

# ALGEBRAIC CURVES IN PARALLEL COORDINATES

## Avoiding the “Over-Plotting” Problem

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Abstract: Until now the representation (i.e. modeling) of curve in Parallel Coordinates is constructed from the point  $\leftrightarrow$  line duality. The result is a “line-curve” which is seen as the envelope of it’s tangents. Usually this gives an unclear image and is at the heart of the “over-plotting” problem; a barrier in the effective use of Parallel Coordinates. This problem is overcome by a transformation which provides directly the “point-curve” representation of a curve. Earlier this was applied to conics and their generalizations. Here the representation, also called dual, is extended to all planar algebraic curves. Specifically, it is shown that the dual of an algebraic curve of degree  $n$  is an algebraic of degree at most  $n(n - 1)$  in the absence of singular points. The result that conics map into conics follows as an easy special case. An algorithm, based on algebraic geometry using resultants and homogeneous polynomials, is obtained which constructs the dual image of the curve. This approach has potential generalizations to multi-dimensional algebraic surfaces and their approximation. The “trade-off” price then for obtaining *planar* representation of multidimensional algebraic curves and hyper-surfaces is the higher degree of the image’s boundary which is also an algebraic curve in  $\parallel$ -coords.

## 1 PARALLEL COORDINATES

Over the years a methodology has been developed which enables the visualization and recognition of multidimensional objects *without loss of information*. It provides insight into multivariate (equivalently multidimensional) problems and lead to several applications. The approach of Parallel Coordinates (abbr.  $\parallel$ -coords) (Inselberg, 1985) is in the spirit of Descartes, based on a coordinate system but differing in an important way as shown in Fig 1. On the Euclidean plane  $\mathbb{R}^2$  (more precisely on the projective plane  $\mathbb{P}^2$ ) with  $xy$ -Cartesian coordinates,  $n$  copies of a real line, labelled  $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_n$ , are placed equidistant and perpendicular to the  $x$ -axis with  $\bar{X}_1$  and  $y$  being coincident. These lines, which have the same orientation as the  $y$ -axis, are the axes of the **Parallel Coordinate** system for the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . A point  $C = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$  is represented by the polygonal line  $\bar{C}$  having vertices at the values  $c_i$  on the  $\bar{X}_i$ -axes. In this way a one-to-one correspondence is established between a points in  $\mathbb{R}^n$  and planar polygonal lines with vertices on the parallel axes. The polygonal line  $\bar{C}$  contains the *complete* lines and not just the segments between adjacent axes.

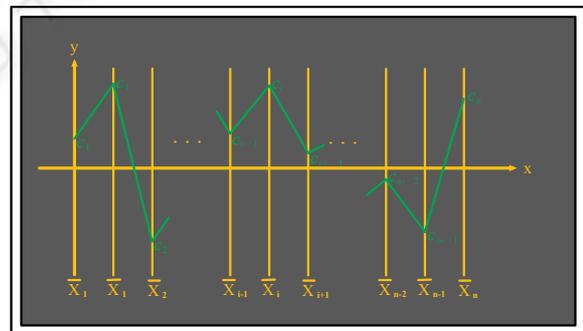


Figure 1: A point  $C = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$  is represented by the polygonal line  $\bar{C}$  (consist of  $n - 1$  sections) with vertices at the  $c_i$  values of the  $\bar{X}_i$  axis for  $i = 1, 2, \dots, n$ .

The restriction to  $\mathbb{R}^2$  provides that not only is a point represented by a line, but that a line is represented by a point. The points on a line are represented by a collection of lines intersect at a single point as can be seen in Fig. 2; i.e. a “pencil” of lines in the language of Projective Geometry. A fundamental *point*  $\leftrightarrow$  *line* duality is induced which is the corner-

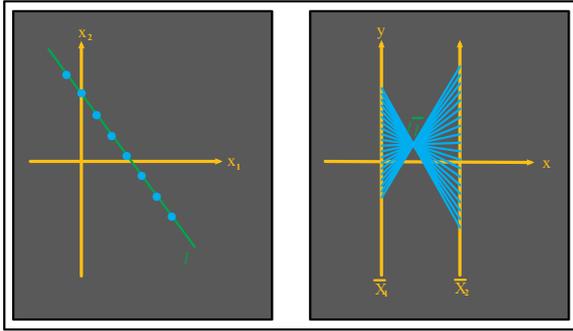


Figure 2: Fundamental *point* ↔ *line* duality.

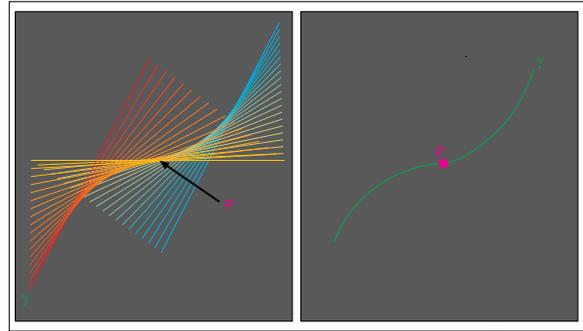


Figure 4: In line-curve (on the left) the tangents cover the image shown on the right as a point-curve.

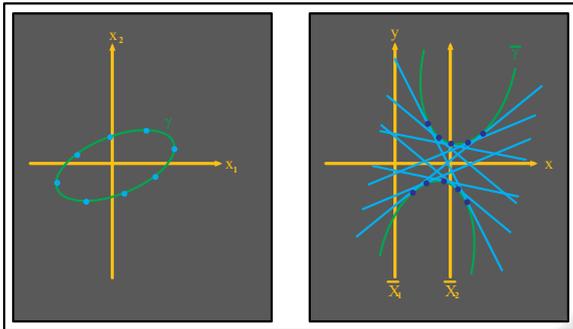


Figure 3: Point-curve mapped into line-curve (envelope of lines).

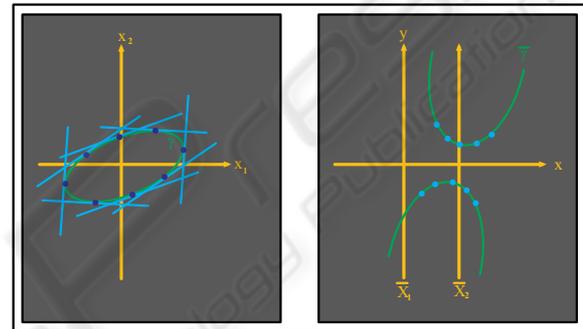


Figure 5: Point-curve mapped into point-curve.

stone of deeper results in  $\|\text{-coords}$ . The multidimensional generalizations for the representation of linear  $p$ -flats in  $\mathbb{R}^n$  (i.e. planes of dimension  $1 \leq p \leq n-1$ ), in terms of indexed points have been obtained. The proper setting for dualities is the projective  $\mathbb{P}^2$  rather than the Euclidean plane  $\mathbb{R}^2$ . A review of the mathematical foundations is available in (Inselberg, 1999).

For non-linear, especially non-convex, objects the representation is naturally more complex. In  $\mathbb{R}^2$  a point-curve, a curve considered as collection of points, is transformed into a *line-curve*; a curve prescribed by its tangent lines as in Fig. 3. The line-curve's envelope, a point-curve, is the curve's image in  $\|\text{-coords}$ . In many cases this yields an image that is difficult to discern. This point requires elaboration in order to motivate and understand some of the development presented here. As posed, the construction of a curve's image involves the sequence of operations:

$$\begin{aligned} \text{point-curve} &\rightarrow \text{line-curve} && \longmapsto \\ \text{point-curve (as the envelope of line-curve)} &&& \end{aligned}$$

Unlike the example shown in Fig. 3, where the line-curve's image is clear, the plethora of overlapping lines obscures parts of the resulting curve-line. This is a manifestation of what is sometimes called "over-plotting" problem in  $\|\text{-coords}$ ; the abundance

of "ink" in the picture covers many underlying patterns. Simply our eyes are not capable of "extracting" the envelope of lots of overlapping lines. There are also computational difficulties involved in the direct computation of the image curve's envelope. In a way this is the "heart" of the "over-plotting" problem considered as a barrier in the effective use of  $\|\text{-coords}$ . This problem can be overcome by skipping the intermediate step and go to an equivalent

$$\text{point-curve} \longmapsto \text{point-curve}$$

transformation which provides **directly** a clear image, as shown in Fig. 5.

This point-to-point mapping (Inselberg Transformation (Inselberg, 1985) ) preserves the curve's continuity properties. The idea is to use the  $\rightarrow$  part of the duality and map the *tangents* of the curve into points as illustrated in Fig. 5. Therefore, a curve is represented by a point-curve (the "dual curve") in the  $\|\text{-coords}$  plane. For a point-curve,  $\gamma$ , defined *implicitly* by

$$\gamma : f(x_1, x_2) = 0. \tag{1}$$

the  $x, y$  coordinates of the point-curve image (i.e.

dual)  $\bar{\gamma}$  are given by

$$\begin{cases} x &= \frac{\partial f / \partial x_2}{(\partial f / \partial x_1 + \partial f / \partial x_2)}, \\ y &= \frac{(x_1 \partial f / \partial x_1 + x_2 \partial f / \partial x_2)}{(\partial f / \partial x_1 + \partial f / \partial x_2)}. \end{cases} \quad (2)$$

It was shown by B. Dimsdale (Dimsdale, 1984) and generalized in (Inselberg, 1985), using this transformation that conics are mapped into conics in 6 different ways (Dimsdale, 1984). Here we develop the extension of the dual image for the family of general algebraic curves. This family of curves is highly significant in many implementations and applications, since these curves can easily and uniquely be reconstructed from a finite collection of their points (for instance using simple interpolation methods (Burden, 1989), (Walter, 1997)). Approximations of such curves can simply be obtained using similar methods.

As will be seen, the dual of an algebraic curve in general has degree higher than the original curve. There are some fringe-benefits, illustrated later, where “special-points” such as self-intersections or inflection-points which are conveniently transformed. But we do not want to get ahead of ourselves. The key reason for this effort is to pave the way for the representation of algebraic curves and more general hypersurfaces, as well as their approximations, in terms of planar regions *without losing information*. To pursue this goal then we need firstly to study the image of curves starting with general algebraic curves.

## 2 TRANSFORMS OF ALGEBRAIC CURVES

### 2.1 The Idea Leading to the Algorithm

In order to represent non-linear relations in  $\|$ -coords it is essential to extend the representational results first to algebraic curves; those described by either, explicitly or implicitly by irreducible polynomials of arbitrary degree. The direct application of eq. (2) turns out to be difficult even for degree 3. There is a splendid way to solve this problem using some ideas and tools from Algebraic Geometry. These involve properties of homogeneous polynomials and Resultant which are explained during the development of the method. For extensive treatments the reader is referred to (Cox, 1997), (Hariss, 1992), (Hodge, 1952) and (Walker, 1978).

Starting with an algebraic curve  $\gamma$  defined by an irreducible polynomial  $f(x_1, x_2) = 0$  in  $\mathbb{R}^2$  its image in  $\|$ -coords is sought. A number of preparatory steps smooth the way for the easier application of the transformations in eq. (2). First the curve  $\gamma \subset \mathbb{R}^2$  is

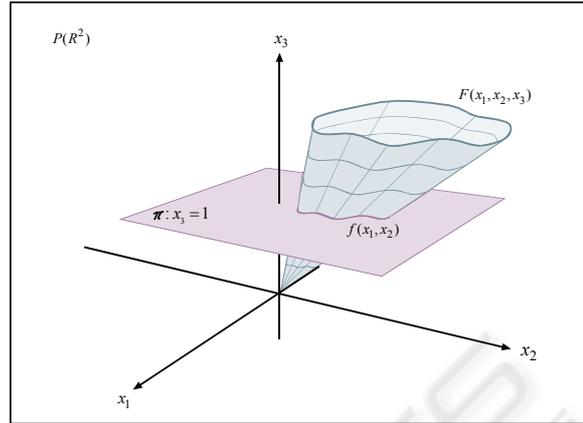


Figure 6: Cone  $F(x_1, x_2, x_3) = 0$  generated by embedding the curve  $f(x_1, x_2) = 0$  in  $\mathbb{P}^2$ .

“raised” to a surface embedded in the projective space  $\mathbb{P}^2$  and only then the corresponding mapping into  $\|$ -coords is applied.

*Step 1* : There exists a one-to-one correspondence (preserving the reducibility of polynomials) between any polynomial  $f(x_1, x_2) = 0$  of degree  $n$  and a homogenous polynomial in the projective plane  $\mathbb{P}^2$ , which is obtained by using homogeneous coordinates. Specifically,

- replace  $x_k$  by  $\frac{x_k}{x_3}$  for  $k = 1, 2$ ,
- multiply the whole polynomial by  $x_3^n$  and simplify.

These multiplications are, of course, allowable for  $x_3 \neq 0$ . The result is a *homogeneous polynomial* (which is also irreducible if  $f = 0$  is) with each term having degree  $n$ . To wit,

$$f(x_1, x_2) = \sum_{i+j=0}^n a_{ij} x_1^i x_2^j \rightarrow F(x_1, x_2, x_3) = \sum_{i+j=0}^n a_{ij} x_1^i x_2^j x_3^{n-i-j}, \quad (3)$$

which describes a surface in  $\mathbb{P}^2$  where the original polynomial curve is embedded as:

$$f(x_1, x_2) = F(x_1, x_2, 1). \quad (4)$$

*Step 2* : The *Gradient* of  $F$  is found and denoted by :

$$\nabla F(x_1, x_2, x_3) = \left( \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3} \right) = (\eta, \xi, \psi).$$

The three derivatives provide the direction numbers of the normal to the tangent plane at the point  $(x_1, x_2, x_3)$ . It is a fundamental property of homogeneous polynomials that

$$x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} + x_3 \frac{\partial F}{\partial x_3} = nF \quad (5)$$

where  $n$  is the degree of  $F$ . In our case  $F = 0$  and hence the equation of the tangent planes is

$$\eta x_1 + \xi x_2 + \psi x_3 = 0. \quad (6)$$

Since the tangent plane at *any* point of the surface goes through the origin it is clear that the surface  $F$  is a **cone** with apex at the origin as shown in Fig. 6.

Step 3 : Substituting for  $x_3$  from eq. (6) in

$$F(x_1, x_2, -(\frac{\eta x_1 + \xi x_2}{\psi})) = 0. \quad (7)$$

provides the intersection of the tangent plane with the cone which is a whole line as shown in Fig. 7. Each one of these lines, of course, goes through the origin and therefore it can be described by any one other of its points and in particular by  $(x_1, x_2, 1)$ . Simplifying the homogeneous coordinates by,

$$(x_1, x_2, -(\frac{\eta x_1 + \xi x_2}{\psi})) = (\psi x_1, \psi x_2, -(\eta x_1 + \xi x_2))$$

results in

$$F(\psi x_1, \psi x_2, -(\eta x_1 + \xi x_2)) = 0. \quad (8)$$

By the way, this is also a homogeneous polynomial in the *five* variables appearing in its argument. It is helpful to understand the underlying geometry. Eq. (8) can be considered as specifying a family of lines through the origin  $(0, 0, 0)$  and through each point  $(x_1, x_2, 1)$  along the curve given by eq. (4). Alternatively, the cone can be considered as being generated by a line, the *generating line* – see Fig. 8, pivoted at the origin and moving continuously along each point  $(x_1, x_2, 1)$  of the curve described by eq. (4). On each one of these lines the direction numbers  $(\eta, \xi, \psi)$  are unique and constant as shown in Fig. 8; i.e. there is a one-to-one correspondence:

$$(x_1, x_2, 1) \leftrightarrow (\eta, \xi, \psi).$$

Eq. (7), and hence its rephrasing eq. (8), contains two equivalent descriptions of the cone, i.e. in terms of the coordinates  $(x_1, x_2, 1)$  and also the direction numbers  $(\eta, \xi, \psi)$ . Hence one can be eliminated and as it turns out it is best to eliminate the  $x_i$ ,  $i = 1, 2$  something which is very conveniently done by means of the *Resultant*. The resultant  $R(F, G)$  of two homogeneous polynomials

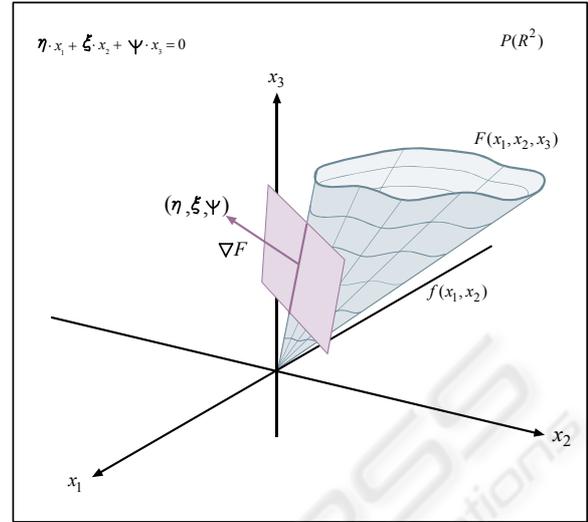


Figure 7: The intersection of the cone  $F(x_1, x_2, x_3) = 0$  with any of its tangent planes is a whole line. Along such a line the direction numbers  $(\eta, \xi, \psi)$  are constant and unique.

$$F(x_1, x_2) = \sum_{i=0}^n a_i x_1^i x_2^{n-i} = 0 \quad \text{and}$$

$$G(x_1, x_2) = \sum_{j=0}^m b_j x_1^j x_2^{m-j} = 0$$

is the polynomial obtained from the determinant of their coefficients matrix:

$$R(F, G) =$$

$$\text{Det} \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & & 0 \\ r & a_0 & a_1 & & & a_n & & \\ & & \ddots & & & & \ddots & \\ b_0 & b_1 & b_2 & \cdots & b_m & & & a_n \\ b_0 & b_2 & & & b_m & & & \\ & & \ddots & & & & \ddots & \\ 0 & & & b_0 & & \cdots & & b_m \end{pmatrix},$$

where the empty spaces are filled by zeros. There are  $m$  lines of  $a_i$  and  $n$  lines of  $b_j$ . When  $F$  and  $G$  are both irreducible so is their resultant. Due to the homogeneity of  $F$  and  $G$ ,

$$R(F, G) = 0 \Leftrightarrow F = 0, G = 0.$$

Let us rewrite eq. (8) as

$$F_{(\eta, \xi, \psi)}(x_1, x_2) = \sum_{i=0}^n a'_i(\eta, \xi, \psi) x_1^i x_2^{n-i} = 0. \quad (9)$$

Appealing again to the homogeneity of  $F$  to obtain the relation

$$x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} = nF$$

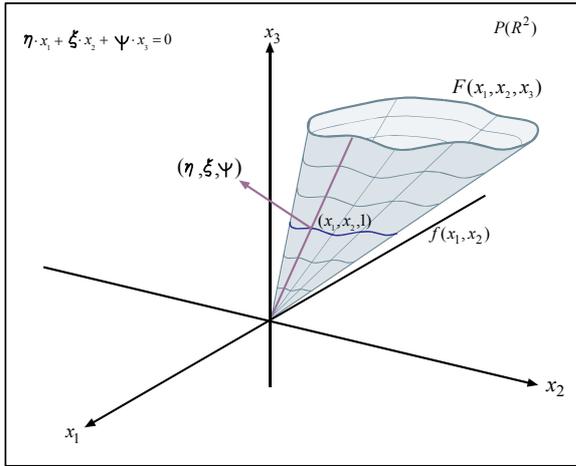


Figure 8: This shows the *generating line* of the cone (which is found from the intersection of the tangent planes with the cone) together with the gradient vector.

which has already been mentioned earlier. Therefore

$$\frac{\partial F}{\partial x_1} = 0, \quad \frac{\partial F}{\partial x_2} = 0 \Rightarrow F = 0$$

and from the property of the resultant of homogeneous polynomials mentioned above

$$R\left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right) = 0 \Leftrightarrow \frac{\partial F}{\partial x_1} = 0, \quad \frac{\partial F}{\partial x_2} = 0.$$

Altogether then

$$R\left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right) = R(\eta, \xi, \psi) = 0 \quad (10)$$

has finite degree since  $F$  and  $R$  are polynomials. It was shown that

$$R = 0 \Leftrightarrow F = 0$$

so not only their zero sets agree, which ensures that they are equivalent polynomials, but they also agree at an infinite number of points. Hence,  $R = 0$  provides the description of the cone  $F = 0$  in terms of the direction numbers  $(\eta, \xi, \psi)$  and is also a homogeneous polynomial. The degree of  $\frac{\partial F}{\partial x_i}$  is  $n - 1$ . This and the structure of the resultant with the four triangular zero portions results in  $R$  being a polynomial of degree at most  $n(n - 1)$ .

*Step 4 :* Since we are interested in the solutions of  $R = 0$  the multiplier in powers of  $\psi$  of  $R$ , if there is one, can be safely neglected due to the homogeneity.

*Step 5 :* Only in this, the final step, the transformation of the curve from the  $x_1x_2$ -plane to the  $xy$ -plane

with parallel coordinates is performed. It is done using equations (2) rewritten, in view of eq. (6) as

$$x = \frac{\xi}{\eta + \xi}, \quad y = -\frac{\psi}{\eta + \xi}. \quad (11)$$

It is important to notice that *both* this and eq. (10) involve *only the derivatives*  $\eta, \xi, \psi$ . Let  $c = \eta + \xi$ , so that  $\xi = cx$ ,  $\eta = c - cx = c(1 - x)$  and  $\psi = -cy$ . Substitution provides the transform of the original curve  $f(x_1, x_2) = 0$  :

$$R(\eta, \xi, \psi) = R(c(1 - x), cx, -cy) =$$

$$c^n R((1 - x), x, -y) = 0 \Rightarrow$$

$$R((1 - x), x, -y) = 0,$$

when  $c \neq 0$ . In the previous step it was pointed out that this polynomial has degree at most  $n(n - 1)$ , the actual degree obtained depends on the presence of singular points in the original algebraic curve  $f = 0$ .

## 2.2 Algorithm

Here the process involved is presented compactly as an algorithm (Izhakian, 2001) whose input is an algebraic curve  $\gamma : f(x_1, x_2) = 0$  and the output is the polynomial which describes  $\bar{\gamma}$ , the curve's image in  $\parallel$ -coords. To emphasize, the algorithm applies to implicit or explicit polynomials of any degree and curves with or without singular points. The formal description of the algorithm is followed by examples which clarify the various stages and their nuances.

For a given irreducible polynomial equation (otherwise apply the algorithm for each of its component separately)  $f(x_1, x_2) = 0$ .

1. Convert to homogeneous coordinates to obtain the transformation of  $f$ , a homogenous polynomial  $F(x_1, x_2, x_3) = 0$ .
2. Substitute
 
$$\begin{aligned} x_1 &\rightarrow \psi x_1, & x_2 &\rightarrow \psi x_2 & \text{and} \\ x_3 &\rightarrow -(\eta x_1 + \xi x_2). \end{aligned}$$
3. Find the resultant of the two derivatives  $F_{x_1}$  and  $F_{x_2}$ .
4. Cancel the multiplier in a power of  $\psi$  of the resultant  $R$  and denote the result by  $R'$ .
5. The output is obtained by the substitution
 
$$\eta = 1 - x, \quad \xi = x, \quad \psi = -y \quad \text{in } R'.$$

### 3 EXAMPLES OF ALGEBRAIC CURVES AND THEIR TRANSFORMS

#### 3.1 Conic Transforms

The algorithm is illustrated with some examples starting with the conics.

$$f(x_1, x_2) =$$

$$\begin{pmatrix} x_1 & x_2 & 1 \end{pmatrix} \begin{pmatrix} A_1 & A_4 & A_5 \\ A_4 & A_2 & A_6 \\ A_5 & A_6 & A_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} =$$

$$A_1x_1^2 + 2A_4x_1x_2 + 2A_5x_1 + A_2x_2^2 + 2A_6x_2 + A_3 = 0.$$

Applying the algorithm in the sequence given above:

*Step 1* : The homogeneous polynomial obtained from  $f$  by using homogeneous coordinates is

$$F(x_1, x_2, x_3) = A_1x_1^2 + 2A_4x_1x_2 + 2A_5x_1x_3 + A_2x_2^2 + 2A_6x_2x_3 + A_3x_3^2 = 0.$$

*Step 2* : Substituting

$$x_1 \rightarrow \psi x_1, \quad x_2 \rightarrow \psi x_2, \quad x_3 \rightarrow -(\eta x_1 + \xi x_2)$$

yields

$$F(\psi x_1, \psi x_2, -(\eta x_1 + \xi x_2)) = (-2A_5\psi\eta + A_1\psi^2 + A_3\eta^2) + x_1^2 + 2(-A_5\psi\xi + A_3\xi\eta + A_4\psi^2 - A_6\psi\eta)x_1x_2 + (A_2\psi^2 - 2A_6\psi\xi + A_3\xi^2)x_2^2 + c_1x_1^2 + 2c_2x_1x_2 + 2c_3x_2^2 = 0,$$

where  $c_i = c_i(\eta, \xi, \psi)$ , for  $i = 1, 2, 3$ .

*Step 3* : Calculate the two derivatives of  $F$  and their resultant:

$$\frac{\partial F}{\partial x_1} = 2c_1x_1 + 2c_2x_2,$$

$$\frac{\partial F}{\partial x_2} = 2c_2x_1 + 2c_3x_2,$$

$$R\left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right) = \text{Det} \begin{pmatrix} 2c_1 & 2c_2 \\ 2c_2 & 2c_3 \end{pmatrix} =$$

$$R(\eta, \xi, \psi) = 4\psi^2[(A_3A_2 - A_6^2)\eta^2 + (A_1A_3 - A_5^2)\xi^2 + (A_1A_2 - A_4^2)\psi^2 + 2(A_5A_6 - A_3A_4)\eta\xi + 2(A_5A_4 - A_1A_6)\xi\psi + 2(A_4A_6 - 2A_5A_2)\eta\psi].$$

*Step 4* : Retain the resultants component which is multiplied by a power of  $\psi$  and let

$$R'(\eta, \xi, \psi) = \frac{R(\eta, \xi, \psi)}{4\psi^2}.$$

*Step 5* : The dual of  $f(x_1, x_2) = 0$  is then given in matrix form by

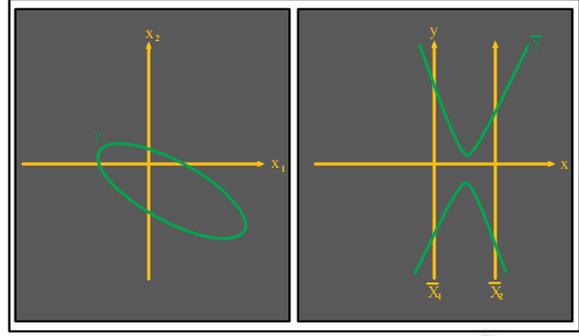


Figure 9: An ellipse is mapped into hyperbola.

$$R'(1 - x, x, -y) =$$

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

where the individual  $a_i$  are :

$$a_1 = A_3(A_1 + A_2 + 2A_4) - (A_5 + A_6)^2,$$

$$a_2 = A_1A_2 - A_4^2,$$

$$a_3 = A_2A_3 - A_6^2,$$

$$a_4 = A_6(A_1 + A_4) - A_5(A_2 + A_4),$$

$$a_5 = A_6^2 + A_5A_6 - A_3(A_2 + A_4),$$

$$a_6 = A_2A_5 - A_4A_6.$$

The result is illustrated for the ellipse in fig 9 and can also be contrasted to the earlier ways for obtaining the transformation

$$\text{conic} \leftrightarrow \text{conic}.$$

#### 3.2 Algebraic Curves of Degree Higher than Two

Next the algorithm is applied to the algebraic curve of 3rd degree

$$f(x_1, x_2) = x_1^3 - x_1^2 - x_2^2 + x_2 - 1 = 0.$$

*Step 1* : The homogeneous polynomial obtained from  $f$  by using homogeneous coordinates is

$$F(x_1, x_2, x_3) = x_1^3 - x_1^2x_3 - x_2^2x_3 + x_2x_3^2 - x_3^3 = 0.$$

*Step 2* : Substituting

$$x_1 \rightarrow \psi x_1, \quad x_2 \rightarrow \psi x_2, \quad x_3 \rightarrow -(\eta x_1 + \xi x_2)$$

yields

$$F(\psi x_1, \psi x_2, -(\eta x_1 + \xi x_2)) = (\psi^3 + \psi^2\eta - \eta^3)x_1^3 +$$

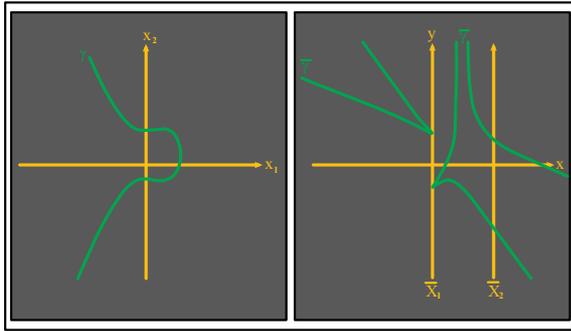


Figure 10: The cubic curve  $x_1^3 - x_1^2 - x_2^2 + x_2 - 1 = 0$  (on the left) is mapped into a six degree curve (on the right). Notice also an instance of the duality *inflection-point*  $\leftrightarrow$  *cusps* [4].

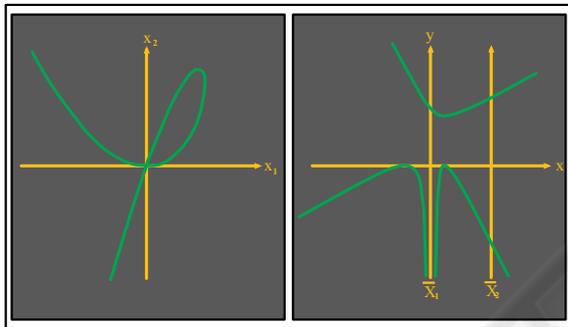


Figure 11: The self intersection point in the curve  $x_1^3 + x_2^2 - 3x_1x_2 = 0$  (on the left), disappears in the image curve  $27y^2 - 54y^2x + 27y^2x^2 - 108y + 270yx - 198yx^2 + 40yx^3 + 108x^3 - 36x^4 - 81x^2 = 0$  (on the right) .

$$\begin{aligned} &(-3\xi\eta^2 + \psi\eta^2 + \psi^2\xi)x_1^2x_2 + \\ &(2\psi\xi\eta + \psi^2\eta - 3\xi^2\eta)x_1x_2^2 + \\ &(-\xi^3 + \psi\xi^2 + \psi^2\xi)x_2^3 = \\ &c_1x_1^3 + c_2x_1^2x_2 + c_3x_1x_2^2 + c_4x_2^3 = 0, \end{aligned}$$

where the  $c_i = c_i(\eta, \xi, \psi)$ , for  $i = 1, \dots, 3$ .

*Step 3* : Calculate the resultant of the two derivatives of  $F$ :

$$\begin{aligned} \frac{\partial F}{\partial x_1} &= 3c_1x_1^2 + 2c_2x_1x_2 + c_3x_2^2, \\ \frac{\partial F}{\partial x_2} &= c_2x_1^2 + 2c_3x_1x_2 + 3c_4x_2^2, \\ R\left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right) &= \\ \text{Det} \begin{pmatrix} 3c_1 & 2c_2 & c_3 & 0 \\ 0 & 3c_1 & 2c_2 & c_3 \\ c_2 & 2c_3 & 3c_4 & 0 \\ 0 & c_2 & 2c_3 & 3c_4 \end{pmatrix} &= \\ R(\eta, \xi, \psi) &= -3\psi^6 R'(\eta, \xi, \psi), \end{aligned}$$

where  $R'$  is polynomial of degree 6.

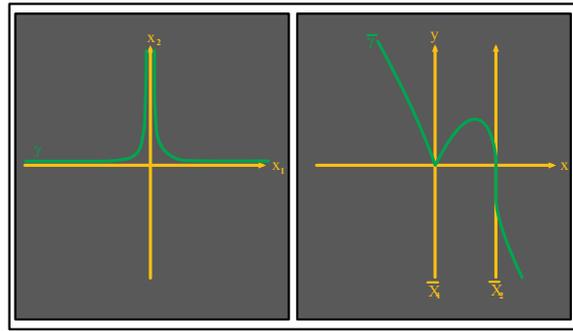


Figure 12:  $x_1^2x_2 - 1 = 0 \mapsto 4y^3 + 27x^3 - 27x^2 = 0$ .

*Step 4* : Discarding the resultant's factor  $(-\psi^6)$  i.e. let

$$R'(\eta, \xi, \psi) = \frac{R(\eta, \xi, \psi)}{-3\psi^6}.$$

*Step 5* : Finally, substituting

$$\eta = 1 - x, \xi = x, \psi = -y$$

results in the 6th-degree curve :

$$\begin{aligned} R'(1 - x, x, -y) &= \\ &-23 + 292y^2x^2 - 422x^2 + 326yx^3 - 146y^2x \\ &+ 610x^3 + 23y^2 - 27y^4x^2 + 54y^4x - 27y^4 \\ &- 22yx^5 - 244y^2x^3 - 66yx^4 - 420yx^2 \\ &- 126y^3x + 232yx + 214x^5 - 499x^4 + 90y^3x^2 \\ &+ 54y^3 + 156x - 50y - 31x^6 + 71y^2x^4 \\ &- 14y^3x^3. \end{aligned}$$

The source and image curve are shown in Fig. 10. In this case  $n = 3$  and the maximum degree  $n(n - 1) = 6$  is attained. Notice that this covers as special cases :

- the point  $\leftrightarrow$  line duality where points (with  $n = 0$ ) are mapped into lines (with  $n = 1$ ) and vice-versa, and
- the conics with  $n = 2$ .

The next two examples show that the image curve may have degree less than  $n(n - 1)$ . Apparently the maximal degree is attained by the image curve in the absence of singularities in the function or it's derivatives. The full conditions relating the image's degree to less than the maximal are not known at this stage. Two such examples are shown in the subsequent figures, Fig. 11 and Fig. 12.

## 4 CONCLUSIONS

Consider the class  $\mathcal{S}$  of hyper-surfaces in  $\mathbb{R}^n$  which are the envelopes of their tangent hyper-planes. As mentioned earlier (Inselberg, 1999) hyper-planes can be represented in  $\|\cdot\|$ -coords by  $n - 1$  indexed points. Hence for a hyper-surface  $\sigma \in \mathcal{S}$  each point  $P \in \sigma$  maps into  $n - 1$  indexed planar points. From this

it follows that the hyper-surface  $\sigma$  can be mapped (i.e. represented) by  $n - 1$  indexed planar regions  $\bar{\sigma}_i$  composed of these points. Restricted classes of hyper-surfaces have been represented in this way and it turns out that the  $\bar{\sigma}_i$  reveal non-trivial properties of the corresponding hyper-surface  $\sigma$ . It has already been proved that for any dimension *Quadratics* (algebraic surfaces of degree 2) are mapped into planar regions whose boundaries are *conics* (Izhakian, 2004). This also includes non-convex surfaces like the "saddle". There are strong evidence supporting the conjecture that algebraic surfaces in general map into planar regions bounded by algebraic curves. This, besides their significance on their own right, is one of the reasons for studying the representation of algebraic curves. In (Inselberg, 2000) families of approximate planes and flats are beautifully and usefully represented in  $\parallel$ -coords. Our results cast the foundations not only for the representation of hyper-surfaces in the class ( $\mathcal{S}$ ), but also their approximations in terms of **planar curved regions**.

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