IMPROVEMENT OF THE DYNAMICS OF THE CONTINUOUS LINEAR SYSTEMS WITH CONSTRAINTS CONTROL

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Abstract: In this work, A time varying control law is proposed for linear continuous-time systems with non Symmetrical constrained control. Necessary and sufficient conditions allowing us to obtain the largest non-symmetrical positively invariant polyhedral set with respect to (w.r.t) the system in the closed loop are given. The asymptotic stability of the origin is also guaranteed. The case of symmetrical constrained control is obtained as a particular case. The performances of our regulator with respect to the results of (Benzaouia and Baddou, 1999) are shown with the help of an example.

1 INTRODUCTION

This paper is devoted to the study of linear continuous-time systems described by (1):

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x \in \Re^n \tag{1}$$

x is the state vector and u is the constrained control, that is:

$$u \in \Omega \subset \Re^m, m \leq n \tag{2}$$

Matrices A and B are constant and satisfy assumption (3):

$$(A, B) \text{ Controllable} \tag{3}$$

Ω is the set of admissible controls defined by (4):

$$\Omega = \left\{ u \in \Re^m / -q_2 \leq u \leq q_1; q_1, q_2 \in \operatorname{int} \Re^m_+ \right\} \tag{4}$$

This is a non-symmetrical polyhedral set, as is generally the case in practical situation.
Practical control systems are often subject to technological and safety constraints, which are translated as bounds on the constraint and state variables. The respect of this constraint can be accomplished by designing suitable feedback law.
In many cases, this can be done by constructing positively invariant domains inside the set of the constraints. The purpose of a regulation law is to stabilise the system while maintaining its state vector in a positively invariant set (Benzaouia and Hmamed, 1993) (Benzaouia and Burgat, 1989). Many approaches have been derived from this concept. Particularly, one which consists on both, using large initialisation domain and respecting the constrained control, (Benzaouia and Baddou, 1999) (Benzaouia and Burgat, 1989) (Benzaoiua, 1988) (Bistoris, 1991) (Wredenhagen and Bélanger, 1994). Recently, a piecewise linear control law has been derived for linear continuous time systems, leading to the use of non-symmetrical Lyapunov functions (Benzaouia and Baddou, 1999). These functions were introduced in (Benzaouia and Burgat, 1989), and are also used in (Benzaouia and Hmamed, 1993). Otherwise, the proposed technique seems to be very long and the problem appears between the size of the initialisation do main and the dynamic of the closed loop system. This justifies the development of this technique by using a time varying regulator. The choose of such regulator has been the subject of many works from which we cite, (Makoudi and Radouane, 1992) (Makoudi and Radouane, 1991) (Anderson and Moore, 1981) in the decentralized control case. Inspired by the work in (Benzaouia and Baddou, 1999), our contribution in the present paper is intended to improve the speed of regulation by setting the modified control law as follows:

$$u(t) = \phi(t)F_0x(t) = F(t)x(t), \; F_0 \in \Re^{mxn} \tag{5}$$

$\operatorname{rang}(F_0) = m$ with $\phi(t) > 0$, $\forall\, t \geq 0$.

Taking into account (5), system (1) becomes a non-stationary system in the following form:

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$$\dot{x}(t) = (A + \phi(t)BF_0)x(t), \ \forall\, t \ge 0 \tag{6}$$

Generally $\phi(t)$ and matrix F_0 must be found that makes the system (6) asymptotically stable and inside the constraints. It is well known that a linear time invariant system is stable if and only if ail eigenvalues of the system matrix have negative real parts (Hahn, 1967). However, this is no longer true for linear time-varying systems. Under the assumption of the non-stationary systems, the eigenvalues method for proving the asymptotic stability is not adequate. An alternative method is the use of matrix measure that means:

$$\mu(A + \phi(t)BF_0) \le -\xi, \ \forall\, t \ge 0, \ \xi \le 0 \tag{7}$$

We will show latter in this work, how to choose the function $\phi(t)$.

Remark: Note that $\mathrm{rang}(F(t)) = m$, because $\mathrm{rang}(F_0) = m$ and $\phi(t) \ne 0$, $\forall\, t \ge 0$.

In the constrained case, we follow the approach proposed in (Gutman and Hagander, 1985) and further developed in (Benzaouia and Hmamed, 1993) (Benzaouia and Burgat, 1989) and (Vassilaki and Bistoris, 1989) and references therein. This approach consists of giving conditions on the choice of the stabilizing regulator (5) such that model (6) remains valid. This is only possible if the state is constrained to evolve in a specified region defined by:

$$D(F(t), q_1, q_2) = \left\{ x \in \Re^n \,/ -q_2 \le F(t)x(t) \le q_1;\, q_1, q_2 \in \mathrm{int}\,\Re^m_+ \right\} \tag{8}$$

Note that this domain is unbounded where $m < n$. In this case, if $x(t) \in D(F(t), q_1, q_2)$ we may get $x(t+\lambda) \in D(F(t), q_1, q_2)$, $\forall\, \lambda \ge 0$. Note that the main property of this set in the stationary case is not valid in our case that is the set $x(t) \in D(F(t), q_1, q_2)$.

In particular, domain $D(I_m, q_1, q_2)$ can be described with function

$$v(z) = \max_i \max\left(\frac{z_i^+}{q_1^i}, \frac{z_i^-}{q_2^i} \right) \tag{9}$$

i.e., $D(I_m, q_1, q_2) = \left\{ z \in \Re^m \,/\, v(z) \le 1 \right\}$.

It follows from above that the main result of this note is to give the necessary and sufficient conditions under which the nonsymmetrical polyhedral domain Ω is positively invariant w.r.t. motions of system 6.

2 PRELIMINARIES

In this section, we present some definitions and useful results for the sequel. Consider a continuous-time non-linear system

$$\dot{z}(t) = f(z(t)), \ \ z \in \Re^m, f(0) = 0 \tag{10}$$

Definitions 2.1: i) Consider a function $v : \Re^m \to \Re^+$ with $v(0) = 0$ and assume that v is directionally differentiable at each direction and define $\dot{v}(z)$ by:

$$\dot{v}(z(t)) = \frac{d^+}{dt}\left[v(z(t)) \right] = \partial v\left[z(t); f(z(t)) \right]$$
$$= \lim_{\varepsilon \to 0^+} \frac{v(z + \varepsilon f(z)) - v(z)}{\varepsilon} \tag{11}$$

$\dot{v}(z)$ is the directional derivative of function v at z in the direction f(z) (Hahn, 1967), with $f(0) = 0$ and $\dot{z}(t) = f(z(t))$. **ii)** If function v is a Lyapunov function of system (10) on a set $\Im \subseteq \Re^m$ then domain D defined by

$D = \{ z \in \Im \,/\, v(z) \le c, c > 0 \}$

is a stability domain of the system.

Lemma 2.2 (Desoer and Vidyasagar, 1975):

Let $A,\ B \in C^{nxn}$, we have:

a) $\mathrm{Re}(\lambda_i(A)) \le \mu(A), \forall i = 1,\dots, n$.

b) $\mu(cA) = c\mu(A), \forall c \ge 0$

c) $\mu(A + cI) = \mu(A) + c$, $\forall c \in \Re$

d) $\mu(A + B) \le \mu(A) + \mu(B)$

e) $\mu : C^{nxn} \to \Re$ is convex on C^{nxn}

$\mu(\lambda A + (1-\lambda)B) \le \lambda\mu(A) + (1-\lambda)\mu(B)$,

$\forall \lambda \in [0,1]$

Definition 2.3 (Benzaoiua, 1988): A differentiable non-zeros vector e(t) is said to be the extended-eigenvector (x-eigenpair) of the nxn matrix G(t), associated with the extended-eigenvalues $\lambda(t)$ (a scalar time function) if it satisfies,

$G(t)e(t) = \lambda(t)e(t) + \dot{e}(t)$

Consider the following continuous non-stationary system,

$$\dot{z}(t) = H(t)z(t),\ z \in \Im \subseteq \Re^m \text{ and } 0 \in \mathrm{Int}\Im \tag{12}$$

The necessary and sufficient condition of function v defined by (9) to be a Lyapunov function for system (12) is given by the following result.

Theorem 2.4

Function $v(z) = \max_i \max\left(\frac{z_i^+}{q_1^i}, \frac{z_i^-}{q_2^i}\right)$ with $q_1 > 0,\ q_2 > 0$, which is continuous positive definite, is a Lyapunov function of system (12) on the set $\Im$ and domain:

$$D(I_m, q_1, q_2) = \left\{ z \in \Re^m / -q_2 \le z \le q_1 \right\} \subseteq \Im$$

is a stability domain of the system if and only if :

$$\tilde{H}(t)q \le 0\ ,\ \forall t \ge 0 \tag{13}$$

$$\tilde{H}(t) = \begin{bmatrix} H_1(t) & H_2(t) \\ H_2(t) & H_1(t) \end{bmatrix}, q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix},\ \forall t \ge 0$$

$$H_1(t) = \begin{cases} h_{ii}(t) & i = j \\ h_{ij}^+(t) & i \ne j \end{cases},\ H_2(t) = \begin{cases} 0 & i = j \\ h_{ij}^-(t) & i \ne j \end{cases}$$

Proof: (If) The same as (Benzaouia and Hmamed, 1993), with:

$$\dot{v}(z) \le \max_i \max\left(\frac{\left(H_1(t)q_1 + H_2(t)q_2\right)_i}{q_1^i} v(z); \frac{\left(H_2(t)q_1 + H_1(t)q_2\right)_i}{q_2^i} v(z) \right)$$

$$\forall t \ge 0 \tag{14}$$

From condition (13), we have:

$$\dot{v}(z) \le 0\ ,\ \forall z \in \Im \subseteq \Re^m$$

Consequently, from (Hahn, 1967), we conclude that domain $D(I_m, q_1, q_2)$ is a stability domain of the system.

(Only if): Assume that function $v(z)$ is a Lyapunov function of system (6) and condition (13) does not hold, i.e., there exist only $i \in [1, m]$ such that,

$$h_{ii}(t)q_1^i + \sum_{j=1,\, j \ne i}^{m} \left[h_{ij}^+(t)q_1^j + h_{ij}^-(t)q_2^j \right] > 0$$

At this step, we follow the proof given in (Benzaouia and Hmamed, 1993).

Remarks

1) When $\phi(t) = 1$, we obtain the result given in (Benzaouia and Hmamed, 1993).

2) It is well known that a stability domain for system (12) is also a positively set for the system

3) The relation (13) is equivalent to the following matrix measure:

$$\mu(H(t)) \le 0\ ,\ \forall t \ge 0 \tag{15}$$

Induced by the vector norm:

$$\|z\| = \max_i \max\left(\frac{z_i^+}{q_1^i}, \frac{z_i^-}{q_2^i} \right) \tag{16}$$

For more detail, see **Appendix 1**.

4) If there exist $\xi > 0$ such that $\mu(H(t)) \le -\xi$, we have:

$$\dot{v}(z) \le -\xi v(z) \tag{17}$$

and then from (Hahn, 1967), system (12) is asymptotically stable.

The symmetrical case is obtained directly by :

Corollary 2.5

Function $v(z) = \max_i \left(\frac{|z_i|}{q^i} \right)$ is a Lyapunov function of system (12) on the set $\Im$ and domain $D(I_m, q, q) = \left\{ z \in \Re^m / -q \le z \le q \right\} \subseteq \Im$ is a stability domain of the system if and only if:

$$\hat{H}(t)q \le 0 \quad \text{with}\ \hat{H} = \begin{cases} h_{ii}(t) & i = j \\ |h_{ij}(t)| & i \ne j \end{cases},\ \forall t \ge 0$$

Proof: Follows readily from Theorem 2.4.

3 MAIN RESULTS

In this section, we apply the results of Theorem 2.4 to the problem of the constrained regulator described in Section I.

Consider system (1) with the feedback law given by (5). The system in the closed loop is then given by (6). Let us make the change of variables,

$$\begin{aligned} z(t) &= \phi(t)F_0 x(t)\ ,\ F_0 \in \Re^{mxn} \\ &= F(t)x(t) \end{aligned} \tag{18}$$

with matrix F_0 given by (5) and (7). It follows that:

$$\begin{aligned} \dot{z}(t) &= \left[\dot{\phi}(t)F_0 + \phi(t)F_0(A + \phi(t)BF_0)\right]x(t) \\ &= \phi(t)F_0\left[\frac{\dot{\phi}(t)}{\phi(t)} I_n + A + \phi(t)BF_0 \right]x(t) \end{aligned}$$

If there exists a matrix $H(t) \in \Re^{mxm}$ such that:

$$F(t)[A + BF(t)] = \left[H(t) - \frac{\dot{\phi}(t)}{\phi(t)} I_n \right] F(t)\ ,\ \forall t \ge 0 \tag{19}$$

Then, the change of variables (18) allows us to transform dynamical system (6) to dynamical non-stationary system (12). The study of the stability of system (6) with $x \in D(F(t), q_1, q_2)$ defined by (8), becomes possible by the use of system (12) and Theorem 2.4, with $z \in \Im = D(I_m, q_1, q_2) = \Omega$.

Before giving the main result, we present all the necessary Lemmas. The first concerns (19), which is to be for every t.

For this, let us define the set $\aleph(F)$ of the matrix F(t) as follows :

$$\aleph(F) = \left\{ x(t) \in \Re^{n} \,/\, F(t)x(t) = 0,\ \forall t \geq 0,\ F(t) \in \Re^{mxn} \right\}$$ I

n the stationary case, $\aleph(F) = Ker(F)$

We note $\hbar(t) = H(t) - \frac{\dot{\phi}(t)}{\phi(t)} I_n$ and $A_0(t) = A + BF(t)$.

Lemma 3.1

If a matrix $H(t) \in \Re^{mxm}$ satisfying (19) exists, then n-m stables extended eigenvectors common to matrices A and $A_0(t)$ belong to $\aleph(F)$.

Proof:

Let a matrix $\hbar(t)$ satisfying equation (19) exists. Consider an extended eigenvector e(t) of matrix $A_0(t)$ corresponding to an extended eigenvalue $\lambda(t)$, (Min-Yen, 1982), i.e:

$$A_0(t)e(t) = \lambda(t)e(t) + \dot{e}(t) \tag{20}$$

Equation (19) allows us to write

$$\begin{aligned} F(t)A_0(t)e(t) &= \lambda(t)(F(t)e(t)) + (F(t)\dot{e}(t)) \\ &= \hbar(t)(F(t)e(t)) \end{aligned} \tag{21}$$

Then $F(t)e(t)$ is an extended eigenvector of matrix $\hbar(t)$ corresponding to the same extended eigenvalue $\lambda(t)$. Matrix $\hbar(t) \in \Re^{mxm}$ could admit only m extended eigenvalues from the set of extended eigenvalues of matrix $A_0(t)$. Let us note $\sigma(A_0(t)) = \Lambda = \Lambda_1 \cup \Lambda_2$, with $\sigma(\hbar(t)) \subset C^m$ and $\Lambda_2 \subset C^{n-m}$.

Where $\sigma(A_0(t))$ ($\sigma(\hbar(t))$) denotes a set of extended eigenvalues of $A_0(t)$ (respectively $\hbar(t)$).

Then, for $\delta \in \Lambda_2$, we should have,

$$A_0(t)w(t) = \delta(t)w(t) + \dot{w}(t) \tag{22}$$

then

$$\begin{aligned} F(t)A_0(t)w(t) &= \delta(t)(F(t)w(t) + (F(t)\dot{w}(t)) \\ &= \hbar(t)(F(t)w(t)) \end{aligned} \tag{23}$$

Implies,

$$F(t)w(t) = 0,\ \forall t \geq t_0 \tag{24}$$

For w satisfying $A_0(t)w(t) = \delta w(t) + \dot{w}(t)$.

Since $A_0(t) = A(t) + B(t)F(t)$, we should also have:

$A(t)w(t) + B(t)F(t)w(t) = \delta(t)w(t) + \dot{w}(t)$

From (24), we obtain $A(t)w(t) = \delta(t)w(t) + \dot{w}(t)$, and then $\Lambda_2 \subset \sigma(A(t))$.

If $\delta = 0$, then from (23), $(d/dt)(F(t)w(t)) = 0$, implies $F(t)w(t) = cste$. In this case, vector w(t) do not belong necessarily to $\aleph(F)$. Further, condition (7) ensures that $\mu(A_0(t)) \leq -\xi$, $\xi > 0$, $\forall t \geq t_0$, using the fact that $Re(\lambda_i(A_0(t)) \leq \mu(A_0(t))$, (Benzaouia, 1994), then, the set of extended eigenvalues of matrix $A_0(t)$ is stable. Consequently, Λ_2 contains n-m stable and non-null extended eigenvalues corresponding to n-m common extended eigenvectors to matrices A(t) and $A_0(t)$ and belonging to $\aleph(F)$.

We now give two lemmas on the $\aleph(F)$ with $F(t) \in \Re^{mxn}$ and $rank(F(t)) = m$.

Lemma 3.2

There exists a matrix $H(t) \in \Re^{mxm}$ satisfying relation (19) if and only if the existence of $t > 0$ such that $x(t) \in \aleph(F)$ implies $x(t+\tau) \in \aleph(F)$, $\forall \tau > 0$, $\forall t$.

Proof: (If): Assume that there exists a matrix $H(t) \in \Re^{mxm}$ satisfying (19) and let $x(0) \in \aleph(F)$, that is,

$$F(0)x(0) = 0 \tag{25}$$

Let us present the solution for system (6) in the following form,

$$x(t) = e^{\int_0^t (A+\phi(\tau)BF_0)d\tau} x(0), \forall t \geq 0 \tag{26}$$

Using the fact that,

$$\begin{aligned} e^{\int_0^t (A+\phi(\tau)BF_0)d\tau} &= \sum_{k=0}^{\infty} \left\{ \int_0^t (A+\phi(t)BF_0)d\tau \right\}^k \\ &= I + \sum_{k=1}^{\infty} \left\{ \int_0^t (A+\phi(t)BF_0)d\tau \right\}^k \end{aligned}$$

By using (19) and the following relation obtained from (19)

$$F_0 e^{\int_0^t (A+\phi(\tau)BF_0)d\tau} = e^{\int_0^t \left[-\frac{\dot{\phi}(\tau)}{\phi(\tau)} I + H(\tau) \right] d\tau} F_0$$

then,

$$F(t)e^{\int_0^t (A+\phi(\tau)BF_0)d\tau}x(0) = e^{\int_0^t \left[-\frac{\dot{\phi}(\tau)}{\phi(\tau)}I+H(\tau)\right]d\tau}F_0\phi(t)x(0)$$

By using (25) and the fact that $\phi(t) \neq 0, \forall t \geq 0$, we obtain

$\phi(t)F_0x(t) = 0, \forall t \geq 0$, i.e., $x(t) \in \aleph(F)$, $\forall t \geq 0$.

(Only if): Assume that the existence of $t > 0$ such that $x(t) \in \aleph(F)$ implies $x(t+\tau) \in \aleph(F)$, $\forall \tau > 0$, and show that condition (19) holds. Let, that is $\phi(t)F_0x(t) = 0, \forall t \geq 0$. It is clear that $(d/dt)(Fx(t)) = 0$ and obviously $\dot{\phi}(t)F_0x(t) + \phi(t)F_0\dot{x}(t) = 0, \forall t \geq 0$.

We obtain:

$$\begin{cases} \phi(t)F_0x(t) = 0 \\ F\left[\frac{\dot{\phi}(t)}{\phi(t)}I + A + \phi(t)BF_0\right]x(t) = 0 \end{cases}, \forall t \geq 0 \quad (27)$$

In this step, we can generalize the results of (Benzaouia and Hmamed, 1993) to the relation (27). This implies the existence of $H(t) \in \Re^{mxm}$ such that (19) is satisfied.

Lemma 3.3

If domain $D(F(t), q_1, q_2)$ is positively invariant w.r.t. system (6), $\forall t \geq 0$, then if $x(t) \in \aleph(F)$, $x(t+\tau) \in \aleph(F)$, $\forall \tau > 0$.

Proof: Let $x(0) \in \aleph(F)$, it is clear that $x(0) \in D(F(t), q_1, q_2)$. From (26), we can deduce

$$F(t)x(t) = F(t)e^{\int_0^t (A+\phi(\tau)BF_0)d\tau}x(0), \ \forall \tau > 0.$$

At this step, we can use the proof given in (Benzaouia and Hmamed, 1993) as the proof remains unchanged. We can deduce that $F(t)x(t) = 0, \forall t \geq 0$.

We are now able to give the main result of this paper.

Theorem 3.4

Domain $D(F(t), q_1, q_2)$ is positively invariant w.r.t system (6) if and only if there exists a matrix $H(t) \in \Re^{mxm}$, such that:

i) $F_0\left[A + \phi(t)BF_0\right] = \left[H(t) - \frac{\dot{\phi}(t)}{\phi(t)}I_n\right]F_0$, $\forall t \geq 0$ (28)

ii) $\tilde{H}(t)q \leq 0$, $\forall t \geq 0$ (29)

with matrix $\tilde{H}(t)$ and vector q are defined by (13).

Proof: The proof is the same as given in (Benzaouia and Hmamed, 1993) and is omitted for brevity.

Remark:

When $\phi(t) = 1$, we obtain the result given in (Benzaouia and Hmamed, 1993).

The symmetrical case is obtained directly by Corollary3.5.

Corollary 3.5

If $q_1 = q_2 = \rho$, domain $D(F(t), q_1, q_2)$ is positively invariant w.r.t system (6) if and only if there exists a matrix $H(t) \in \Re^{mxm}$, such that:

i) $F_0\left[A + \phi(t)BF_0\right] = \left[H(t) - \frac{\dot{\phi}(t)}{\phi(t)}I_n\right]F_0$, $\forall t \geq 0$

ii) $\hat{H}(t)q \leq 0$, $\forall t \geq 0$.

matrix $\hat{H}$ is given in Corollary 2.4.

The result of this Theorem is based on the existence of a matrix $H(t) \in \Re^{mxm}$ satisfying (19). A necessary and sufficient condition of the existence of a matrix $H(t)$ is giving by the following Theorem.

Theorem 3.6

There exists a matrix $H(t) \in \Re^{mxm}$ solution of (19) or (28), where $F_0 \in \Re^m$ and $rang(F_0) = m$, $m \leq n$ if and only if :

$$rang\begin{bmatrix} F_0 \\ F_0A \end{bmatrix} = m \quad (30)$$

Proof: We change only matrix A by $\left(\frac{\dot{\phi}(t)}{\phi(t)}I + A\right)$ in the proof given in (Porter, 1977) and by observing that:

$$\begin{bmatrix} F_0 \\ F_0\left(\frac{\dot{\phi}(t)}{\phi(t)}I + A\right) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \frac{\dot{\phi}(t)}{\phi(t)}I & I \end{bmatrix}\begin{bmatrix} F_0 \\ F_0A \end{bmatrix}$$

The proof remains unchanged.

In order to ensure a rate of increase of the system dynamics, one should impose to matrix H(t) :

$\tilde{H}(t)q \leq -\varepsilon q$, $\forall t \geq 0$

where ε is a positive real number ($\varepsilon \geq 0$).

Comments

Conditions (28) and (29) guarantee that domain $D(F(t), q_1, q_2)$ defined by (8) is positively invariant w.r.t system (1)-(7), despite the existence of non-symmetrical constraints on the control, but these

conditions are very difficult to verify, because we can not compute the matrix $H(t)$ for all t. Then, we propose to employ only $H(0)$ and $H(\infty)$ to handle such situation.
Before proving the Proposition 3.7, we first need the following assumptions about the function $\phi(t)$:

(a) $\phi(t) > 0$, $\forall t \geq 0$

(b) $\phi(t)$ is a nondecreasing function.

(c) $\frac{\dot{\phi}(t)}{\phi(t)} \leq \frac{\dot{\phi}(0)}{\phi(0)}\left(\frac{\phi(t)-\phi(\infty)}{\phi(0)-\phi(\infty)}\right)$, $\forall t \geq 0$.

Remarks

1) From assumption (a) and (b), we have:
$0 < \phi(0) \leq \phi(t) \leq \phi(\infty)$, $\forall t \geq 0$
It follows that,
$$0 \leq \frac{\phi(t)-\phi(\infty)}{\phi(0)-\phi(\infty)} \leq 1, \quad \forall t \geq 0$$

2) From (b), we have $\dot{\phi}(t) \geq 0$, $\forall t \geq 0$, then from (a), we can conclude that:
$$\frac{\dot{\phi}(t)}{\phi(t)} \geq 0, \quad \forall t \geq 0$$

3) Giving the inequality (c), and taking its limit as t tends to infinity, one has:
$$\lim_{t\to\infty} \frac{\dot{\phi}(t)}{\phi(t)} \leq \lim_{t\to\infty} \frac{\dot{\phi}(0)}{\phi(0)}\left(\frac{\phi(t)-\phi(\infty)}{\phi(0)-\phi(\infty)}\right)$$
It is clear that: $\frac{\dot{\phi}(\infty)}{\phi(\infty)} \leq 0$, $\forall t \geq 0$.
Combining this condition and the condition giving by Remark2, (i.e. $\frac{\dot{\phi}(t)}{\phi(t)} \geq 0$, $\forall t \geq 0$), this implies that: $\frac{\dot{\phi}(\infty)}{\phi(\infty)} = 0$. From assumption (a), one has $\dot{\phi}(\infty) = 0$. This suffices to conclude that: $\phi(\infty) = cste = K$.

Proposition 3.7

The polyhedral set defined by (8) is a positively invariant w.r.t. system (6) if and only if there exists $H(0)$ and $H(\infty)$ such that:

$$F_0\left[A + \phi(0)BF_0\right] = \left[H(0) - \frac{\dot{\phi}(0)}{\phi(0)} I_n\right]F_0 \quad (31)$$

$$F_0\left[A + \phi(\infty)BF_0\right] = \left[H(\infty) - \frac{\dot{\phi}(\infty)}{\phi(\infty)} I_n\right]F_0 \quad (32)$$

$$\tilde{H}(0)q \leq 0 \quad (33)$$

$$\tilde{H}(\infty)q \leq 0 \quad (34)$$

Proof:

(IF) It follows from (31), (32) and (19) that:

$$F_0A = H(0)F_0 - F_0\left[\frac{\dot{\phi}(0)}{\phi(0)} I_m + \phi(0)BF_0\right] \quad (35)$$

$$= H(\infty)F_0 - F_0\left[\frac{\dot{\phi}(\infty)}{\phi(\infty)} I_m + \phi(\infty)BF_0\right] \quad (36)$$

$$= H(t)F_0 - F_0\left[\frac{\dot{\phi}(t)}{\phi(t)} I_m + \phi(t)BF_0\right] \quad (37)$$

Then the full rankness of the matrix F_0 leads to the following equation,

$$H(\infty) - \frac{\dot{\phi}(\infty)}{\phi(\infty)} I - \phi(\infty)F_0B = H(0) - \frac{\dot{\phi}(0)}{\phi(0)} I - \phi(0)F_0B \quad (38)$$

Then,

$$H(\infty) = H(0) - \left[\frac{\dot{\phi}(0)}{\phi(0)} - \frac{\dot{\phi}(\infty)}{\phi(\infty)}\right]I - \left[\phi(0) - \phi(\infty)\right]F_0B \quad (39)$$

From (37) and (38), we have:

$$H(t) = \left(1 - \left(\frac{\phi(t)-\phi(\infty)}{\phi(0)-\phi(\infty)}\right)\right)H(\infty) + \left(\frac{\phi(t)-\phi(\infty)}{\phi(0)-\phi(\infty)}\right)H(0)$$
$$+\left[\frac{\dot{\phi}(t)}{\phi(t)} - \left(1-\left(\frac{\phi(t)-\phi(\infty)}{\phi(0)-\phi(\infty)}\right)\right)\frac{\dot{\phi}(\infty)}{\phi(\infty)} - \left(\frac{\phi(t)-\phi(\infty)}{\phi(0)-\phi(\infty)}\right)\frac{\dot{\phi}(0)}{\phi(0)}\right]I_m \quad (40)$$

We note:

$$e(t) = \frac{\phi(t)-\phi(\infty)}{\phi(0)-\phi(\infty)} \quad (41)$$

Then,

$$\mu(H(t)) = \mu((1-e(t))H(\infty) + eH(0) + c(t)I_m) \quad (42)$$

where:

$$c(t) = \frac{\dot{\phi}(t)}{\phi(t)} - (1-e(t))\frac{\dot{\phi}(\infty)}{\phi(\infty)} - e(t)\frac{\dot{\phi}(0)}{\phi(0)} \leq 0 \quad (43)$$

and e(t) is giving by (41).
By applying Lemma 2.2 ©, we have,

$$\mu(H(t)) = \mu((1-e(t))H(\infty) + e(t)H(0)) + c(t) \quad (44)$$

$\phi(t)$ is chosen to satisfy (a), (b) and (c), then by applying Lemma 2.2 to equation (44), we obtain,

$$\mu(H(t)) \leq (1-e(t))\mu(H(\infty)) + e(t)\mu(H(0)) + c(t) \quad (45)$$

where c(t) is giving by (40) and e(t) by (41).
Furthermore,

$$\mu(H(t)) \leq \max(\mu(H(0), \mu(H(\infty)) + c(t), \; c(t) \leq 0 \quad (46)$$

It follows that if (33) and (34) holds, from the above results, one should obtain $\tilde{H}(t)q \leq 0$, $\forall t \geq 0$.

(Only if): We assume that the polyhedral (8) is positively invariant w.r.t. system (6). By using Theorem3.4, there exists $H(t) \in \Re^{mxm}$ such that:

$$F_0\left[A+\phi(t)BF_0\right]=\left[H(t)-\frac{\dot\phi(t)}{\phi(t)}I_n\right]F_0$$

$$\tilde H(t)q \le 0 \quad \forall t \ge 0$$

In particular, for $t = 0$ and $t \to \infty$, we obtain:

$$F_0\left[A+\phi(0)BF_0\right]=\left[H(0)-\frac{\dot\phi(0)}{\phi(0)}I_n\right]F_0$$

$$F_0\left[A+\phi(\infty)BF_0\right]=\left[H(\infty)-\frac{\dot\phi(\infty)}{\phi(\infty)}I_n\right]F_0$$

$$\tilde H(0)q \le 0$$

$$\tilde H(\infty)q \le 0$$

Remarks

1) The symmetrical case is easily deduced.
2) In order to augment the system dynamics, one should impose to matrices $H(0)$ and $H(\infty)$:

$$\tilde H(0)q \le -\varepsilon q \tag{47}$$

$$\tilde H(\infty)q \le -\varepsilon q \tag{48}$$

where ε is a positive number $\varepsilon \ge 0$.

Comments:

When the regulator F_0 is changed to $F_\infty = \phi(\infty)F_0$, the eigenvalues of $H(\infty)$ will be placed in a region of the left half-complex space, which makes them more stables than the eigenvalues of $H(0)$. Furthermore, the control law increases the gain as the trajectory converges towards the origin.

$\phi(t)$ is chosen to satisfy assumptions (a), (b) and (c). This means that the dynamics amelioration cannot be made with enough liberty.

4 APPLICATION

The assumption (a), (b) and (c) institute the class of regulator, which permit to achieve the desired performance. In particular, we can choose $\phi(t)$ in the form:

$$\phi(t) = 1+\beta(1-e^{-\alpha t}),\ \alpha,\beta \ge 0$$

It is clear that the assumption a)-c) are satisfied.

The aim of this kind of regulator is to permit to start with a slow dynamics very close to the regulator with the gain F_0 and to force this dynamics to increase until it reaches the one of the regulator with the gain $(1+\beta)F_0$ at asymptotic behaviour. In addition, this permits the boundless of the time-varying control gain $\phi(t)$.

In this case, equation (31) and (32) become the following:

$$F_0(A+BF_0)=\left[H(0)-\alpha\beta I_m\right]F_0$$

$$F_0[A+(1+\beta)BF_0]=H(\infty)F_0$$

with :

$$H(t) = I(1-e^{-\alpha t})H(\infty)+e^{-\alpha t}H(0) + \left[\frac{\alpha\beta e^{-\alpha t}}{1+\beta(1-e^{-\alpha t})}-\alpha\beta e^{-\alpha t}\right]I$$

and

$$H(0) = H(\infty)+\alpha\beta I-\beta F_0B$$

Two parameters must be found to satisfy assumption (a), (b) and (c) with:

$$\frac{\dot\phi(t)}{\phi(t)}=\frac{\alpha\beta e^{-\alpha t}}{1+\beta(1-e^{-\alpha t})},\ \frac{\dot\phi(0)}{\phi(0)}=\alpha\beta,\ \frac{\dot\phi(\infty)}{\phi(\infty)}=0.$$

From (45), we have:

$$\mu(H(t)) \le (1-e^{-\alpha t})\mu(H(\infty))+e^{-\alpha t}\mu(H(0)) - \frac{\alpha\beta^2 e^{-\alpha t}(1-e^{-\alpha t})}{1+\beta(1-e^{-\alpha t})},\ \forall t \ge 0$$

In order to recapitulate all the steps required to satisfy our purpose, we present the following algorithm.

Algorithm

Step0: Verify that A possesses (n-m) stable eigenvalues. When it is not the case, we proceed to an augmentation of the vector entries without losing assumption (3a), this technique is given in (Benzaouia and Burgat, 1989 - a).

Step1: Give ε, $\alpha,\beta \ge 0$ and a matrix $H(0)$ such that;

$$\tilde H(0)q \le -\varepsilon q$$

Step2: Solve equation (31) by using the inverse procedure detailed in (Benzaouia, 1994) to obtain F_0.

Step3: Solve equation (32) to obtain $H(\infty)$.

Step4: If $\tilde H(\infty)q \le -\varepsilon q$ holds, then use α, β and F_0 to realize a time-varying regulator. If not, we return to step1.

5 COMPUTER SIMULATION

In this section, we present several numerical examples illustrating the performance of the proposed regulator.

Example1

Consider the second order system (1) given by:

$$A = \begin{bmatrix} 1 & -2 \\ -3 & -0.5 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad q = [13 \quad 5]^T .$$

$\sigma(A) = \{-2.3120, 2.8117\}$

We choose, $\alpha = 0.21$, $\beta = 3$ and $\varepsilon = 0.3$ and let: $H(0) = -0.39$.

The resolution of equation (31) gives:

$F_0 = [-9.674 \quad 5.8422]$

and then :

$F_\infty = (1+\beta)F_0 = [-38.69 \quad 23.36]$

According to (32), $H(\infty)$ is given by: $H(\infty) = -12.4554$ and

$\tilde{H}(\infty)q = [-161.9202 \quad -62.2770]^T \le -\varepsilon q$

We obtain the desired results given by:

$\sigma(A + BF_0) = \{-2.3120, -0.9998\}$

$\sigma(A + (1+\beta(1-e^{-\alpha t}))BF_0)_{t=10} = \{-2.3120, -11.0358\}$

$\sigma(A + BF_\infty) = \{-2.3120, -12.5152\}$

Note that the eigenvalues -2.3120 is common to A and $A + BF_0$

According to the result given in (Benzaouia and Baddou, 1999), we choose N=3 and H_0 such that $F_0A + F_0BF_0 = H_0F_0$ and $\tilde{H}_0 q \le -\varepsilon q$, which implies from (31) that $H_0 = H(0) - \alpha\beta I = -1.02$.

From (Benzaouia and Baddou, 1999), if we choose $\alpha_{[3]} = 1.01$, we obtain the following results, with:

$\sigma(A + BF_0) = \{-0.9998, -2.3120\}$

$\sigma(A + (\alpha_{[3]})BF_0) = \{-1.0581 \ , -2.312\}$.

$\sigma(A + (\alpha_{[3]})^2 BF_0) = \{-1.0968 \ , -2.3120\}$.

$\sigma(A + (\alpha_{[3]})^3 BF_0) = \{-1.1359 \ , -2.312\}$.

Finally, the dynamics amelioration is guaranteed by the choice of this regulator. The state and the control components for time varying control, piece-wise control (Benzaouia and Baddou, 1999) and for a fixed gain chosen to be F_0, the initial gain is represented in figure4 and figure5 respectively.

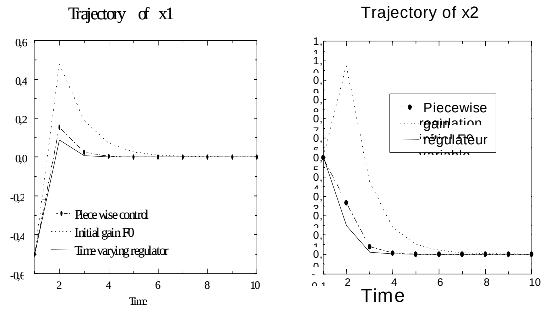

Figure 4: Space state

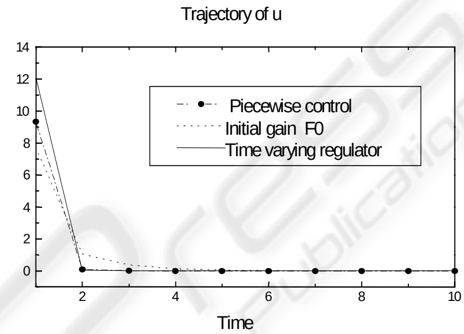

Figure 5: Control evolution

Example2

Consider the system (1) with:

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 0.45 & -4 & 4 \\ 2 & -0.9 & 15 \end{bmatrix}, B = \begin{bmatrix} 1 & 0.2 \\ 0.3 & -1 \\ 0 & -0.5 \end{bmatrix}$$

Matrix A is unstable,
i.e, $\sigma(A) = \{1.5046, 14.2937, -3.7983\}$.

$q = [37 \quad 26 \quad 25.8 \quad 7]^T$.

We choose $\alpha = 0.1$ and $\beta = 2$.

Let:

$$H(0) = \begin{bmatrix} -0.4 & 0 \\ 0 & -0.3 \end{bmatrix}$$

By applying the algorithm, the resolution of equation (31) gives:

$$F_0 = \begin{bmatrix} -4.5443 & 1.9614 & -17.4774 \\ 3.9817 & -1.7706 & 31.0186 \end{bmatrix}$$

and

$$F_\infty = (1+\beta)F_0 = \begin{bmatrix} -13.6329 & 5.8842 & -52.4322 \\ 11.9451 & -5.3118 & 93.0558 \end{bmatrix}$$

If we choose $\varepsilon = 0.1$, according to (32), we obtain:

$$H(\infty) = \begin{bmatrix} -8.5118 & 11.7369 \\ 6.9010 & -26.3847 \end{bmatrix}$$

With : $\sigma(H(\infty)) = \{-4.7653, -30.1312\}$ and :

$$\tilde{H}(\infty)q = [-9.772 \quad -430.6652 \quad -137.4461 \quad -6.6471]^T \le -\varepsilon q$$

Finally, we obtain the following results:

$$\sigma(A + BF_0) = \{-0.6, -0.5, -3.7984\}$$

$$\sigma(A + (1+\beta(1-e^{-\alpha t}))BF_0)_{t=10} = \{-19.2358, -3.2275, -3.7984\}$$

$$\sigma(A + (1+\beta)BF_0) = \{-30.1312, -4.7653, -3.7984\}$$

Note that -3.7984 is a common eigenvalues of A, $A + BF_0$ and $A + (1+\beta)BF_0$.

Furthermore,

$$\mathrm{Re}(\lambda_i(H(\infty)) \le \mathrm{Re}(\lambda_i(H(0)) , \; i = 1, \ldots, m$$

Which means that in the control, the dominant eigenvalues of $H(\infty)$ is more stable than the eigenvalues of $H(0)$.

According to the result given in (Benzaouia and Baddou, 1999), we choose $N = 3$ and a diagonal matrix H_0 such that $F_0 A + F_0 BF_0 = H_0 F_0$ and $\tilde{H}_0 q \le -\varepsilon q$, which implies from (31) that:

$$H_0 = H(0) - \alpha\beta I = \begin{bmatrix} -0.6 & 0 \\ 0 & -0.5 \end{bmatrix}$$

From (Benzaouia and Baddou, 1999), we obtain $0 < \alpha_{[3]} \le 1.0260$, if we choose $\alpha_{[3]} = 1.025$, we obtain the following results, with:

$$\sigma(A + BF_0) = \{-0.6, -0.5, -3.7984\}$$

$$\sigma(A + (\alpha_{[3]})BF_0) = \{-0.6361, -0.9117, -3.7983\} .$$

$$\sigma(A + (\alpha_{[3]})^2 BF_0) = \{-0.6955, -1.3119, -3.7983\}$$

$$\sigma(A + (\alpha_{[3]})^3 BF_0) = \{-0.7546, -1.7247, -3.7983\}$$

Then, compared to the results given in (Benzaouia and Baddou, 1999), the dynamics amelioration with a time-varying regu1ator is guaranteed and is better than that derived in (Benzaouia and Baddou, 1999). The state and the control components for time varying control, piece-wise contro1 (Benzaouia and Baddou, 1999) and for a fixed gain chosen to be F_0, the initial gain is represented in figure2 and figure3 respectively.

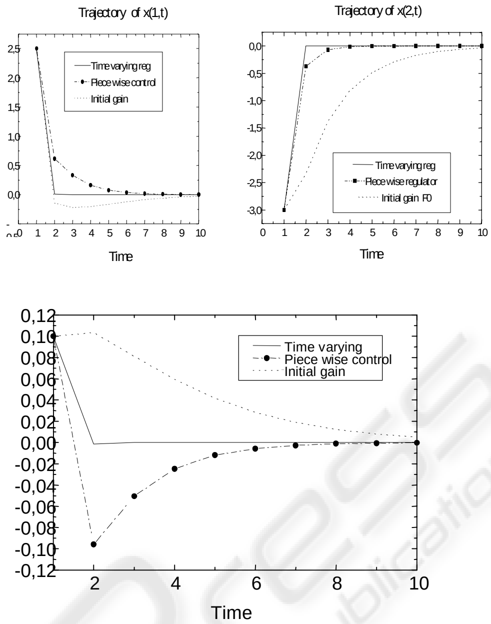

Figure 2: Space state

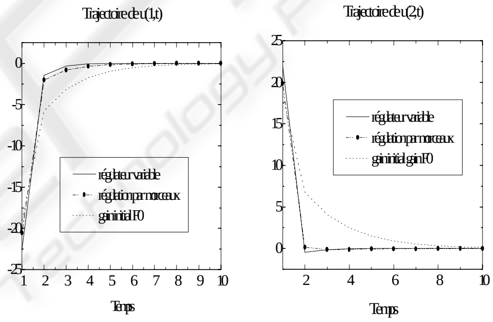

Figure 3: Control Evolution

6 CONCLUSION

In this paper, a time varying regulator is derived for linear continuous time systems. Necessary and sufficient conditions for domain $D(F(t), q_1, q_2)$ to be a positively invariant set w.r.t. system (6) are given. The proposed technique guarantees the admissibility of the control and enables system in the closed loop to admit the largest non-symmetrical constrained control. The asymptotic stability of the origin is also guaranteed. The results have been shown to be better than the literature ones.

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APPENDIX I

Matrix norm $\|M\|_\infty$

The matrix norm given by the vector norm:

$$\|z\|_\infty = \max_i \max\left(\frac{z_i^+}{q_1^i}, \frac{z_i^-}{q_2^i}\right)$$

is giving by

$$\|M\|_\infty = \max_{\|z\|_\infty = 1} \|Mz\|_\infty$$

then :

$$\|Mz\|_\infty = \max_i \max\left(\frac{(Mz)_i^+}{q_1^i}, \frac{(Mz)_i^-}{q_2^i}\right)$$

For this, we use the result of (Benzaouia and Burgat, 1989 – b, c)

$$\|Mz\|_\infty = \max_i \max\left(\frac{(M^+q_1)_i + (M^-q_2)_i}{q_1^i}, \frac{(M^-q_1)_i + (M^+q_2)_i}{q_2^i}\right)\|z\|_\infty$$

Thus, $\|M\|_\infty =$

$$\max_i \max\left[\sum_{j=1}^m \frac{q_1^j}{q_1^i} m_{ij}^+ + \frac{q_2^j}{q_1^i} m_{ij}^-; \sum_{j=1}^m \frac{q_1^j}{q_2^i} m_{ij}^- + \frac{q_2^j}{q_2^i} m_{ij}^+\right]$$

APPENDIX II

NOTATIONS: If x is a vector of $\Re^n$ then:

$x_i^+ = \sup(x_i, 0)$ and $x_i^- = \sup(-x_i, 0)$, $i = 1, \ldots, n$

We will further note the following: for two vectors x, y of $\Re^n$:

$x \le y$ (Respectively, $x < y$) if $x_i \le y_i$ (respectively, $x_i < y_i$) $i = 1, \ldots, n$.

I_n is the identity matrix of $\Re^{nxn}$; $\sigma(A)$ denotes the spectrum of matrix A; $\mathrm{Re}(\lambda)$ the real part of the eigenvalue λ and $\lambda_i(A)$ the ith eigenvalue of A. $\mu(A)$ the measure of A , $\mathrm{Int}(\Re_+^m)$ is the interior of $\Re_+^m$, whereas ∂D denotes the boundary of D. $\mathrm{Ker}\ F$ is the null space of matrix F.