LQG CONTROL UNDER AMPLITUDE AND VARIANCE CONSTRAINTS

A. Królikowski†, D. Horla, T. Kubiak
Poznań University of Technology,
Institute of Control and Information Engineering,
ul. Piotrowo 3A, 60-965 Poznań, POLAND.

Keywords: LQG, Amplitude constraint, Variance constraint.

Abstract: In this paper, the amplitude and variance-constrained LQG control is considered for a plant given by discrete-time ARMAX model. The minimization of constrained quadratic cost is approached by Kalman filter, approximation of the probability density function (pdf) of the state by the Gaussian one and by by tuning of the Lagrange multiplier. The obtained optimization algorithm is simulated for second-order stable plant model and different constraints.

1 INTRODUCTION

Control input constraints are ubiquitous in many control applications, therefore including them in a control system design is of practical importance. Hard-limit input constraint and variance or mean-square input constraint are of the most frequent occurrence in industrial control processes. Neglecting these constraints in the controller design may lead to performance deterioration or even instability of the control system. Specifically, the unstable open-loop systems in the presence of constrained control signal can not be globally stabilizable.

The problem addressed in this paper is the LQG control of ARMAX plant in the presence of simultaneous amplitude and variance constrained input. The constrained control problem is approached using the Kalman filter and approximation of the pdf of the Kalman filter output by the Gaussian pdf. Analysis and computer simulations of second-order systems are given.

It should be noted that in the literature the considered LQG control problem is treated mostly for separate control constraints, see for example in (Królikowski, 1997, Mäkilä, 1982, Mäkilä et at, 1984, Toivonen, 1983).

2 PROBLEM FORMULATION

The plant is given by a discrete-time ARMAX model

$$A(q^{-1})y_t = B(q^{-1})u_t + C(q^{-1})e_t, \tag{1}$$

where A, B, C are polynomials in the backward shift operator q^{-1}, i.e.,$A = 1 + a_1q^{-1} + \cdots + a_{na}q^{-na}$, $B = b_1q^{-1} + \cdots + b_{nb}q^{-nb}$, $C = 1 + c_1q^{-1} + \cdots + c_{nc}q^{-nc}$, y_t is the output, u_t is the control input, and $\{e_t\}$ is assumed to be a sequence of independent random variables with zero mean and variance σ_e^2.
Consider the stationary cost function

$$J_1 = E[y_t^2 + q_u u_t^2] = \sigma_y^2 + q_u\sigma_u^2, \tag{2}$$

where the output and input variances $E[y_t^2]$, $E[u_t^2]$ are denoted as σ_y^2 and σ_u^2, respectively, and $q_u \geq 0$.
The amplitude and variance constraints imposed on the control input are given as follows

$$|u_t| \leq \alpha, \tag{3}$$
$$\sigma_u^2 \leq c^2. \tag{4}$$

It is known that ARMAX model (1) has an equivalent innovation state space representation

$$x_{t+1} = F\underline{x}_t + \underline{g}u_t + \underline{k}_e e_t, \tag{5}$$
$$y_t = \underline{h}^T\underline{x}_t + e_t, \tag{6}$$

for $na = nb = nc = n$, where the corresponding vectors are $\underline{g} = (b_1, \ldots, b_n)^T$, $\underline{k}_e = (c_1 - a_1, \ldots, c_n -$

Królikowski A., Horla D. and Kubiak T. (2005).
LQG CONTROL UNDER AMPLITUDE AND VARIANCE CONSTRAINTS.
In *Proceedings of the Second International Conference on Informatics in Control, Automation and Robotics*, pages 292-296
Copyright © SciTePress

$a_n)^T$, $\underline{h} = (1, 0, \ldots, 0)^T$, and

$$F = \begin{bmatrix} -a_1 & 1 \ldots & 0 \\ \cdot & \cdot \ldots & 0 \\ -a_{n-1} & \cdot \ldots & 1 \\ -a_n & \cdot \ldots & 0 \end{bmatrix}.$$

The associated Kalman filter is

$$\underline{\hat{x}}_{t+1} = F\underline{\hat{x}}_t + \underline{g}u(t) + \underline{k}\tilde{y}_t, \tag{7}$$

where $\underline{k}$ is the stationary gain vector, and $\tilde{y}_t = y_t - \underline{h}^T \underline{\hat{x}}_t$ with variance $\sigma^2_{\tilde{y}} = \underline{h}^T P_k \underline{h} + \sigma^2_e$. The matrix P_k is the solution to the following Riccati equation

$$\begin{aligned} P_k &= FP_kF^T - \\ &\quad -(FP_k\underline{h} + \sigma^2_\xi \underline{k}_\xi)(FP_k\underline{h} + \sigma^2_e \underline{k}_e)^T \times \\ &\quad \times(\underline{h}^T P_k \underline{h} + \sigma^2_e)^{-1} + \underline{k}_e \underline{k}_e^T \sigma^2_e. \end{aligned} \tag{8}$$

The goal of the control is to minimize the loss function J_1 under the given structure of the controller specified by the feedback gain vector $\underline{f}$ in the case of the Kalman filter-based controller subject to the amplitude and variance constrints (3), (4). Thus, the constrained control law has a form

$$u_t = sat(\underline{f}^T \underline{\hat{x}}_t; \alpha), \tag{9}$$

where sat denotes a saturation function and $\underline{\hat{x}}_t$ is the output of the Kalman filter (7).

3 CONTROL UNDER AMPLITUDE CONSTRAINT

Consider now the cost function

$$J = E[\underline{x}_t^T Q_x \underline{x}_t + q_u u_t^2] = trQ_xR_x + q_u\sigma^2_u, \tag{10}$$

where $R_x = E\underline{x}_t\underline{x}_t^T$, $R_x = R_{\hat{x}} + P_k$ and $R_{\hat{x}} = E\underline{\hat{x}}_t\underline{\hat{x}}_t^T$. If the weight matrix Q_x is such that $Q_x = \underline{h}\underline{h}^T$ then it is easy to see that the cost function J (10) can be considered as an alternative formulation for J_1 (2) w.r.t. minimization.

Using any stabilizing feedback control law, the following stationary equation for $R_{\hat{x}}$ resulting from (7) can be derived

$$\begin{aligned} R_{\hat{x}} &= FR_{\hat{x}}F^T + FR_{\hat{x}u}\underline{g}^T + \underline{g}R^T_{\hat{x}u}F^T + \\ &\quad +\sigma^2_u \underline{g}\underline{g}^T + \sigma^2_{\tilde{y}}\underline{k}\underline{k}^T, \end{aligned} \tag{11}$$

where $R_{\hat{x}u} = E\underline{\hat{x}}_t u_t$. The approximate expressions for σ^2_u and $R_{\hat{x}u}$ under the constrained control law (9) are (Toivonen, 1983):

$$\sigma^2_u = \sigma^2 g_1(\sigma), \quad R_{\hat{x}u} = R_{\hat{x}}\underline{f}g_2(\sigma), \tag{12}$$

where

$$\sigma^2 = \underline{f}^T R_{\hat{x}} \underline{f} \tag{13}$$

and $g_1(\sigma) = erf(\alpha\sigma^{-1}2^{-\frac{1}{2}}) - \alpha\sigma^{-1}2^{\frac{1}{2}}ierfc(\alpha\times \times\sigma^{-1}2^{-\frac{1}{2}})$, $g_2(\sigma) = erf(\alpha\sigma^{-1}2^{-\frac{1}{2}})$. Introducing (12), (13) into (11) one obtains an equation that enables iterative calculation of $R_{\hat{x}}$. The corresponding cost function (10) takes then the form

$$\begin{aligned} J(\underline{f}) &= tr(Q_x + q_u g_1(\sigma)\underline{f}\underline{f}^T)R_{\hat{x}} + \\ &\quad +trQ_xP_k = \\ &= J_f(\underline{f}) + trQ_xP_k. \end{aligned} \tag{14}$$

Using the gradient of $J_f(\underline{f})$ the following iterative algorithm for calculating the feedback gain $\underline{f}$ in the control law (9) can be proposed (Tovoinen, 1983)

$$\underline{f}^{(k+1)} = \underline{f}^{(k)} + \alpha_k \underline{s}^{(k)}, \tag{15}$$

where α_k is the step length and

$$\underline{s}^{(k)T} = d^{(k)} \left(\frac{\partial J_f}{\partial \underline{f}}\right)^{(k)T} \left(R^{(k)}_{\hat{x}}\right)^{-1} \tag{16}$$

for the gradient given as

$$\left(\frac{\partial J_f}{\partial \underline{f}}\right)^{(k)T} = \underline{e}^{(k)T} R^{(k)}_{\hat{x}}. \tag{17}$$

Calculations for k-th iteration are performed for $\underline{f}^{(k)}$. Expressions for $d^{(k)}, \underline{e}^{(k)}$ are given as follows (Tovoinen, 1983):

$$\begin{aligned} d^{(k)} &= -\frac{1}{2}\left[\left(g_1(\sigma^{(k)}) + h_1(\sigma^{(k)})\sigma^{2(k)}\right) \times \right. \\ &\quad \left. \times \left(\underline{g}^T S^{(k)} \underline{g} + q_u\right)\right]^{-1}, \\ \underline{e}^{(k)T} &= 2\left[\left(g_1(\sigma^{(k)}) + h_1(\sigma^{(k)})\sigma^{2(k)}\right) \times \right. \\ &\quad \times \left(\underline{g}^T S^{(k)} \underline{g} + q_u\right) \underline{f}^{(k)T} + \\ &\quad + g_2(\sigma^{(k)})\underline{g}^T S^{(k)} F + \\ &\quad \left. + 2h_2(\sigma^{(k)})\underline{g}^T S^{(k)} F R^{(k)}_{\hat{x}} \underline{f}^{(k)} \underline{f}^{(k)T}\right], \end{aligned}$$

where $S^{(k)}$ is a positive definite solution of the equation

$$\begin{aligned} S^{(k)} &= F^T S^{(k)} F + Q_x + \\ &\quad +\underline{f}^{(k)}\left[\left(g_1(\sigma^{(k)}) + h_1(\sigma^{(k)})\sigma^{2(k)}\right) \times \right. \\ &\quad \times\left(\underline{g}^T S^{(k)} \underline{g} + q_u\right) + \\ &\quad \left. + 2h_2(\sigma^{(k)})\underline{g}^T S^{(k)} F R^{(k)}_{\hat{x}} \underline{f}^{(k)}\right] \underline{f}^{(k)T} + \\ &\quad +g_2(\sigma^{(k)})(F^T S^{(k)} \underline{g}\underline{f}^{(k)T} + \underline{f}^{(k)} \underline{g}^T S^{(k)} F). \end{aligned}$$

As an initial iteration for calculation of $R^{(k)}_{\hat{x}}$ one can take for example $R^{(0)}_{\hat{x}} = \Sigma_e = \underline{k}\underline{k}^T\sigma^2_e$, and $\underline{f}^{(0)}$ where $\underline{f}^{(0)}$ results from the standard unconstrained solution of LQG problem. It is convenient to take the same value of $\underline{f}^{(0)}$ as an initial iteration in (15). It can be shown (Tovoinen, 1983) that there is a constant $a > 0$ such that for every $\alpha_k \in (0, a)$ it holds

$$J_f(\underline{f}^{(k+1)}) < J_f(\underline{f}^{(k)}),$$

if $(\frac{\partial J_f}{\partial \underline{f}})^{(k)}) \neq 0$. Thus, the proper choice of step α_k assures the convergence of the algorithm.

4 CONTROL UNDER VARIANCE CONSTRAINT

In the case of variance constraint given by the inequality (4) the associated Lagrangian is

$$L = J + \lambda(\sigma_u^2 - c^2) \tag{18}$$

or alternatively, the Lagrangian L can be rewritten

$$L = trQ_x R_x + (q_u + \lambda)\sigma_u^2, \tag{19}$$

where $\lambda \geq 0$ is the Lagrange multiplier. The Kuhn-Tucker necessary conditions for the constrained minimum of L are

$$\frac{\partial L}{\partial \lambda} \leq 0, \qquad \frac{\partial L}{\partial \underline{f}} = 0. \tag{20}$$

The optimal variance constrained control strategy can be computed by solving the conditions (20). In practice, this is done iteratively, as it will be shown in Section 5.

The controller to be designed is of the form

$$u_t = \underline{f}^T \hat{\underline{x}}_t, \tag{21}$$

where $\underline{f}$ follows from appropriate Riccati equation and $\hat{\underline{x}}_t$ is the Kalman filter output. The minimization of the Lagrangian (19) w.r.t. all admissible u_t is closely related to the minimization of the loss function J subject to the constraint (4). If $u_t^* = \underline{f}^{*T} \hat{\underline{x}}_t$ minimizes the Lagrangian (19), and the inequality constraint (4) and complementary condition

$$\lambda(\sigma_u^2 - c^2) = 0 \tag{22}$$

are fulfilled at u_t^*, then u_t^* is also an optimal control signal for variance-constrained control problem.

A major problem is the determination of appropriate estimates for the Lagrange multiplier λ such that the conditions (4) and (22) are satisfied for u_t^*. In practice this is done iteratively where each iteration step k consists of solving a standard LQG problem, i.e. of minimizing the Lagrangian (19) with $\lambda = \lambda^{(k)}$ and of updating the Lagrange multiplier according to a suitable algorithm. A realization of this algorithm needs the appropriate equations for $R_{\hat{x}}$ and σ_u^2, (see eqns.(25), (26)).

An iterative algorithm for updating the Lagrange multiplier $\lambda^{(k)}$ proposed in (Mäkilä, 1982, Mäkilä et at, 1984) can be combined with an algorithm described in Section 3 to yield the algorithm given below.

5 SIMULTANEOUS AMPLITUDE AND VARIANCE CONSTRAINTS

First, it can be observed that the amplitude constraint α (3) restricts itself the input variance because $\sigma_u^2 \leq \alpha^2$. Taking into account (4) and assuming $c^2 = \gamma\sigma_e^2$ one obtains

$$\gamma \leq \frac{\alpha^2}{\sigma_e^2}. \tag{23}$$

This means that if for a given amplitude constraint α, a given variance constraint has a form $\gamma \geq \frac{\alpha^2}{\sigma_e^2}$ then it is automatically fulfilled and optimization of the feedback gain can only be performed wrt amplitude constraint as shown in Section 3. On the other hand, if for a given α, a given variance constraint is such that $\gamma < \frac{\alpha^2}{\sigma_e^2}$ then a problem may have an optimization sense according to the problem formulated in Section 2. The proposed algorithm consists of the following steps:

step 1: Take $\lambda^{(0)} > 0, h_0 = 1, 0 < \alpha_0 < 1$.

step 2: Calculate $\underline{f}^{(k)}$ according to the method given in Section 3 for

$$q_u^{(k)} = q_u + \lambda^{(k)}. \tag{24}$$

step 3: Calculate $R_{\hat{x}}^{(k)}$ according to eqn. (11) taking into account (12), (13), i.e.

$$\begin{aligned} R_{\hat{x}}^{(k)} &= FR_{\hat{x}}^{(k)}F^T + \\ &+ (FR_{\hat{x}}^{(k)}\underline{f}^{(k)}\underline{g}^T + \underline{g}\underline{f}^{T(k)}R_{\hat{x}}^{T(k)}F^T) \times \\ &\times g_2(\sigma^{(k)}) + \underline{g}\underline{g}^T\underline{f}^{T(k)}R_{\hat{x}}^{(k)}\underline{f}^{(k)}g_1(\sigma^{(k)}) + \\ &+ \underline{k}\underline{k}^T\sigma_{\tilde{y}}^2 \end{aligned} \tag{25}$$

and

$$\sigma_u^{2(k)} = \underline{f}^{T(k)}R_{\hat{x}}^{(k)}\underline{f}^{(k)}g_1(\sigma^{(k)}), \tag{26}$$

$$\sigma^{2(k)} = \underline{f}^{T(k)}R_{\hat{x}}^{(k)}\underline{f}^{(k)}. \tag{27}$$

step 4: Check out the value (22), i.e.

$$\psi^{(k)} = \lambda^{(k)}(\sigma_u^{2(k)} - c^2). \tag{28}$$

If $\psi^{(k)}$ is sufficiently close to zero, according to some prescribed criterion then STOP, otherwise go to step 5.

step 5: If $k = 0$, then go to step 6, otherwise update h_k (if positive) according to

$$h_k = h_{k-1} + \frac{\Delta\lambda^{(k)} + h_{k-1}\Delta\psi^{(k)}}{\Delta\psi^{(k)}}, \tag{29}$$

where $\Delta\lambda^{(k)} = \lambda^{(k)} - \lambda^{(k-1)}$, $\Delta\psi^{(k)} = \psi^{(k)} - \psi^{(k-1)}$ and $\psi^{(k)}$ is given by (27).

step 6: Update the multiplier $\lambda^{(k)}$ according to

$$\lambda^{(k+1)} = \lambda^{(k)} + sat(\beta_k h_k \psi^{(k)}; a\lambda^{(k)}), \quad (30)$$

where $0 < a < 1$.

step 7: Calculate β_{k+1} according to

$$\beta_{k+1} = \beta_k(\gamma_0 - \beta_k)(\gamma_0 - 1)^{-1}, \quad (31)$$

where $\gamma_0 > 1$. Take $k \rightarrow k + 1$ and go to step 2.

It should be noted that in the case of tight constraints the problem may not have a solution, i.e. the set of feedback gains for which the cost function has finite values can be empty.

6 SIMULATION RESULTS

Consider the ARMAX plant described by the following stable model $A = 1 + 1.8q^{-1} - 0.9q^{-2}$, $B(q^{-1}) = q^{-1}$, $C(q^{-1}) = 1$ where the noise variance is set at $\sigma_e^2 = 1.0$.
The performance of the iterative algorithm given in Section 5 is illustrated in Figs.1,2 for constraints $\alpha = 3.0$ and $c^2 = 2.0$, initial value $q_u = 0.01$ and $Q_x = (1,0)^T(1,0)$, $\lambda^{(0)} = 1.0$, $\alpha_0 = 0.5$, $\gamma_0 = 5.0$, $a = 0.06$. The corresponding plots for $\alpha = 3.0$ and $c^2 = 3.0$ are shown in Figs.3,4. It can be seen that the input variances attain their constraint values. It is worthy to notice that the condition (23) is fulfilled for both values of constraint c^2. The plots of signals for $\alpha = 3.0$ and $c^2 = 3.0$ are shown in Fig.5, where one can see that the control signal attains sometimes its constraint.

7 CONCLUSIONS

The algorithm solving the amplitude and variance-constrained LQG control problem is given for plant described by ARMAX model. For unstable open-loop systems there is a lower bound of variance constraint which can be imposed on the control signal to preserve closed-loop stability, however imposing hard amplitude constraint is not allowable.
For the self-tuning control implementation the estimates $\hat{F}_t$, $\underline{\hat{g}}_t$, $\underline{\hat{k}}_{e,t}$ can be easily obtained from on-line estimation of the ARMAX model parameters $a_1, \ldots, a_{na}, b_1, \ldots, b_{nb}, c_1, \ldots, c_{nc}$.

REFERENCES

Królikowski, A. (1997). Self-tuning control under variance and amplitude constraints, *Prepr. 11th IFAC Symp.on Syst.Ident.*, SYSID'97, Kitakyushu, 8 - 11 July 1997, Vol.1, pp.345-350.

Mäkilä, P.M. (1982). Constrained liner quadratic gaussian control for process application, *Academic dissertation*, Report 82-6, Process Control Laboratory, Abo, Finland.

Mäkilä, P.M., T. Westerlund and H.T. Toivonen (1984). Constrained linear quadratic gaussian control with process applications. *Automatica*, 20(1), pp.15-29.

Toivonen H.T. (1983). Suboptimal control of discrete stochastic amplitude constrained systems, *Int.J.Control*, Vol.37, No.3, pp.493-502.

Toivonen, H.T. (1983). Variance constrained self tuning control, *Automatica*, 19(4), pp.415-418.

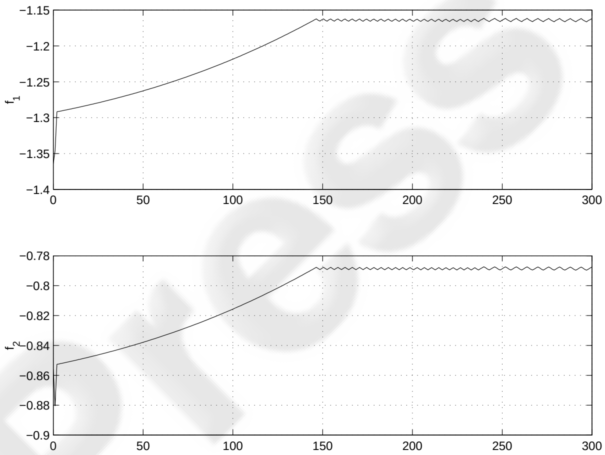

Figure 1: Plots of feedback gains f_1, f_2; $c^2 = 2.0$

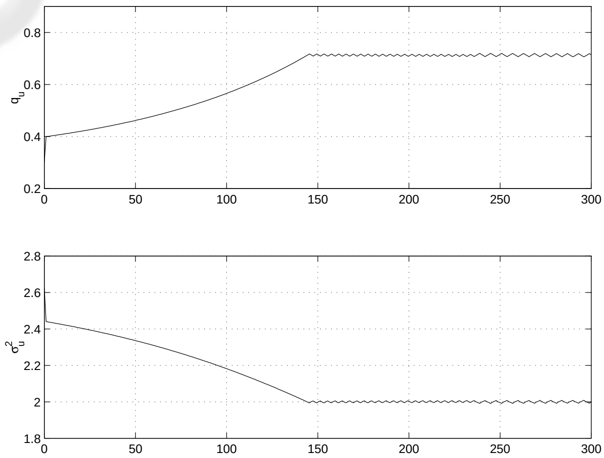

Figure 2: Plots of the weight $q_u^{(k)}$ and variance σ_u^2; $c^2 = 2.0$

SciTePress

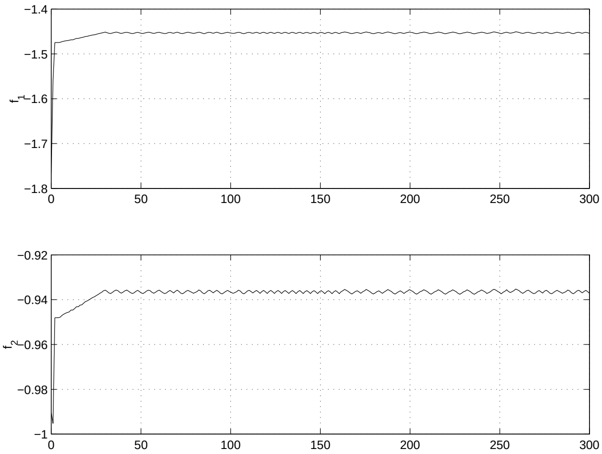

Figure 3: Plots of feedback gains f_1, f_2; $c^2 = 3.0$

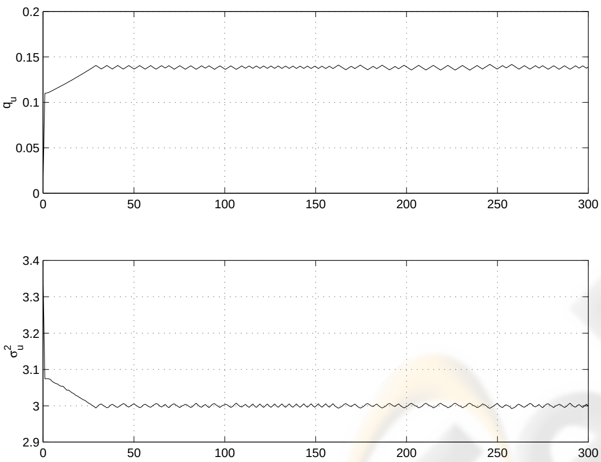

Figure 4: Plots of the weight $q_u^{(k)}$ and variance σ_u^2; $c^2 = 3.0$

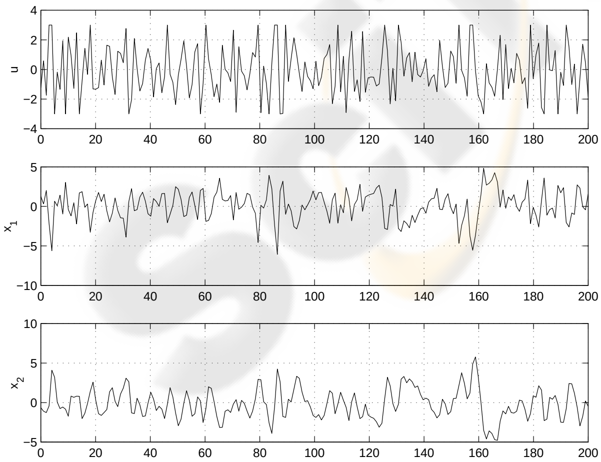

Figure 5: Plots of signals for $c^2 = 3.0$ and $\alpha = 3.0$