STATE TRANSFORMATION FOR EULER-LAGRANGE SYSTEMS

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Abstract: The transformation of an Euler-Lagrange system into a state affine system in order to solve some interesting problem as the design of observer, the output tracking control, is considered in this paper. A necessary and a sufficient condition is given as well as a method to compute this transformation.

1 INTRODUCTION

Euler- Lagrange systems with n generalized configuration coordinates $q = (q_1, ..., q_n)^\top$ are described by equations of the form

$$\begin{aligned} \dot{q} &= v, \\ M(q)\dot{v} + C(q,\ v)v + V(q) &= \tau, \end{aligned} \tag{1}$$

where $M(q)$ denotes the inertia matrix, while $C(q,\ v)v$, with $v = \dot{q} = (\dot{q}_1, ..., \dot{q}_n)^\top$ the generalized velocities, denotes the centrifugal and Coriolis forces, $V(q)$ consists of the gravity terms and τ is the vector of input torques. This celebrated family of systems has been the subject of an important literature over half a century, because the equations of many physical devices belong to this family: (M. Spong and Vidyasagar, 1989). When these systems are fully-actuated, they are globally feedback linearizable. But feedback linearization can be performed only when all the variables are measured. Unfortunately in practice, very often the variables of velocity cannot be measured. Therefore, the global output feedback stabilization of these systems with $y = q$ as output is challenging from a practical point of view. But, from a theoretical point of view, it is one of the most difficult problems in the field of nonlinear control: Indeed, the matrix $C(q, v)v$ is a nonaffine function of the unmeasured part of the state v: this fact precludes from applying most of the classical techniques. For instance, the methods of (L. Praly and Z.P. Jiang, 1993), (J.P. Gauthier and I. Kupka, 1994) and (J-B. Pomet, R. M. Hirschorn, W. A. Cebuhar, 1993). For more explanations on the obstacles which are due to the presence of terms nonaffine with respect to the unmeasured variables, see the introduction of (F. Mazenc and J.C. Vivalda, 2002).

Most recently, in (Besancon, G. 2000) an elegant alternative for one-degree-of -freedom systems was reported. The author presented a reduced order observer converging exponentially . This observer is based upon a global nonlinear change of coordinates which makes the system affine in the unmeasured part of the state. This is crucial to define a very simple controller to solve the problem of tracking trajectory. So a very natural question arises: which conditions ensure that an Euler -Lagrange systems (1) can be transformed, with the help of a change of coordinates, into some structure affine in the unmeasured part of the state.

This question has been addressed in (Besancon, G. 2000) and (Loria, A and Pantely, E, 1999). However the questions of existence and computation of the required solution were not answered. In the present paper, we address these question: we show that this problem can be brought back to the resolution of a set of partial differential equation for which an explicit solution is given.

The paper is organized as follows. The main result is stated and proved in Section 3. Section 4 is devoted to example. Section 5 contains concluding remarks.

Preliminary

Euler-lagrange systems are such that:

- The matrix $M(q)$ is symmetric positive definite for all q.

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- The equality $\dot{M}(q) = C^{\top}(q,\ v) + C(q,\ v)$. holds.

Also, throughout the paper,

- $M_n(\mathbb{R})$ denotes the set of n-square real matrices.
- $GL_m(\mathbb{R})$ denotes the set of n-square real invertible matrices.
- For $S \in M_n(\mathbb{R})$ symmetric positive definite $S^{\frac{1}{2}}$ denotes the *square root* of S.

2 PROBLEM STATEMENT

We consider the family of Euler-Lagrange systems described by the equations (1) where the output is $q = (q_1\ \dots\ q_n)^{\top} \in \mathbb{R}^n$, and the input is $\tau = (\tau_1\ \dots\ \tau_n)^{\top} \in \mathbb{R}^n$. The unmeasured part of the state is $v = (\dot{q}_1\ \dots\ \dot{q}_n)^{\top}$

As it is pointed in the introduction the difficulty to stabilize or to construct observers for system (1) mainly stems from the fact that Coriolis and centrifugal forces vector in (1), has a quadratic growth in the generalized velocities v, which are not measured. The global change of coordinates introduced in (Besancon, G. 2000) for one-degree-of freedom (i.e. $n = 1$) systems overcomes this problem by rewriting the dynamics with functions which are linear in the unmeasured velocities. As it is discussed in (Besancon, G. 2000), the design procedure might be extended to the case of systems with more degrees of freedom, as soon as the same kind of change of coordinates can be found, that is to say if we can select an invertible matrix $\mathrm{T}(q)$ such that

$$\frac{d\,\mathrm{T}(q)}{dt} = \mathrm{T}(q)M^{-1}(q)C(q,v) \tag{2}$$

Condition (2) is necessary but not sufficient. A necessary and sufficient condition is the existence of a nonsingular matrix T such that

$$\frac{d\,\mathrm{T}(q)}{dt}v = \mathrm{T}(q)M^{-1}(q)C(q,v)v \tag{3}$$

Remak 2.1. *Since matrix $M(q)$ is positive definite, there exists an uniformly bounded matrix $\Delta(q)$ such that $M(q) = \Delta^{\top}(q)\Delta(q)$. It can be easily checked that in the case one degree of freedom, $\Delta(q)$ is a solution of* (3) *and* (2). *But in the general case, this decomposition of $M(q)$ does not provide a solution of* (3) *or* (2).

Looking for some more general factorization of $M(q)$, as investigated in (Loria, A and Pantely, E, 1999) for robot manipulators, may lead to solutions.

Remak 2.2. *One can remark that if (2) admits a solution, then (3) admits also (the same) solution. But the converse is false as shown in the following example.*

Consider the following inertia matrix $M(q)$

$$M(q) = \begin{pmatrix} e^{-q_2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the Christoffel symbols of the first kind (M. Spong and Vidyasagar, 1989), matrix C is given by

$$C(q,v) = \frac{1}{2}e^{-q_2}\begin{pmatrix} -v_2 & -v_1 \\ v_1 & 0 \end{pmatrix}$$

and an easy calculation shows that the matrix

$$\mathrm{T}(q) = \begin{pmatrix} e^{-q_2} & 0 \\ \frac{1}{2}q_1e^{-q_2} & 1 \end{pmatrix} \tag{4}$$

satisfies equation (3), but not (2). In fact, equation (2) does not admit any solution (as we will see later).

3 MAIN RESULT

In this section we present the main contribution of the paper, namely the solution of problem (2) or (3). We begin with the study of the equation (2). We provide necessary and sufficient conditions for the existence of a change of coordinates as well as method to compute this solution (Corollary 3.3).

Theorem 3.1. *Consider the nonlinear system (1). Equation (2) admits a solution if and only if*

$$\frac{\partial C_i}{\partial q_j} - \frac{\partial C_j}{\partial q_i} = C_j^{\top}M^{-1}C_i - C_i^{\top}M^{-1}C_j, \tag{5}$$

for all $1 \leq i,j \leq n$. Where the matrices C_i are such as

$$C(q,\ v) = \sum_{i=1}^{i=n} C_i(q)v_i \tag{6}$$

To establish Theorem 3.1, we need to prove the following preliminary lemma, and its corollary.

Lemma 3.2. *(Isidori, 1989) Let $x_1, \dots, x_m$ denote coordinates of a point x in $\mathbb{R}^m$ and $y_1, \dots, y_n$ coordinates of a point y in $\mathbb{R}^n$. Let $M^1, \dots, M^m$ be smooth functions*

$$M^i : \mathbb{R}^m \longrightarrow \mathbb{R}^{n\times n} \tag{7}$$

such that

$$\frac{\partial M^i}{\partial x_k} - \frac{\partial M^k}{\partial x_i} + M^iM^k - M^kM^i = 0. \tag{8}$$

Consider the set of partial differential equations

$$\frac{\partial y(x)}{\partial x_i} = M^i(x)y(x),\ 1 \leq i \leq m. \tag{9}$$

Given a point $(x^0, y^0) \in \mathbb{R}^m \times \mathbb{R}^n$, there exist a neighborhood U of x^0 and a unique smooth function $y(x)$ which satisfies (9) *and is such that $y(x^0) = y^0$*

Corollary 3.3. *Let $M_1(q),\ldots,M_n(q)$ be smooth functions*

Consider the set of partial differential equations

$$\frac{\partial \mathrm{T}}{\partial q_i}(q)=\mathrm{T}(q)M_i(q), \qquad \forall i=1,\ldots n. \quad (10)$$

Given any matrix $\mathrm{T}_0\in GL_m(\mathbb{R}), q_0\in\mathbb{R}^n$, there exists a unique smooth matrix $\mathrm{T}(q)$ which satisfies (10) and is such that $\mathrm{T}(q_0)=\mathrm{T}_0$ if and only if the functions $M_1(q),\ldots,M_n(q)$ satisfy the conditions

$$\forall i<j\leq n;\ \ M_jM_i-M_iM_j=\frac{\partial M_j}{\partial q_i}-\frac{\partial M_i}{\partial q_j}. \quad (11)$$

Proof. Necessity:

The necessity follows from the assumption that a solution $\mathrm{T}(q)$ exists. Then from the property

$$\frac{\partial^2\mathrm{T}}{\partial q_i\partial q_j} = \frac{\partial^2\mathrm{T}}{\partial q_j\partial q_i} \quad (12)$$

one has

$$\frac{\partial(\mathrm{T}\ M_j)}{\partial q_i} = \frac{\partial(\mathrm{T}\ M_i)}{\partial q_j} \quad (13)$$

Expanding the derivatives on both sides we obtain

$$\mathrm{T}\ (M_iM_j+\frac{\partial M_j}{\partial q_i}) = \mathrm{T}\ (M_jM_i+\frac{\partial M_i}{\partial q_j}) \quad (14)$$

which, due to the fact that $\mathrm{T}(q)$ is invertible, yields the condition (11).

Sufficiency:

- Using Lemma 3.2.
  Let $\mathrm{T}_0\in GL_m(\mathbb{R})$ and note $\mathrm{T}_0^{-1}=(\Gamma_0^1,....,\Gamma_0^n)$, Γ_0^i being the columns of T_0^{-1}.
  Conditions (11) ensure the existence of a family of functions Γ^k such that for all k we have

  $$\frac{\partial\Gamma^k}{\partial q_i}=-M_i\Gamma^k, \qquad \Gamma^k(q_0)=\Gamma_0^k \quad (15)$$

  The matrix $\Gamma=(\Gamma^1,...,\Gamma^n)$ satisfies

  $$\frac{\partial\Gamma}{\partial x_i}=-M_i\Gamma, \qquad \Gamma(q_0)=\mathrm{T}_0^{-1}. \quad (16)$$

  Since $\Gamma(q_0)=\mathrm{T}_0^{-1}$ which is non singular, we conclude that there exists neighborhood U of q_0 such that Γ is non singular and as a solution of (10), we take the matrix $\mathrm{T}(q)=\Gamma^{-1}(q)$.

  The previous proof gives a condition of existence, but not a method allowing construction of the solution. However, the control implementation needs the knowledge of a matrix $\mathrm{T}(q)$.

  In the sequel we will give another proof of the corollary 3.3, based on a reasoning by induction which provides an explicit solution of (2). Moreover this solution is defined on the whole domain of definition of the matrices M_i.
- By induction.

By induction on n we show that if (11) holds, then we have the following property $\mathcal{P}(n)$.

$\mathcal{P}(n)$: $n\geq 1$, there exists an invertible matrix $\mathrm{T}(q)$ such that equations (10) holds.

1. For $n=1$: Equation (10) becomes

   $$\frac{\partial\mathrm{T}(q_1)}{\partial q_1} = \mathrm{T}'(q_1)=\mathrm{T}(q_1)M(q_1) \quad (17)$$

   which admits solutions for all matrix $M_1\in\mathbb{M}_m(\mathbb{R})$.
2. For $n=2$: Let $M_1(q_1,q_2), M_2(q_1,q_2)\in\mathbb{M}_m.(\mathbb{R})$ Denote by $\Phi_{q_2}(q_1)$ the solution of the non autonomous differential equation

   $$\frac{d\Phi_{q_2}}{dq_1}(q_1) = \Phi_{q_2}M_1 \quad (18)$$

   Now, let us construct a particular solution $\mathrm{T}(q)$ of the system (10) in the form $\mathrm{T}(q)=\Psi(q_2)\Phi_{q_2}(q_1)$. Since $\mathrm{T}(q)$ is a solution of (10), one can prove that

   $$\frac{\partial\mathrm{T}}{\partial q_2} = \frac{d\Psi(q_2)}{dq_2}\Phi_{q_2}+\Psi(q_2)\frac{\partial\Phi_{q_2}}{\partial q_2} \quad (19)$$

   It follows that

   $$\frac{d\Psi(q_2)}{dq_2} = \Psi(q_2)(\Phi_{q_2}M_2\Phi_{q_2}^{-1}-\frac{\partial\Phi_{q_2}}{\partial q_2}\Phi_{q_2}^{-1})$$

   Let

   $$K(q) = \Phi_{q_2}M_2\Phi_{q_2}^{-1}-\frac{\partial\Phi_{q_2}}{\partial q_2}\Phi_{q_2}^{-1} \quad (20)$$

   Then $\Psi(q_2)$ exists if we have $\frac{\partial K(q)}{\partial q_1}=0.$

   Now

   $$\begin{aligned}\frac{\partial K(q)}{\partial q_1} &= \Phi_{q_2}M_1M_2\Phi_{q_2}^{-1}+\Phi_{q_2}\frac{\partial M_2}{\partial q_1}\Phi_{q_2}^{-1}\\ &\quad -\Phi_{q_2}M_2M_1\Phi_{q_2}^{-1}-\frac{\partial^2\Phi_{q_2}}{\partial q_1q_2}\Phi_{q_2}^{-1}\\ &\quad +\frac{\partial\Phi_{q_2}}{\partial q_2}M_1\Phi_{q_2}\\ &= \Phi_{q_2}(M_1M_2+\frac{\partial M_2}{q_1}-M_2M_1\\ &\quad -\frac{\partial M_1}{q_2})\Phi_{q_2}^{-1}\\ &= 0\end{aligned} \quad (21)$$

   So $\mathcal{P}(2)$ is true.

3. Assume that $\mathcal{P}(n)$ is true, where $n1$.
   Now using the induction hypothesis, we must prove that $\mathcal{P}(n+1)$ is true ?
   Let $M_1,\ldots,M_{n+1} \in \mathbb{M}_m(\mathbb{R})$ such that

$$M_j M_i - M_i M_j = \frac{\partial M_j}{\partial q_i} - \frac{\partial M_i}{\partial q_j} \qquad (22)$$

   The induction hypothesis implies that there exists an invertible matrix $\mathrm{T}_{q_{n+1}}(q_1,q_2,\ldots,q_n)$ such that $\dfrac{\partial \mathrm{T}_{q_{n+1}}}{\partial q_i} = \mathrm{T}_{q_{n+1}} M_i, \forall i = 1,\ldots,n.$
   Let us show that there are solutions of the form $\mathrm{T} = \Psi_1(q_{n+1})\mathrm{T}_{q_{n+1}}$.
   Observe that

$$\begin{aligned}\frac{\partial \mathrm{T}}{\partial q_i} &= \Psi_1(q_{n+1})\frac{\partial \mathrm{T}_{q_{n+1}}}{\partial q_i}\\ &= \Psi_1(q_{n+1})\mathrm{T}_{q_{n+1}}M_1 = \mathrm{T}M_1,\end{aligned}$$

   and

$$\begin{aligned}\frac{\partial \mathrm{T}}{\partial q_{n+1}} &= \frac{d\Psi_1}{dq_{n+1}}\mathrm{T}_{q_{n+1}} + \Psi\frac{\partial \mathrm{T}_{q_{n+1}}}{\partial q_{n+1}}\\ &= \mathrm{T}M_{n+1} = \Psi_1 \mathrm{T}_{q_{n+1}} M_{n+1}\end{aligned}$$

   So we get

$$\frac{d\Psi_1}{dq_{n+1}} = \Psi_1(\mathrm{T}_{q_{n+1}}M_{n+1} - \frac{\partial \mathrm{T}_{q_{n+1}}}{\partial q_{n+1}})\mathrm{T}^{-1}_{q_{n+1}} \qquad (23)$$

   $\Psi_1(q_2)$ exists if the term $(\mathrm{T}_{q_{n+1}}M_{n+1} - \dfrac{\partial \mathrm{T}_{q_{n+1}}}{\partial q_{n+1}})\mathrm{T}^{-1}_{q_{n+1}}$ is independent of $q_1, q_2, \ldots, q_n$.
   Easy calculations show that for $i = 1, 2, \ldots, n$; we have

$$\frac{\partial[\mathrm{T}_{q_{n+1}}M_2\mathrm{T}^{-1}_{q_{n+1}} - \dfrac{\partial \mathrm{T}_{q_{n+1}}}{\partial q_{n+1}}\mathrm{T}^{-1}_{q_{n+1}}]}{\partial q_i} = 0 \qquad (24)$$

   This concludes the proof.

$\square$

Proof of Theorem 3.1

Returning to the problem of the existence of the solution of (2), note that we have

$$\dot{\mathrm{T}} = \sum_{i=1}^{i=n} \frac{\partial \mathrm{T}}{\partial q_i} v_i.$$

Moreover the matrix $C(q,\ v)$ is linear with respect to v_i (M. Spong and Vidyasagar, 1989). This yields for all q

$$C(q,v) = \sum_{i=1}^{i=n} C_i(q) v_i \qquad (25)$$

where matrices C_i are such that, $C_i + C_i^\top = \dfrac{\partial M}{\partial q_i}$.

Therefore one can deduce easily that T satisfies

$$\frac{\partial \mathrm{T}}{\partial q_i} = \mathrm{T}M^{-1}C_i. \qquad (26)$$

According to Corollary 3.3, we deduce that a solution of (2) exists if and only if $\dfrac{\partial \mathrm{T}}{\partial q_i} = \mathrm{T}M_i, \forall i = 1, \ldots n$ where $M_i = M^{-1}C_i$.

Now,

$$\begin{aligned}\frac{\partial M_j}{\partial q_i} - \frac{\partial M_i}{\partial q_j} &= -M^{-1}(C_i + C_i^\top)M^{-1}C_j\\ &\quad + M^{-1}\frac{\partial C_j}{\partial q_i} - M^{-1}\frac{\partial C_i}{\partial q_j}\\ &\quad + M^{-1}(C_j + C_j^\top)M^{-1}C_i\\ &= M_jM_i - M_iM_j\\ &\quad + M^{-1}(\frac{\partial C_j}{\partial q_i} - \frac{\partial C_i}{\partial q_j}\\ &\quad - C_i^\top M^{-1}C_j + C_j^\top M^{-1}C_i).\end{aligned}$$

It follows that a necessary and sufficient condition for the existence of $\mathrm{T}(q)$ such that (2) is given by:

$$\frac{\partial C_i}{\partial q_i} - \frac{\partial C_j}{\partial q_i} = C_j^\top M^{-1}C_i - C_i^\top M^{-1}C_j\ .$$

Which allows as to conclude. $\square$

The preceding theorem gives an algebraic characterization of a family of Euler-Lagrange systems which can be transformed, with the help of a change of coordinates into some structure, affine in the unmeasured part of the state $v = \dot{q}$.

The following gives another characterization of such a solution of the differential equation (2),

Theorem 3.4. *Consider an Euler-Lagrange system* (1). *The following conditions are equivalent .*

1. *There exists a matrix* $\mathrm{T}(q)$ *such that (2) holds.*
2. *There exists a matrix* $N(q)$ *such that* $M(q) = N^\top(q)N(q)$ *and* $N^\top(q)\dot{N}(q) = C(q,v)$.
3. *There exists a function* $\Theta(q) : \mathbb{R}^n \to \mathbb{R}^n$ *and* $N(q)$ *nonsingular such that* $M(q) = N^\top(q)N(q)$ *and* $Jacobian(\Theta) = N(q)$.

Proof. Due to the space limitation we only give the proof of the two first properties.

Suppose that (2) admits a solution.

Calculating $\dfrac{d\bar{M}}{dt}$, where $\bar{M} = \mathrm{T}^{\top -1}M\mathrm{T}^{-1}$.

We have

$$\begin{aligned} \frac{d\bar{M}}{dt} &= -\mathrm{T}^{\top -1}(C' M^{-1}\mathrm{T}^{\top})\mathrm{T}^{\top -1}M\mathrm{T}^{\top -1} \\ &\quad -\mathrm{T}^{\top -1}M\mathrm{T}^{-1}(\mathrm{T}M^{-1}C)\mathrm{T}^{-1} \\ &\quad +\mathrm{T}^{\top -1}\frac{dM}{dt}\mathrm{T}^{-1} \\ &= -\mathrm{T}^{\top -1}(C^{\top} + C - \frac{dM}{dt})\mathrm{T}^{-1} \\ &= 0 \end{aligned} \tag{27}$$

since $C^{\top} + C = \dfrac{dM}{dt}$.

So, $\bar{M} = \mathrm{T}^{\top -1}M\mathrm{T}^{-1} = S$, where S is a symmetric positive definite matrix .

Let $N = S^{\frac{1}{2}}\mathrm{T}$, then we obtain

$$M(q) \quad = \quad N^{\top}(q)N(q) \tag{28}$$

and

$$N(q)\frac{d}{dt}N(q) \quad = \quad C(q, v). \tag{29}$$

For sufficiency assume that there exists $N(q)$ such that $M(q) = N^{\top}(q)N(q)$ and $N^{\top}(q)\dot{N}(q) = C(q, v)$.

So we get, $N^{\top}(q)\dfrac{\partial N}{\partial q_i}(q) = C_i(q)$.

One can check easily that conditions (31) are satisfied, which concludes the proof. □

Now, we present our final result on reduction of Euler-Lagrange systems.

Let us consider the problem of finding T such that (3) is satisfied.

Observe first that the matrix $M^{-1}C(q,\ v)v$ is quadratic in v with coefficients depending only on q. i.e there exists R_i such that

$$M^{-1}(q)C(q, v)v = \sum_{i=1}^{n} v_i R_i v \tag{30}$$

then one can easily show the following

Theorem 3.5. *Consider an Euler-Lagrange system* (1). *A necessary and sufficient condition for* (3) *to admit a solution is that there exist R_i such that*

$$\forall i < j \leq n; \quad R_j R_i - R_i R_j = \frac{\partial R_j}{\partial q_i} - \frac{\partial R_i}{\partial q_j}. \tag{31}$$

Proof. The proof follows from similar reasoning as in Theorem 3.1 and Corollary 3.3 gives an explicit solution of the problem. □

Remak 3.6. *One can remark that when $n = 1$, a solution always exists (See (Besancon, G. 2000)).*

In the case of higher order system, the problem (2) *can admit no solution. Indeed, taking again the example of the Remark 2.2, a solution exists if and only if $e^{-q_2} = 0$ which is impossible.*

But if we take

$$R_1 = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}e^{-q_2} & 0 \end{pmatrix}, R_2 = \begin{pmatrix} \frac{-1}{2}e^{-q_2} & 0 \\ 0 & 0 \end{pmatrix}, \tag{32}$$

one can check easily that conditions (30) *and* (31) *are satisfied. So* (3) *admits a solution.*

Observe also that there is no function $\Theta(q)$ such that $Jacobian(\Theta) = \mathrm{T}(q)$.

4 EXAMPLE

As an example, consider the inverted pendulum. The dynamics obtained by Euler-Lagrange formulation are:

$$\begin{aligned} (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta &= \tau_1, \\ ml\ddot{x}\cos\theta + ml^2\ddot{\theta} - mlg\sin\theta &= 0, \end{aligned}$$

where $(M,\ x)$ are mass and position of the cart which is moving horizontally, $(m,\ l;\ \theta)$ are mass, length and angular derivation from the upward vertical position for the pendulum which is pivoting around a point fixed on the cart. We denote the state vector $(\theta\ \ x\ \ \dot{\theta}\ \ \dot{x})^{\top}$ as $(q_1\ \ q_2\ \ v_1\ \ v_2)^{\top}$. The output is $y = (q_1, q_2)^{\top}$.

The inertia matrix is:

$$M(q) = \begin{pmatrix} a_1 & a_2\cos q_2 \\ a_2\cos q_2 & a_3 \end{pmatrix}$$

with $a_1 = M + m$, $a_2 = ml$ and $a_3 = ml^2$.

Using Christoffel symbols, we obtain

$$\begin{aligned} C_1 &= (0) \\ C_2 &= \begin{pmatrix} 0 & -a_2\sin(q_2) \\ 0 & 0 \end{pmatrix} \end{aligned}$$

One can check easily that condition (11) are verified so according to Theorem 3.1, equation (2) admits a solution.

Let us consider $\Phi_{q_2}(q_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as a solution of the differential equation

$$\frac{d\Phi_{q_2}}{dq_1}(q_1) \quad = \quad \Phi_{q_2}M_1 = 0. \tag{33}$$

Let

$$\Psi(q_2) \quad = \quad \begin{pmatrix} \Psi_{11}(q_2) & \Psi_{12}(q_2) \\ \Psi_{21}(q_2) & \Psi_{22}(q_2) \end{pmatrix} \tag{34}$$

the solution of the differential equation

$$\begin{aligned}\frac{d\Psi(q_2)}{dq_2} &= \Psi(q_2)(\Phi_{q_2}M_2\Phi_{q_2}^{-1} - \frac{\partial\Phi_{q_2}}{\partial q_2}\Phi_{q_2}^{-1}) \\ &= \Psi(q_2)M^{-1}C_2\end{aligned} \tag{35}$$

Using Mathematica 4.1, one can check easily that a solution of Equation (35) with initial condition $\Psi(0) = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is

$$\Psi(q_2) = \begin{pmatrix} 1 & \frac{a_2\beta(0)\cos(q_2) - a_2\beta(q_2)}{a_1\beta(0)} \\ 0 & \frac{\beta(q_2)}{\beta(0)} \end{pmatrix} \tag{36}$$

where $\beta(q_2) = \sqrt{a_1a_3 - a_2^2\ \cos(q_2)^2}$.

Thanks to the Corollary 3.3, a solution $\mathrm{T}(q_2)$ of Equation (2) with $\mathrm{T}(0) = \mathrm{T}_0 = I_2$, is

$$\begin{aligned}\mathrm{T}(q_2) &= \Psi(q_2)\Phi_{q_2}(q_1) \\ &= \begin{pmatrix} 1 & \frac{a_2\beta(0)\cos(q_2) - a_2\beta(q_2)}{a_1\beta(0)} \\ 0 & \frac{\beta(q_2)}{\beta(0)} \end{pmatrix}\end{aligned} \tag{37}$$

Therefore, the following change of coordinates

$$\Theta_1 = q_1 + \int_0^{q_2} \frac{a_2\beta(0)\cos(s) - a_2\beta(s)}{a_1\beta(0)}ds,$$

$$\Theta_2 = \int_0^{q_2} \frac{\beta(s)}{\beta(0)}ds,\ p = T(q)\dot{q},$$

transforms the dynamics of the Cart-Pendulum into a double integrator

$$\dot{\Theta} = p, \tag{38}$$

$$\dot{p} = \mathrm{T}(q)M^{-1}(q)\tau = u. \tag{39}$$

Clearly This system is linear in the unmeasured part of the state and a high gain observer can be constructed (M. Mabrouk, F. Mazenc and J.C. Vivalda, 2004).

5 CONCLUSION

A necessary and a sufficient condition for determining a state change of coordinate which transform an Euler-Lagrange system into an affine system in the unmeasured part of state was given. Obviously in the case of one degree of freedom , a solution always exists. A case of higher order system, is for instance, that of the cart-pendulum system (F. Mazenc and J.C. Vivalda, 2002), the Tora system (Z. P. Jiang and I. Kanellakopoulos, 2000) and the overhead crane (B. d Andra-Novel and J. Lvine, 1990). We conjecture the result several others problems in nonlinear control.

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