \\ \mathrm{Im}\, c'\phi_1 \\ \vdots \\ \mathrm{Im}\, c'\phi_{M_\beta} \end{bmatrix}, \quad \Phi = \begin{bmatrix} \mathrm{Re}\, c'\gamma_0 \\ \mathrm{Re}\, c'\gamma_1 \\ \vdots \\ \mathrm{Re}\, c'\gamma_{M_\beta} \\ \mathrm{Im}\, c'\gamma_0 \\ \mathrm{Im}\, c'\gamma_1 \\ \vdots \\ \mathrm{Im}\, c'\gamma_{M_\beta} \end{bmatrix}, \quad \varepsilon_{\mathrm{B}} = \begin{bmatrix} \mathrm{Re}\, c'E_0 \\ \mathrm{Re}\, c'E_1 \\ \vdots \\ \mathrm{Re}\, c'E_{M_\beta} \\ \mathrm{Im}\, c'E_0 \\ \mathrm{Im}\, c'E_1 \\ \vdots \\ \mathrm{Im}\, c'E_{M_\beta} \end{bmatrix}$$

$\xi \in \Re^{2(M_\beta+1)}$, $\Phi \in \Re^{(2M_\beta+2)\times n_b}$ and $\varepsilon_{\mathrm{B}} \in \Re^{2(M_\beta+1)}$

Note that the row dimension of ξ, Φ and ε_{B} is $2(M_\beta+1)$. Based on this regression model and assuming linearly independent regressors and zero-mean Gaussian residuals ε_{B} with a nonsingular covariance matrix $W_\beta = E\{\varepsilon_{\mathrm{B}}\varepsilon_{\mathrm{B}}'\}$, the estimate of θ_β can be obtained by

$$\hat{\theta}_\beta = \left(\Phi' W_\beta{}^{-1}\Phi\right)^{-1}\Phi' W_\beta{}^{-1}\xi \tag{34}$$

Note that $\hat{\theta}_\beta$ is estimated by the Weighted Least Squares (WLS) using the $\hat{\alpha}$ estimates, which is accurately estimated in the modified *alpha*-stage.

3.4 Weighting Matrix in the Modified Beta Stage

If we follow the same procedure in section 3.2, we get the block diagonal covariance matrix for the $2(M_\beta+1)$ dimensional residual vector ε_{B} in the modified *beta*-stage:

$$W_\beta = \frac{\sigma^2}{2}\begin{bmatrix} C_\beta P_\beta P_\beta{}^H C_\beta' + c_0{}^2\hat{A}[0]^2 e_1 e_1' & \Theta \\ \Theta & C_\beta P_\beta P_\beta{}^H C_\beta' - c_0{}^2\hat{A}[0]^2 e_1 e_1' \end{bmatrix} \tag{35}$$

where $P_\beta P_\beta{}^H = diag(\hat{A}[0]^2, |\hat{A}[1]|^2, \cdots, |\hat{A}[M_\beta+n]|^2)$, $|\hat{A}[m]|^2 = |\hat{A}(im\omega_0)|^2$ and $\hat{A}[0]=1$. C_β is a $(M_\beta+1)\times(M_\beta+n+1)$ real matrix which has the same pattern as (26) and P_β is a function of parameter $\hat{\alpha}$, which is estimated in the modified *alpha*-stage. Note that the weighting matrix for the modified *beta*-stage, W_β, needs to be computed only once to estimate the b_i numerator parameters.

3.5 Selection of ω_0 with Modified Algorithm

The highest harmonics required of the output data over the data intervals $[t_\alpha, t_\alpha + T]$ and $[t_\beta, t_\beta + T]$, are the $(M_\alpha + n)$th and $(M_\beta + n)$th harmonics respectively, cf. (22) and (33). For a strictly proper rational transfer function, i.e., $n > n_b$, and assuming the same ratio for the selection of M_α and M_β in ((19), (32)) is chosen, then $(M_\alpha + n)$ is bigger than $(M_\beta + n)$ and the corresponding highest frequency is $(M_\alpha + n)\omega_0$. It follows that ω_0 should be chosen as

$$\omega_0 = \frac{\omega_{BW}}{(M_\alpha + n)} \tag{36}$$

With this choice, both frequency models in (22) and (33) cover the system bandwidth ω_{BW}. Also, this choice assures adherence to the equality (20) made earlier as a condition on selecting M_{BW}.

All modes of a system might not be exited by a low frequency sinusoid as used in Pearson's alpha stage algorithm. But by applying this one sinusoid with a frequency that is just outside bandwidth, all high frequency system information within the system bandwidth could be obtained. This is another great advantage of the modified algorithm.

4 SIMULATION RESULTS

An 8th order system with 4th-order in the numerator, as shown in the following, was used to evaluate and compare the performance of the Pearson's decoupling algorithm (Pearson, 1998) and the modified decoupling algorithm devised in this study:

$$H(s) = \frac{s^4 + 2s^3 + 5s^2 + 4s + 0.1}{(s + 0.03 \pm i1.2)(s + 0.002 \pm i3)(s + 0.01 \pm i0.5)(s + 0.01 \pm i0.8)} \tag{37}$$

For the above specific system, its step response will take about 400 seconds to reach steady state, which is a lightly damped case. The data were collected during the system transient state, mostly during the first 50 sec. The system bandwidth is 3.38 $[rad/\sec]$. 2048 data of input/output were sampled for T sec, where T varies with m_α in each algorithm.

Fig. 1 shows the Bode diagram of the system used for simulation. The noise-to-signal ratio (NSR), which characterizes the percent additive noise on the output is defined as

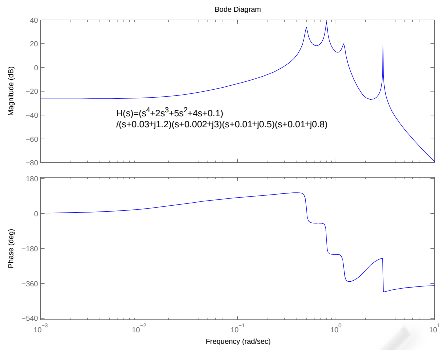

Figure 1: Bode diagram of the system

$$NSR = 100\% \frac{\sqrt{\int_0^T [n(t)]^2\,dt}}{\sqrt{\int_0^T [y_0(t)]^2\,dt}} = \frac{\|n(t)\|_2}{\|y_0(t)\|_2}.100\%$$

where $y_0(t)$ is a noise free signal, and $n(t)$ is an additive noise sequence. As for a true parameter θ_j, its ensemble average $\hat{\theta}_j$ and the number of parameters L, a composite normalized bias error (CNB) and standard deviation (CNSTD) are defined as:

$$\text{CNB} = \sqrt{\frac{1}{L}\sum_{j=1}^{L}\left(\frac{\hat{\theta}_j - \theta_j}{\theta_j}\right)^2}\,\% \qquad \text{CNSTD} = \sqrt{\frac{1}{L}\sum_{j=1}^{L}\left(\frac{\sigma_j}{\theta_j}\right)^2}\,\%$$

where σ_j is the standard deviation of the estimate of the true θ_j. These will be used to measure the accuracy of the different algorithms.

In Pearson's algorithm, we used the routine SOLVE in Symbolic Math Toolbox of MATLAB to solve the algebraic equations for the extraction of b_i 's parameters in the numerator from *beta* parameters,

$\beta^R[m_k] = \operatorname{Re} B(im_k\omega_0)$, $\beta^I[m_k] = \operatorname{Im} B(im_k\omega_0)$, $k = 1, 2, \cdots$.

For instance, 5 b_i 's of the example system in equation (37) are shown;

$$\begin{aligned}
b_0 &= \frac{3}{2}\beta^R(\omega_0) - \frac{3}{5}\beta^R(2\omega_0) + \frac{1}{10}\beta^R(3\omega_0) \\
b_1 &= \frac{8\beta^I(\omega_0) - \beta^I(2\omega_0)}{6\omega_0} \\
b_2 &= \frac{13\beta^R(\omega_0) - 16\beta^R(2\omega_0) + 3\beta^R(3\omega_0)}{24\omega_0{}^2} \\
b_3 &= \frac{2\beta^I(\omega_0) - \beta^I(2\omega_0)}{6\omega_0{}^3} \\
b_4 &= \frac{5\beta^R(\omega_0) - 8\beta^R(2\omega_0) + 3\beta^R(3\omega_0)}{120\omega_0{}^4}
\end{aligned} \tag{38}$$

Note that the parameter b_i 's are inversely proportional to $\omega_0{}^i$, and ω_0 is usually a small number for high resolution. Thus the computed b_i 's from the β 's have wide distribution. This is the reason why Pearson's *beta*-stage produces large composite STD. This is the disadvantage of "Quasi-Harmonic operation". To improve this large STD problem, the modified *beta*-stage is suggested with a simulation example in the next section.

In this section, we will compare the modified decoupling algorithm denoted by $MOD\alpha\beta$, which uses the modified *alpha*-stage and modified *beta*-stage, with Pearson's algorithm, denoted by *HAR*, and an intermediate algorithm denoted by $MOD\alpha$, which uses the modified *alpha*-stage and Pearson's *beta*-stage. In the experiment setup, we focus on adding the same noise level for the different algorithms. The system bandwidth ω_{BW} is 3.38 $[rad/\sec]$ and the sampling rate is around 45 Hz. 500 Monte Carlo runs were made for each NSR under the initial condition fixed at zeros. Here we will explain simulation setups for three different algorithms.

1) Input parameters for the Pearson's algorithm: For the estimation of denominator parameters in the *alpha*-stage, $C_\alpha = 1 + j1$, $m_\alpha = 1$ and $M_\alpha = 2n = 16$ were chosen, so $\omega_0 = 0.1352$ $[rad/\sec]$ and the observation time interval is $T = 46.47$ [sec], and $u_\alpha(t) = C_\alpha e^{i\omega_0 t} + C_\alpha^* e^{-i\omega_0 t}$ was used for a probing signal in the *alpha*-stage. For the estimation of numerator parameters in the *beta*-stage, three harmonics were applied to the system one by one to estimate 3 sets of β parameters and they are given by: $u_{\beta 1}(t) = C_\alpha e^{i\omega_0 t} + C_\alpha^* e^{-i\omega_0 t}$, $u_{\beta 2}(t) = C_\alpha e^{i2\omega_0 t} + C_\alpha^* e^{-i2\omega_0 t}$, $u_{\beta 3}(t) = C_\alpha e^{i3\omega_0 t} + C_\alpha^* e^{-i3\omega_0 t}$.

2) Input parameters for the $MOD\alpha$ algorithm: $m_\alpha = 25$, $M_\alpha = 2n = 16$ and $C_\alpha = 1 + j1$ were chosen, so the probing signal in the modified *alpha*-stage is $u_\alpha(t) = C_\alpha e^{i25\omega_0 t} + C_\alpha^* e^{-i25\omega_0 t}$. For the *beta*-stage, the same 3 harmonic inputs as in Pearson's *beta*-stage were used. $\omega_0 = 0.1408$ $[rad/\sec]$ and $T = 44.61$ [sec] were used both in the *alpha* and *beta* stage. Notice that ω_0 and T are a little different with those of Pearson's algorithm [4] because the computation methods of ω_0 for both algorithms are different.

3) Input parameters for the $MOD\alpha\beta$ algorithm: Here, $m_\alpha = 25$, $M_\alpha = 2n = 16$ and $C_\alpha = 1 + j1$ were chosen, and $u_\alpha(t) = C_\alpha e^{i25\omega_0 t} + C_\alpha^* e^{-i25\omega_0 t}$ was applied for the modified *alpha*-stage and $u_\beta(t) = 0.1119 \sin\left(t^2/75\right)$ for the modified *beta*-stage, which produced the same output norm as $u_\alpha(t)$ to ensure the same level of noise can be added in the *alpha* and *beta* stage.

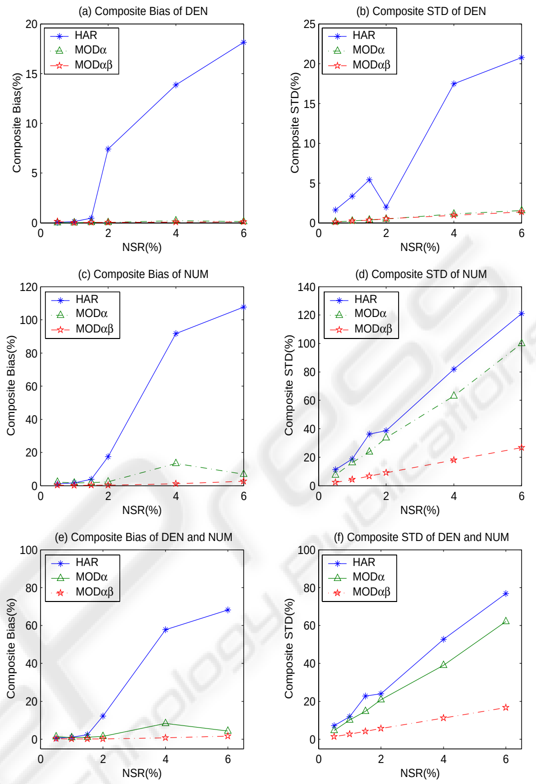

Figure 2: CNB and CNSTD of Pearson's algorithm and the modified algorithm

Fig. 2 shows the composite bias and STD for three different algorithms. For the composite bias of the denominator shown in Fig. 2(a), the bias for Pearson's algorithm is as small as that for the modified *alpha*-stage algorithm when the NSR is less than 1.5 %, but the composite bias and composite STD of the denominator sharply increase to 17% and 21%, respectively, as the NSR increases from 2% to 6%. In other words, Pearson's *alpha*-stage algorithm is very sensitive to noise.

The composite biases and the composite STDs of the denominator for $MOD\alpha$ and $MOD\alpha\beta$ are almost the same because they both use the modified *alpha*-stage algorithm, see Fig. 2(a) and (b). The modified *alpha*-stage shows excellent performance over the Pearson's *alpha*-stage. The $MOD\alpha$ shows better performance than Pearson's algorithm in *beta*-stage even though the two algorithms use the same Pearson's *beta*-stage. That results from the fact that the $MOD\alpha$ uses a weighting matrix in Pearson's *beta*-stage based on the accurately estimated denominator parameters by the modified *alpha*-

stage. The composite bias of the numerator was greatly reduced by the $MOD\alpha$, but the composite STD of the numerator was not much improved by the $MOD\alpha$, see Fig. 2(c) and (d). In Fig. 2(c) and (d), the $MOD\alpha\beta$ shows better performance for the numerator than the $MOD\alpha$ both in composite bias and composite STD aspects. This means that the modified *beta*-stage improves not only standard deviation but also bias. The composite bias of the numerator for Pearson's algorithm is very large as we expected. But it is greatly reduced by the modified *beta*-stage algorithm. Even though the modified *beta*-stage algorithm reduces the composite bias and composite STD of the numerator, those values are larger than the denominator's.

From Fig. 2(a) ~ (d), we can know that both the modified *alpha*-stage and modified *beta*-stage algorithm have decreased the bias and standard deviation at each NSR. Fig. 2(e) and (f) show the composite bias and composite STD of all parameters including the denominator and numerator. The $MOD\alpha\beta$, the proposed algorithm, produces the lowest bias and standard deviation among the three algorithms.

5 CONCLUDING REMARKS

We have presented a new parameter decoupling algorithm for the transfer function identification on the basis of Pearson's algorithm using harmonic and non-harmonic signals. We have also shown with simulation examples that these algorithms offer significant improvement in estimation performance and computation burden over existing methods.

In the new algorithm, we apply a harmonic sinusoid with one high frequency component outside the system bandwidth in the *alpha*-stage, so that we can use the lower indexed Fourier coefficients for the denominator estimation. Also, a one step estimation algorithm was adopted using a sweep sine input as probing signal for the numerator parameters in *beta*-stage. By using one step estimation algorithm, the computation burden was decreased and the estimation performance was increased. Clearly, simulation results show that the modified parameter decoupling algorithm is much better than Pearson's algorithm.

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