CONTROLLING THE LORENZ SYSTEM WITH DELAY

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Abstract: A generalized method for adaptive control, synchronization of chaos and parameter identification in systems governed by ordinary differential equations and delay-differential equations is developed. The method is based on the Lagrangian approach to fluid dynamics. The synchronization error, defined as a norm of the difference between the state variables of two similar and coupled systems, is treated as a scalar fluid property advected by a fluid particle in the vector field of the controlled response system. As this error property is minimized, the two coupled systems synchronize and the time variable parameters of the driving system are identified. The method is applicable to the field of secure communications when the variable parameters of the driver system carry encrypted messages. The synchronization method is demonstrated on two Lorenz systems with variable parameters. We then apply the method to the synchronization of hyperchaos in two modified Lorenz systems with a time delay in one the state variables.

1 INTRODUCTION

In this paper we develop a generalized method for controlling chaos in dynamical systems and for synchronizing two coupled chaotic or hyperchaotic systems. The systems are described by ordinary differential equations with initial conditions or delay-differential equations with initial functions. The independent parameters appearing in the equations can be constant or variable. The synchronization error , defined as a norm of the difference between the state variables of the two chaotic systems, is viewed as a fluid property advected by a marker particle moving along a trajectory in the vector field of the response system. The controlled parameters of the response system are varied continuously such as to minimize the synchronization error while the two chaotic systems evolve in time. For the initial value problem, we derive a system of differential equations governing the evolution of the controlled parameters required for synchronization. For the case of the initial function problem, we derive a system of delay-differential equations governing the controlled parameters for synchronizing the response system. In both cases, the fluid dynamical approach mentioned above is used to develop the equations for the controlled parameters.

The method is demonstrated by studying several examples of chaotic and hyperchaotic systems. The synchronization method is demonstrated on two Lorenz systems with variable parameters. We then apply the method to the synchronization of hyperchaos in two modified Lorenz systems with a time delay in one the state variables.

The possibility of synchronizing two chaotic physical systems has attracted considerable attention in recent years (Boccaletti, Farini and Arecchi, 1997). A major motivation is the potential of applying the synchronization methods to the field of secure communications by chaotic masking or scrambling of messages (Goedgebuer, Larger and Porte, 1998), (Yang, 2004). Other important applications are in the field of nonstationary time series analysis and system identification (Parlitz, 1996), (Crispin, 2002). So far, most of the attention has been directed towards the study of stationary chaotic systems, that is systems with constant parameters. To date, the possibility of using systems with nonstationary parameters for secure communications has received less attention, although such systems are good candidates for secure communications,

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especially when the nonstationary parameters rather than the state variables are used to hide the secret messages (Crispin, 1998).

Several methods for the control of chaos in dynamical systems have been proposed in recent years (Boccaletti et al., 2000). The conjecture of chaos control by means of perturbations of an accessible parameter is based on an inherent property of chaotic systems, namely, their sensitivity to small perturbations in the parameters (Parlitz, Junge and Kocarev, 1996), (Carr and Schwartz, 1994), (Ott and Spano, 1995).

Model reference adaptive control methods have been suggested for chaos suppression and synchronization (Aguirre and Billings, 1994). For example, a model reference method for the adaptive control of chaos in dynamical systems with periodic forcing has been proposed by (Crispin and Ferrari, 1996). Adaptive control and synchronization of chaos in discrete time systems has been studied by (Crispin, 1997). Another method for controlling chaos in dynamical systems is by introducing parametric forcing and adaptive control, see for example (Crispin, 2000). A more recent method based on an analogy from fluid dynamics has been described by (Crispin, 2002). Other applications include parameter estimation (Parlitz, Junge and Kocarev, 1996), synchronization of chaotic systems with variable parameters (Crispin, 1998) and the control of chaos in fluids (Crispin, 1999).

Many physical, physiological and biological systems display time delay in their dynamics. Nonlinear dynamical systems with time delay can have periodic orbits or very complex dynamics depending upon the range of values of the time delay and the independent parameters of the system. This complex behavior has attracted a lot of interest in the study of time delayed systems from the mathematical point of view (Kolmanovskii and Myshkis, 1992) as well as from the physical and physiological points of view, see for example (Losson, Mackey and Longtin, 1993), (Mansour and Longtin, 1998). The use of time delay in feedback control systems has also been proposed, see for example (Hegger et al., 1998), (Goedgebuer, Larger and Porte, 1998) and (Just et al., 1998). Control of chaos in systems with time delay has also been studied in (Mansour and Longtin, 1998).

2 THE FLUID DYNAMICAL APPROACH

Consider two similar dynamical systems described by ordinary differential equations. We define two dynamical systems as similar if the right hand sides of the equations are represented by the same function $f(x(t), p(t))$, except that the independent parameters $p(t)$ or $q(t)$ can be represented either by different or the same functions of time. We first present the method for the initial value problem and then we consider the case when one of the state variables has a time delay.

$$d\boldsymbol{x}/dt = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{p}(t)) \tag{2.1}$$

$$d\boldsymbol{y}/dt = \boldsymbol{f}(\boldsymbol{y}(t), \boldsymbol{q}(t))$$

Here t is time, $\boldsymbol{x} \in \mathrm{R}^n$ are the state variables of the driver and $\boldsymbol{y} \in \mathrm{R}^n$ are the state variables of the response system. The parameters $\boldsymbol{p}(t) \in \mathrm{R}^k$ and $\boldsymbol{q}(t) \in \mathrm{R}^k$ are independent time variable parameters of the respective systems and $\boldsymbol{f}$:R^nx $\mathrm{R}^k \mapsto \mathrm{R}^n$ is a nonlinear vector function of the state variables. It is assumed that the initial values of the state variables $\boldsymbol{x}(0) = \mathrm{x}_0$ and $\boldsymbol{y}(0) = \mathrm{y}_0$ for $t = 0$ are not necessarily the same. Similarly the initial values of the parameters $\boldsymbol{p}(0) = \boldsymbol{p_0}$ and $\boldsymbol{q}(0) = \boldsymbol{q_0}$ of the driver and response systems are different. Since chaotic systems are sensitive to initial conditions, the driver and response systems will not synchronize, unless the response system is controlled and forced to synchronize with the driver system using some kind of coupling, such as a transmitted scalar signal. For instance, a single scalar signal $s(t)$, which is a function of the state $\boldsymbol{x}(t)$, can be transmitted by the driver and used to enslave the response system (Tamasevicius and Cenys, 1997), (Peng, Ding and Yang, 1996).

$$s(t) = h(\boldsymbol{x}(t)) \tag{2.3}$$

As stated above, the purpose of this paper is to propose a generalized method of control, stabilization, synchronization and parameter identification of chaotic systems in the more general case where the parameters $\boldsymbol{p}(t)$ of the driver system vary as a function of time. In the context of secure communications, this means that it would be possible to encode a message in one of the parameters of the driver system rather then in a state variable, as has been proposed so far. Once a variable parameter is identified by the response system using the proposed generalized method, the encoded message can be recovered. The method allows more flexibility in masking information in chaos. The message can be encoded in a state variable or in a time variable parameter. The useful information can also be split into two messages, where one message is modulated by a state variable and a second message modulated by a parameter. Synchronization of the state variables of an eavesdropping response system with the state variables of the driver will be difficult because of the sensitivity to small variations in the parameters of the system, in addition to the sensitivity to initial conditions and the divergence of nearby trajectories in chaotic systems. Also, synchronization can be achieved even when the parameters $\boldsymbol{p}(t)$ and $\boldsymbol{q}(t)$ are initially substantially different. This is accomplished by controlling the response system $\boldsymbol{y}$ such that the parameters $\boldsymbol{p}(t)$ of the driver system $\boldsymbol{x}$ are eventually identified, that is,

$$\lim_{t\to\infty} |\boldsymbol{p}(t) - \boldsymbol{q}(t)| = 0 \qquad (2.4)$$

In order to achieve synchronization, the dynamics of the response system parameters $\boldsymbol{q}(t)$ need to be determined. In other words, the differential equations governing the evolution of the response parameters $\boldsymbol{q}(t)$ need to be derived for dynamical systems of the form of Eqs.(2.1-2.2).

The proposed fluid dynamical approach is based on the Lagrangian description of fluid motion. It follows the motion of a fluid particle as it moves in the velocity vector field $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{p})$ created by a fluid flow (Lamb, 1995), (Milne-Thomson, 1968). According to this approach, the equations of motion of two marker particles advected in the fluid flows described by the vector fields $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{p}) = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{p})$ and $\boldsymbol{w}(\boldsymbol{y},\boldsymbol{q}) = \boldsymbol{f}(\boldsymbol{y},\boldsymbol{q})$, are given by Eq.(2.1-2.2), where the right hand sides are to be interpreted as the local velocity vectors $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{p})$ and $\boldsymbol{w}(\boldsymbol{y},\boldsymbol{q})$ at any given point $\boldsymbol{x} \in \mathrm{R}^n$ and $\boldsymbol{y} \in \mathrm{R}^n$ of the two flow fields, respectively, i.e,

$$d\boldsymbol{x}/dt = \boldsymbol{w}(\boldsymbol{x},\boldsymbol{p}) = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{p}) \qquad (2.5)$$

$$d\boldsymbol{y}/dt = \boldsymbol{w}(\boldsymbol{y},\boldsymbol{q}) = \boldsymbol{f}(\boldsymbol{y},\boldsymbol{q})$$

In fluid dynamics, the vector fields $\boldsymbol{w}(\boldsymbol{x},\boldsymbol{p}), \boldsymbol{w}(\boldsymbol{y},\boldsymbol{q}) \in \mathrm{R}^n$ and the state variables $\boldsymbol{x},\boldsymbol{y} \in \mathrm{R}^n$ have a dimension $n \leq 3$. Here the analogy is extended to vector fields of higher dimensions. Consider the time variation of a scalar property $J(\boldsymbol{x},\boldsymbol{y})$ of the flow along a trajectory of the response system as it evolves in the state space $\boldsymbol{y} \in \mathrm{R}^n$. The rate of change of this scalar property is due to two contributions: a local contribution due to its time variation, plus a contribution which is due to the rate of change of the property as it is advected along the trajectory in the state space. The total rate of change is then given by the substantial derivative DJ/Dt of the scalar property J, which is the derivative following the flow:

$$DJ/Dt = \partial J/\partial t + \boldsymbol{f}(\boldsymbol{y}(t),\boldsymbol{q})\nabla J \qquad (2.6)$$

$$\nabla = (\partial/\partial y_1, \partial/\partial y_2, ..., \partial/\partial y_n)^t$$

Here the product $\boldsymbol{f}(\boldsymbol{y},\boldsymbol{q})\nabla J$ is a scalar or dot product. Consider a scalar property J based on the Euclidean distance between the state vectors $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$J = \frac{1}{2}|\boldsymbol{y}-\boldsymbol{x}|^2 = \frac{1}{2}\sum_{i=1}^{n}(y_i - x_i)^2 = \frac{1}{2}\sum_{i=1}^{n} e_i^2 \quad (2.7)$$

where $e_i = y_i - x_i$ are components of the error vector function $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{x}$. Using Eq.(2.7), the local component of the derivative of J with respect to time is given by:

$$\partial J/\partial t = \sum_{i=1}^{n} e_i \, de_i/dt \qquad (2.8)$$

whereas the components of the gradient ∇J are given by

$$\partial J/\partial y_i = y_i - x_i = e_i \qquad (2.9)$$

Using Eqs.(2.5), (2.6), (2.8) and (2.9), the substantial derivative of J is written as:

$$DJ/Dt = \sum_{i=1}^{n} e_i[de_i/dt + f_i(\boldsymbol{y}(t),\boldsymbol{q})] \qquad (2.10)$$

and since $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{x}$ and

$$d\boldsymbol{e}/dt = d\boldsymbol{y}/dt - d\boldsymbol{x}/dt = \boldsymbol{f}(\boldsymbol{y},\boldsymbol{q}) - \boldsymbol{f}(\boldsymbol{x},\boldsymbol{p})$$

Eq.(2.10) reduces to:

$$DJ/Dt = \sum_{i=1}^{n} e_i[2f_i(\boldsymbol{y}(t),\boldsymbol{q}) - f_i(\boldsymbol{x}(t),\boldsymbol{p})] \quad (2.11)$$

Eq.(2.11) defines the rate of change of the positive scalar property J in terms of the state variables $\boldsymbol{x}$ and $\boldsymbol{y}$ of the driver and response systems, the independent driver parameters $\boldsymbol{p}$ and the controllable parameters $\boldsymbol{q}$ of the response system. The question now is how should the control vector $\boldsymbol{q}$ be varied such as to continuously minimize J? As the driver and response systems evolve, the substantial derivative should be continuously decreased in order to achieve control and synchronization, as can be seen from Eq.(2.11), where perfect synchronization is reached when $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{x} = \boldsymbol{0}$, and from Eq.(2.7), J reaches the minimum $J = 0$. A possible control law is to vary the control vector $\boldsymbol{q}$ such as to decrease the substantial derivative DJ/Dt, that is, to continuously change the control $\boldsymbol{q}$ in a direction opposite to the gradient of DJ/Dt with respect to the control $\boldsymbol{q}$:

$$d\boldsymbol{q}/dt = -G' \nabla_{\boldsymbol{q}}(DJ/Dt) =$$

$$= -G' \nabla_{\boldsymbol{q}}\{\sum_{i=1}^{n} e_i[2f_i(\boldsymbol{y}(t),\boldsymbol{q}) - f_i(\boldsymbol{x}(t),\boldsymbol{p})]\} \qquad (2.12)$$

where

$$\nabla_{\boldsymbol{q}} = (\partial/\partial q_1, \partial/\partial q_2, \, ..., \partial/\partial q_k)^t$$

and G' is a kxk matrix of control gains. Since $f_i(\boldsymbol{x}, \boldsymbol{p})$ does not depend on the control $\boldsymbol{q}$,it follows that

$$\nabla_{\boldsymbol{q}} f_i(\boldsymbol{x}(t), \boldsymbol{p}) = 0$$

,

and Eq.(2.12) becomes:

$$d\boldsymbol{q}/dt = -G \, \nabla_{\boldsymbol{q}} \, [\sum_{i=1}^{n} e_i f_i(\boldsymbol{y}(t), \boldsymbol{q})] \qquad (2.13)$$

where $G = 2G'$.

3 SYNCHRONIZING THE LORENZ SYSTEM

We now apply the method to a dynamical system without delay, the chaotic Lorenz system. Consider the case of synchronization between two Lorenz systems with variable parameters, where the driver system is given by:

$$dx_1/dt = \sigma(x_2 - x_1)$$

$$dx_2/dt = p_1(t)x_1 - p_2(t)x_2 - p_3(t)x_1x_3 \qquad (3.1)$$

$$dx_3/dt = x_1x_2 - bx_3$$

and the transmitted coupling signal for synchronization is chosen as the single variable:

$$s(t) = h(\boldsymbol{x}(t)) = x_2(t) \qquad (3.2)$$

The response system is defined by the following Lorenz system with variable parameters. It is driven by the transmitted signal $s(t) = x_2(t)$.

$$dy_1/dt = \sigma(s(t) - y_1)$$

$$dy_2/dt = q_1(t)y_1 - q_2(t)y_2 - q_3(t)y_1y_3 \qquad (3.3)$$

$$dy_3/dt = y_1s(t) - by_3$$

The next step is to derive a system of differential equations governing the evolution of the controlled parameters $q_1(t)$, $q_2(t)$ and $q_3(t)$. As the response system evolves and synchronizes with the driver system, the parameters $q_1(t)$, $q_2(t)$ and $q_3(t)$ will follow the original parameters $p_1(t)$, $p_2(t)$ and $p_3(t)$ of the driver system. According to Eq.(2.13) of the previous section, the first step is to develop the term $\sum_{i=1}^{3} e_i f_i(\boldsymbol{y}, \boldsymbol{q})$ for the response system (3.3). Using the right hand sides of (3.3), we have:

$$\sum_{i=1}^{3} e_i f_i(\boldsymbol{y}, \boldsymbol{q}) = (y_1 - x_1)[\sigma(s(t) - y_1)] +$$

$$+(y_2 - x_2)[q_1(t)y_1 - q_2(t)y_2 - q_3(t)y_1y_3] +$$

$$+(y_3 - x_3)(y_1s(t) - by_3) \qquad (3.4)$$

The gradient with respect to the parameters $\boldsymbol{q}$ is given by:

$$\nabla_{\boldsymbol{q}} \, [\sum_{i=1}^{3} e_i f_i(\boldsymbol{y}, \boldsymbol{q})] =$$

$$= [(y_2 - x_2)y_1 \, , \, -(y_2 - x_2)y_2 \, , \, -(y_2 - x_2)y_1y_3]^T \qquad (3.5)$$

The differential equations governing the evolution of the controlled parameters $\boldsymbol{q}$ are given by:

$$dq_1/dt = -G_{11}(y_2 - s(t))y_1 \, + G_{12} \, (y_2 - s(t))y_2 +$$

$$+G_{13} \, (y_2 - s(t))y_1y_3$$

$$dq_2/dt = -G_{21}(y_2 - s(t))y_1 \, + G_{22} \, (y_2 - s(t))y_2 +$$

$$+G_{23} \, (y_2 - s(t))y_1y_3 \qquad (3.6)$$

$$dq_3/dt = -G_{31}(y_2 - s(t))y_1 \, + G_{32} \, (y_2 - s(t))y_2 +$$

$$+G_{33} \, (y_2 - s(t))y_1y_3$$

For example, consider the case where only one parameter, say $p_1(t)$ is to be identified, whereas the other two parameters $p_2(t)$ and $p_3(t)$ are known constants. If the synchronized systems are used for secure communication, the parameter $p_1(t)$ can be used to carry a hidden message, which can be identified by the response system. In this case, Equations (3.4-3.6) reduce to:

$$\sum_{i=1}^{3} e_i f_i(\boldsymbol{y}, \boldsymbol{q}) = (y_1 - x_1)[\sigma(s(t) - y_1)] +$$

$$+(y_2 - x_2)[q_1(t)y_1 - p_2(t)y_2 - p_3(t)y_1y_3] +$$

$$+(y_3 - x_3)(y_1 y_2 - b y_3) \qquad (3.7)$$

$$\nabla_{\boldsymbol{q}} [\sum_{i=1}^{3} e_i f_i(\boldsymbol{y}, \boldsymbol{q})] = [(y_2 - x_2)y_1 \, , \, 0 \, , \, 0]^T \qquad (3.8)$$

$$dq_1/dt = -G_{11}(y_2 - s(t))y_1 \qquad (3.9)$$

$$dq_2/dt = 0$$

$$dq_3/dt = 0$$

We now show results of a computer simulation with the following values of the parameters:

$$\sigma = 10 \;\; b = 8/3 \qquad (3.10a)$$

$$p_1(t) = r_0 + \delta r \, sin \, \omega t \qquad (3.10b)$$

$$\delta r = 2 \;\; \omega = 2\pi/10 \;\; r_0 = 28 \qquad (3.10c)$$

$$q_2 = p_2 = 1 \;\; q_3 = p_3 = 1 \qquad (3.10d)$$

together with the initial condition:

$$q_1(0) = r_0 = 28 \qquad (3.10e)$$

The results of this example are given in Figures 1-3. Fig.1 shows the chaotic state variables of the driver system. The chaotic attractor is shown in Fig.2. Similar results are obtained for the response system as it synchronizes with the driver system. Fig.3 displays the synchronized state variables $y_1 = x_1, y_2 = x_2$ and $y_3 = x_3$, all three eventually converging to straight lines as shown in the figure. The identified signal $q_1(t) - r$ converges to the driver signal $q_1(t) - r = \delta r \, sin \, \omega t$ and is also shown in the figure.

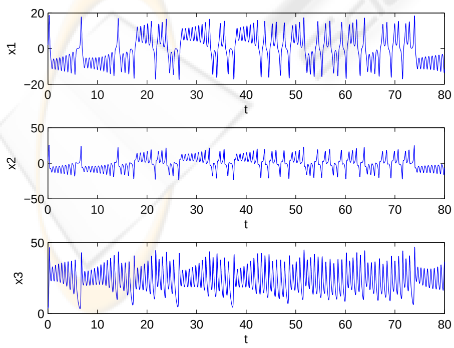

Figure 1: The chaotic state variables of the driver Lorenz system with variable parameter.

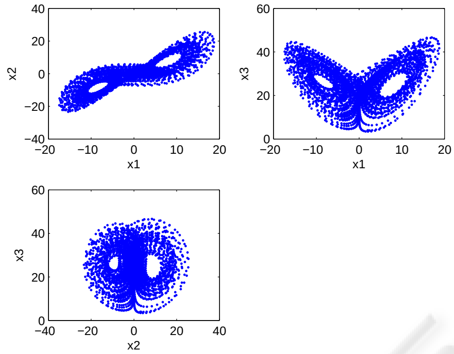

Figure 2: The chaotic attractor of the driver Lorenz system with variable parameter. The response system has a similar attractor.

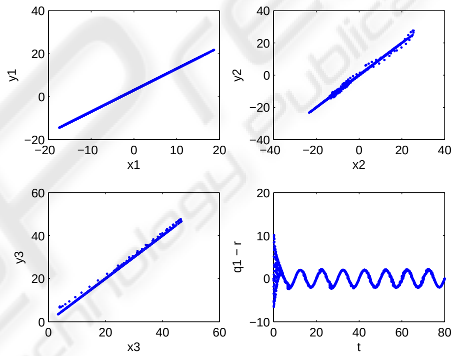

Figure 3: Synchronization between the driver and response Lorenz systems with variable parameters and the parameter q_1(t) of the response system as it identifies the driver signal p_1(t).

4 SYNCHRONIZING LORENZ SYSTEM WITH DELAY

In this section we apply the method to a hyperchaotic dynamical system, the Lorenz system with a time delay in one of the state variables. Consider the case of synchronization between two Lorenz systems with time delay. We treat the case where the state variable $x_1(t - T)$ is delayed by a time delay T. Here the driver system is given by:

$$dx_1/dt = \sigma(x_2(t) - x_1(t - T))$$

$$dx_2/dt = p_1(t)x_1(t - T) - p_2(t)x_2(t) - \\ -p_3(t)x_1(t - T)x_3(t)$$

$$dx_3/dt = x_1(t - T)x_2(t) - bx_3(t) \qquad (4.1)$$

Suppose the transmitted signal for synchronization is chosen as the single variable:

$$s(t) = h(\boldsymbol{x}(t)) = x_2(t) \tag{4.2}$$

The response system is defined by the following nonstationary and delayed Lorenz system, where the transmitted scalar signal $s(t) = x_2(t)$ is used to drive the system.

$$dy_1/dt = \sigma(s(t) - y_1(t-T))$$

$$dy_2/dt = q_1(t)y_1(t-T) - q_2(t)y_2(t) - \\ -q_3(t)y_1(t-T)y_3(t)$$

$$dy_3/dt = y_1(t-T)s(t) - by_3(t) \tag{4.3}$$

The next step is to derive a system of differential equations governing the evolution of the controlled parameters $q_1(t)$, $q_2(t)$ and $q_3(t)$. As the response system evolves and synchronizes with the driver system, the parameters $q_1(t)$, $q_2(t)$ and $q_3(t)$ will follow the original parameters $p_1(t)$, $p_2(t)$ and $p_3(t)$ of the driver system. According to Eq.(2.13) of section 2, the first step is to develop the term $\sum_{i=1}^{3} e_i f_i(\boldsymbol{y}, \boldsymbol{q})$ for the response system (4.3).

$$\sum_{i=1}^{3} e_i f_i(\boldsymbol{y}, \boldsymbol{q}) = (y_1(t-T) - x_1)[\sigma(s(t) - y_1(t-T))] +$$

$$+(y_2 - x_2)[q_1(t)y_1(t-T) - q_2(t)y_2 - q_3(t)y_1(t-T)y_3] +$$

$$+(y_3 - x_3)[y_1(t-T)s(t) - by_3] \tag{4.4}$$

The gradient with respect to the parameters $\boldsymbol{q}$ is given by:

$$\nabla_{\boldsymbol{q}} [\sum_{i=1}^{3} e_i f_i(\boldsymbol{y}, \boldsymbol{q})] =$$

$$= [(y_2 - x_2)y_1(t-T)\ ,\ -(y_2 - x_2)y_2\ ,\ -(y_2 - x_2)y_1(t-T)y_3]^t \tag{4.5}$$

The differential equations governing the evolution of the controlled parameters $\boldsymbol{q}$ are given by:

$$dq_1/dt = -G_{11}(y_2 - s(t))y_1(t-T) + G_{12}\,(y_2 - s(t))y_2 +$$

$$+G_{13}\,(y_2 - s(t))y_1(t-T)y_3$$

$$dq_2/dt = -G_{21}(y_2 - s(t))y_1(t-T) + G_{22}\,(y_2 - s(t))y_2 +$$

$$+G_{23}\,(y_2 - s(t))y_1(t-T)y_3$$

$$dq_3/dt = -G_{31}(y_2 - s(t))y_1(t-T) + G_{32}\,(y_2 - s(t))y_2 +$$

$$+G_{33}\,(y_2 - s(t))y_1(t-T)y_3 \tag{4.6}$$

For example, consider the case where only one parameter, say $p_1(t)$ is to be identified, whereas the other two parameters $p_2(t)$ and $p_3(t)$ are known constants. If the synchronized systems are used for secure communication, the parameter $p_1(t)$ can carry a hidden message, which can be identified by the response system. In this case, Equations (4.4-4.6) reduce to:

$$\sum_{i=1}^{3} e_i f_i(\boldsymbol{y}, \boldsymbol{q}) = (y_1 - x_1)[\sigma(s(t) - y_1)] +$$

$$+(y_2 - x_2)[q_1(t)y_1 - p_2 y_2 - p_3 y_1 y_3] + (y_3 - x_3)(y_1 y_2 - by_3) \tag{4.7}$$

$$\nabla_{\boldsymbol{q}} [\sum_{i=1}^{3} e_i f_i(\boldsymbol{y}, \boldsymbol{q})] = [(y_2 - x_2)y_1(t-T),\ 0\ ,\ 0]^T \tag{4.8}$$

$$dq_1/dt = -G_{11}(y_2 - s(t))y_1(t-T) \tag{4.9}$$

$$dq_2/dt = 0$$

$$dq_3/dt = 0$$

Here we show results of a computer simulation with the following values of the parameters:

$$\sigma = 10\ \ b = 8/3\ \ T = 0.1\ \ G11 = 10 \tag{4.10a}$$

$$p_1(t) = r_0 + \delta r\, sin\, \omega t \tag{4.10b}$$

$$\delta r = 2\ \ \omega = 2\pi/10\ \ r_0 = 28 \tag{4.10c}$$

$$q_2 = p_2 = 1\ \ q_3 = p_3 = 1 \tag{4.10d}$$

together with the initial condition:

$$q_1(0) = r_0 = 28 \tag{4.10e}$$

The results of this example are given in Figures 4-6. Fig.4 shows the hyperchaotic state variables of the driver system. A comparison with Fig.1 above, it can be seen that the state variables in Fig.4 display a more complex type of chaos, because of the time delay.

Comparing with the attractor of Fig. 2 above, it is apparent that the hyperchaotic attractor shown in Fig.5 displays a more complex behavior. Similar results are obtained for the response system as it synchronizes with the driver system. Fig.6 displays the synchronized state variables $y_1 = x_1, y_2 = x_2$ and $y_3 = x_3$, all three eventually converging to straight lines as shown in the figure. The identified signal $q_1(t) - r$ converges to the driver signal $q_1(t) - r = \delta r \, sin \, \omega t$ and is also shown in the figure.

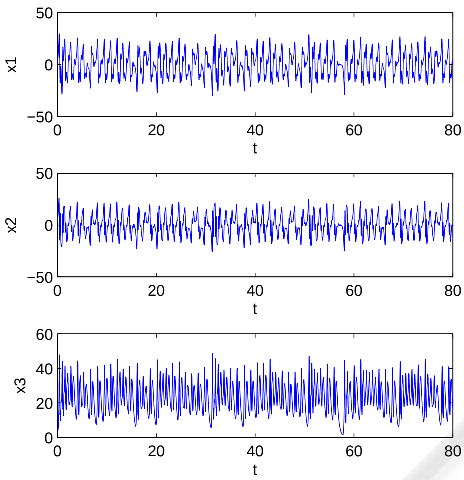

Figure 4: The hyperchaotic state variables of the driver Lorenz system with delay and variable parameter.

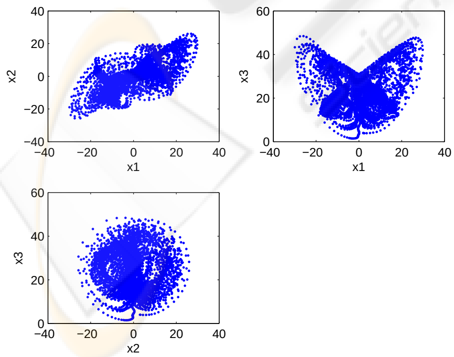

Figure 5: The hyperchaotic attractor of the driver Lorenz system with delay and variable parameter. The response system has a similar attractor.

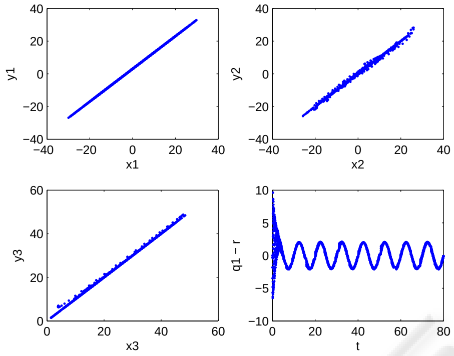

Figure 6: Synchronization between the hyperchaotic driver and response Lorenz systems with delay and variable parameters and the parameter $q_1(t)$ of the response system as it identifies the driver signal $p_1(t)$.

5 CONCLUSIONS

A generalized method for adaptive control, synchronization of chaos and parameter identification in systems governed by ordinary differential equations and delay-differential equations has been presented. The method is based on the Lagrangian approach to fluid dynamics. The synchronization error is treated as a scalar fluid property advected in the vector field of the controlled response system. Upon minimizing this error, the two coupled systems synchronize and the time variable parameters of the driving system are identified. The method was used to synchronize two Lorenz systems with variable parameters. The method was also applied to the synchronization of hyperchaos in two modified Lorenz systems with a time delay in one the state variables. Some implications of using the method in the field of secure communications where the transmitted information is masked by chaos or hyperchaos have been discussed.

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