AN UNIFIED THEORY FOR STEERABLE AND QUADRATURE FILTERS

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Abstract: In this paper, a complete theory of steerable filters is presented which shows that quadrature filters are only a special case of steerable filters. Although there has been a large number of approaches dealing with the theory of steerable filters, none of these gives a complete theory with respect to the transformation groups which deform the filter kernel. Michaelis and Sommer (Michaelis and Sommer, 1995) and Hel-Or and Teo (Teo and Hel-Or, 1996; Teo and Hel-Or, 1998) were the first ones who gave a theoretical justification for steerability based on Lie group theory. But the approach of Michaelis and Sommer considers only Abelian Lie groups. Although the approach of Hel-Or and Teo considers all Lie groups, their method for generating the basis functions may fail as shown in this paper. We extend these steerable approaches to arbitrary Lie groups, like the important case of the rotation group $SO(3)$ in three dimensions.
Quadrature filters serve for computing the local energy and local phase of a signal. Whereas for the one dimensional case quadrature filters are theoretically well founded, this is not the case for higher dimensional signal spaces. The monogenic signal (Felsberg and Sommer, 2001) based on the Riesz transformation has been shown to be a rotational invariant generalization of the analytic signal. A further generalization of the monogenic signal, the 2D rotational invariant quadrature filter (Köthe, 2003), has been shown to capture richer structures in images as the monogenic signal.
We present a generalization of the rotational invariant quadrature filter based on our steerable theory. Our approach includes the important case of 3D rotational invariant quadrature filters but it is not limited to any signal dimension and includes all transformation groups that own an unitary group representation.

1 INTRODUCTION

Steerable filters and quadrature filters are well established methods in signal and image processing. Steerability is at least implicitly used when computing directional derivatives as this is the central operation in differential motion estimation. Quadrature filters are the choice for computing the local energy and local phase of a signal. In this paper we present a complete theory of steerable filters and derive a group invariant quadrature filter approach based on our steerable filter theory.

Although a large number of approaches dealing with the theory of steerable filters have been published (Danielsson, 1980; Freeman and Adelson, 1991; Perona, 1995; Simoncelli et al., 1992; Simoncelli and Farid, 1996; Michaelis and Sommer, 1995; Teo and Hel-Or, 1996; Teo and Hel-Or, 1998; Yu et al., 2001), none of these provides a complete and closed theory. Such a theory shall describe the general requirements which are necessary for a filter kernel to be a steerable filter. The benefit from this work is a deeper understanding of the concepts of steerable filters and enables the user to construct steerable filters for every Lie group transformation. None of the previously published approaches gives a general solution of the following problem: If one is confronted with a certain filter kernel and an arbitrary Lie group, what are the approbate basis functions to steer the filter kernel and how are the interpolation functions to be computed.

The steerable approach of Michaelis and Sommer (Michaelis and Sommer, 1995) give a solution to this problem in the case of Abelian Lie groups, whereas the approach of Teo and Hel-Or (Teo and Hel-Or, 1996; Teo and Hel-Or, 1998) handles all Lie group transformations. But the latter approach may fail as we show in section 3.4. In contrast these approaches

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(Michaelis and Sommer, 1995; Teo and Hel-Or, 1996; Teo and Hel-Or, 1998) which either do not cover the case of non-Abelian groups (Michaelis and Sommer, 1995) or do not work for all filter kernels (Teo and Hel-Or, 1996; Teo and Hel-Or, 1998), our approach uses the full power of Lie group theory to generate the minimum number of basis functions also for non-Abelian compact Lie groups. It is a direct extension of the approach of (Michaelis and Sommer, 1995) in which the basis functions are generated by the eigenfunctions of the generators of the Lie group. In our approach a Casimir operator is used to generate the basis functions also for non-Abelian compact Lie groups. For non-Abelian, non-compact groups, we show that polynomials serve as appropriate basis functions.

The quadrature filter is a well established method in signal processing and low-level image processing for computing the local energy and local phase of a signal. Whereas the local signal is an estimate of the local intensity structure, the local phase provides information about the local shape of the signal. In the 1D case the quadrature filter is well defined by an even part, a bandpass filter, and an odd part, the Hilbert transformation of the bandpass filter. The filter output is the analytic signal, a representation of the signal from which the local energy and local phase can easily be computed. Large efforts have been made to generalize the analytic signal (Bülow and Sommer, 2001; Granlund and Knutsson, 1995) to the 2D case by projecting the scalar valued Hilbert transformation in the two dimensional space. The drawback of all of these methods are that they are not rotational invariant. The Riesz transformation has been shown to be a rotational invariant generalization of the Hilbert transformation generalizing the analytic signal to the monogenic signal (Felsberg and Sommer, 2001). A further generalization of the monogenic signal is the 2D rotational invariant quadrature filter (Köthe, 2003), based on rotated *steerable filters*, which is able to capture richer structures from an image than the monogenic signal.

Many interesting computer vision and image processing applications, like motion estimation, are not restricted to the two dimensional case. We present a generalization of the rotational invariant quadrature filter (Köthe, 2003) with respect to the signal dimension and the transformation group. This includes the important case of a 3D rotational invariant quadrature filter, but it is not limited to any signal dimension. Also other transformations than the rotation is considered, like shearing, which is the correct transformation group when describing motion in space time.

2 STEERABLE FILTERS

Let g denote an element of an arbitrary Lie group $\mathcal{G}$ and $\mathbf{x} \in \mathbb{R}^N$ the coordinate vector of an N dimensional signal space. We define a steerable filter $h_g(\mathbf{x})$ as the impulse response whose deformed version, with respect to the Lie group, equals the linear combination of a finite set of basis filters $\{b_j(\mathbf{x})\}$, $j = 1, 2, ..., M$. Furthermore, only the coefficients $\{a_j(g)\}$, denoted as the interpolation functions, depend on the Lie group element

$$h_g(\mathbf{x}) = \sum_{j=1}^{M} a_j(g)\, b_j(\mathbf{x}) \ . \tag{1}$$

Applying the deformed filter to a signal $s(\mathbf{x})$ is equivalent to the linear combination of the individual impulse responses of the basis filters

$$h_g(\mathbf{x}) * s(\mathbf{x}) = \sum_{j=1}^{M} a_j(g)(b_j(\mathbf{x}) * s(\mathbf{x})) \ . \tag{2}$$

Questions arising with steerable functions are:

- Under which conditions can a given function $h_g(\mathbf{x})$ be steered?
- How can the basis functions $b_j(\mathbf{x})$ be determined?
- How many basis functions are needed to steer the function $h_g(\mathbf{x})$?
- How can the interpolation functions $a_j(g)$ be determined?

In the last decade, several steerable filter approaches have been developed trying to answer these questions, but all of them, except for the approach of Teo and Hel-Or (Teo and Hel-Or, 1996; Teo and Hel-Or, 1998), tackle only a special case, either for the filter kernel or for the corresponding transformation group.

3 AN EXTENDED STEERABLE APPROACH BASED ON LIE GROUP THEORY

In the following section, we present our steerable filter approach based on Lie group theory covering all recent approaches developed so far. It delivers for Abelian Lie groups and for compact non-Abelian Lie groups the minimum required number of basis functions and the corresponding interpolation functions. In order to complete the steerable approach the case of non-Abelian, non-compact Lie groups has to be considered separately. After presenting our concept, its relation to recent approaches is discussed and some examples are presented.

3.1 Conditions for Steerability

In the following we show the steerability of all filter kernels $h : \mathbb{R}^N \rightarrow \mathbb{C}$ which are expandable according to a finite number M of basis function $\mathcal{B} = \{b_j(\mathbf{x})\}$ of a subspace V :=span$\{\mathcal{B}\} \subset L^2$ of all quadratic integrable functions. Since every element of L^2 can arbitrary exactly be approximated by a finite number of basis functions, we consider, at least approximately, all quadratic integrable filter kernels. The problem of approximating such a function by a smaller number of basis functions when allowing a certain error has been examined in (Perona, 1995) and is not topic of this paper. With the notation of the inner product $\langle \cdot, \cdot \rangle$ in L^2 and the Fourier coefficients $c_j = \langle h(\mathbf{x}), b_j(\mathbf{x}) \rangle$ the expansion of $h(\mathbf{x})$ reads

$$h(\mathbf{x}) = \sum_{j=1}^{M} c_j b_j(\mathbf{x}) \ . \tag{3}$$

Furthermore, every basis function $b_j(\mathbf{x}) \in V$ shall belong to an invariant subspace $U \subseteq V$ with respect to a Lie group $\mathcal{G}$ transformation.

Then, $h(\mathbf{x})$ is steerable with respect to $\mathcal{G}$.

We have assigned the preconditions such that this statement can be easily verified. Let $D(g)$ denote the representation of $g \in \mathcal{G}$ in the function space V and $\mathbf{D}(g)$ the representation of $\mathcal{G}$ in the N-dimensional signal space. It is easy to verify that the transformed function $\mathcal{D}(g)h(\mathbf{x})$ equals the linear combination of the transformed basis functions

$$\begin{aligned} \mathcal{D}(g)h\left(\mathbf{x}\right) &= h\left(\mathbf{D}(g)^{-1}\mathbf{x}\right) && (4)\\ &= \sum_{j=1}^{M} c_j b_j(\mathbf{D}(g)^{-1}\mathbf{x}) && (5)\\ &= \sum_{j=1}^{M} c_j \mathcal{D}(g) b_j(\mathbf{x}) \ . && (6) \end{aligned}$$

Since every basis function $b_j(\mathbf{x})$ is, per definition, part of an invariant subspace, the transformed version $\mathcal{D}(g)b_j(\mathbf{x})$ can be expressed by a linear combination of the subspace basis. Let denote $m(j)$ the mapping of the index j of the basis function $b_j(\mathbf{x})$ onto the lowest index of the basis function belonging to the same subspace and $d(j)$ the mapping of the index of the basis function $b_j(\mathbf{x})$ onto the dimension d_j of its invariant subspace. The transformed basis function $\mathcal{D}(g)b_j(\mathbf{x})$ can be expressed, with the previous definition of $m(j)$ and $d(j)$, and the coefficients of the linear combination $w_{jk}(g)$ as

$$\mathcal{D}(g)b_j(\mathbf{x}) = \sum_{k=m(j)}^{m(j)+d(j)-1} w_{jk}(g) b_k(\mathbf{x}) \ . \tag{7}$$

Inserting equation (7) into equation (6) yields

$$\mathcal{D}(g)h(\mathbf{x}) = \sum_{i=1}^{M} c_i \sum_{k=m(j)}^{m(j)+d(j)-1} w_{jk}(g) b_k(\mathbf{x}) \ . \tag{8}$$

The double sum can be written such that all coefficients belonging to the same basis function are grouped together, where L denotes the number of invariant subspaces in V

$$\begin{aligned} \mathcal{D}(g)h(\mathbf{x}) = & \sum_{b_k \in U_1} b_k(\mathbf{x}) \sum_{w_{jk} \in U_1} c_j w_{jk}(g) && (9)\\ & + \sum_{b_k \in U_2} b_k(\mathbf{x}) \sum_{w_{jk} \in U_2} c_j w_{jk}(g) + \ldots \\ & + \sum_{b_k \in U_L} b_k(\mathbf{x}) \sum_{w_{jk} \in U_L} c_j w_{jk}(g) \ . \end{aligned}$$

Thus, in order to steer the function h we have to consider all basis functions spanning the L subspaces.

3.2 The Basis Functions

The next question arising is how to obtain the appropriate basis functions. We require the basis functions to span finite dimensional invariant subspaces. Furthermore, the invariant subspaces are desired to be as small as possible in order to lower computational costs. Group theory provides the solution of this problem and the functions fulfilling these requirements are, per definition, the basis of an irreducible representation of the Lie group. This has already pointed out by Michaelis and Sommer (Michaelis and Sommer, 1995) and a method for generating such a basis for Abelian Lie groups has been proposed. We extend this method for the case of non-Abelian, compact Lie groups. The case of non-Abelian, non-compact groups is discussed in subsection 3.2.2.

3.2.1 Basis Functions for Compact Lie Groups

The invariant space spanned by an irreducible basis cannot be decomposed further into invariant subspaces and thus, forming a minimum number of basis functions for the steerable function. Michaelis and Sommer showed that such a basis is given by the eigenfunctions of the generators in case of Abelian Lie groups. Since the generators of a non-Abelian groups do not commutate and thus have no simultaneous eigenfunctions, the method does not work in this case any more. But their framework can be extended with a slight change to compact non-Abelian groups. Instead of constructing the basis functions from the simultaneous eigenfunctions of the generators of the group, the basis function can also be constructed by the eigenfunctions of a *Casimir* operator $\mathcal{C}$ of the corresponding Lie group. In order to define the Casimir operator we first have to introduce the Lie bracket, or commutator, of two operators

$$[\mathcal{D}(a), \mathcal{D}(b)] = \mathcal{D}(a)\mathcal{D}(b) - \mathcal{D}(b)\mathcal{D}(a) \ . \qquad (10)$$

Operators commutating with all representations of the group elements are denoted as Casimir operators

$$[\mathcal{C}, \mathcal{D}(g)] = 0 \quad \forall g \in \mathcal{G} \ . \qquad (11)$$

Let $\{b_m(\mathbf{x})\}, m = 1, ..., d_\alpha$ denote the set of eigenfunctions of $\mathcal{C}$ corresponding to the same eigenvalue α. Then, every transformed basis function $\mathcal{D}(g)b_i(\mathbf{x})$ is an eigenfunction with the same eigenvalue α

$$\begin{aligned} \mathcal{C}\mathcal{D}(g)b_i(\mathbf{x}) &= \mathcal{D}(g)\mathcal{C}b_i(\mathbf{x}) \qquad (12) \\ &= \mathcal{D}(g)\alpha b_i(\mathbf{x}) \\ &= \alpha\mathcal{D}(g)b_i(\mathbf{x}) \ . \end{aligned}$$

Thus, $\{b_m(\mathbf{x})\}$ forms a basis of a d_α dimensional invariant subspace U_α. Any transformed element of this subspace can be expressed by a linear combination of basis functions of this subspace

$$\mathcal{D}(\mathbf{u})b_m(\mathbf{x}) = \sum_{j=1}^{d_\alpha} w_{mj} b_j(\mathbf{x}) \ . \qquad (13)$$

Thus, we have found a method for constructing invariant subspaces also for non-Abelian groups. A Casimir operator is constructed by a linear combination of products of generators of the corresponding Lie group where n denotes the number of generators

$$\mathcal{C} = \sum_{ij} f_{ij} \mathcal{L}_i \mathcal{L}_j, \quad i, j = 1, ..., n \ . \qquad (14)$$

The coefficients f_{ij} are solved by the constraints

$$[\mathcal{C}, \mathcal{L}_k] = 0, \quad k = 1, ..., n \ . \qquad (15)$$

If the Casimir operator of a compact group is self-adjoint with a discrete spectrum, the eigenfunctions constitute a complete orthogonal basis of the corresponding function space. It is a well known fact from functional analysis that in this case all eigenfunctions belonging to the same eigenvalue span a finite dimensional subspace. If furthermore the considered group the Casimir operator has the symmetry of the group $\mathcal{G}$, i.e. there exists no operation which does not belong to the group and under which the Casimir operator is invariant, then the eigenfunctions are basis functions of irreducible representation (Wigner, 1959). After computing one eigenfunction $b_1(\mathbf{x})$ corresponding to the eigenvalue α we can construct all other basis functions of this invariant subspace by applying all possible combinations of generators of the Lie group to $b_1(\mathbf{x})$. The sequence of generators is stopped when the resulting function is linear dependent from the ones which have already been constructed. This equals to the method for constructing the basis functions proposed by Teo and Hel-Or (Teo and Hel-Or, 1998) except for the fact that they propose to apply this procedure directly to the steerable function $h(\mathbf{x})$.

3.2.2 Basis Function for Non-Compact Lie Groups

Since only Abelian Lie groups and compact non-Abelian Lie groups are proofed to own complete irreducible representations, i.e. the representation space falls into invariant subspaces, we have to treat the case of non-Abelian, non-compact groups separately. Since we do only require an invariant subspace and not an entirely irreducible representation we can easily construct such a space from a polynomial basis of the space of square integrable functions. The order of a polynomial term does not change by an arbitrary Lie group transformation and thus the basis of a polynomial term constitute a basis for a steerable filter. In order to steer an arbitrary polynomial we have to determine the terms of different order. The sum of the basis functions of the corresponding invariant subspaces are a basis for the steerable polynomial.

We can now construct for every Lie group transformation the corresponding basis for a steerable filter. For Abelian groups and compact groups we chose the basis from the eigenfunction of the Casimir operator whereas for all other groups we choose a polynomial basis. The next section addresses the question how to combine these basis functions in order to steer the resulting filter kernel with respect to any Lie group transformation.

3.3 The Interpolation Functions

The computation of the interpolation functions $\{a_j(g)\}$ can already be deduced from equ.(9). In order to obtain the interpolation function corresponding to the basis function $b_m(\mathbf{x})$ the transformed version of the original filter kernel $h(\mathbf{x})$ has to be projected onto $b_m(\mathbf{x})$

$$\begin{aligned} a_m(g) &= \langle \mathcal{D}(g)h(\mathbf{x}), b_m(\mathbf{x}) \rangle \qquad (16) \\ &= \left\langle \mathcal{D}(g) \sum_{n=1}^{M} c_n b_n(\mathbf{x}), b_m(\mathbf{x}) \right\rangle \\ &= \sum_{n=1}^{M} c_n \langle \mathcal{D}(g) b_n(\mathbf{x}), b_m(\mathbf{x}) \rangle \ . \end{aligned}$$

The relation between $\{c_k\}_1^M$ and $\{a_k\}_1^L$ is a linear map $\mathbf{P} \in \mathbb{R}^{M \times L}$ with the matrix elements

$$(\mathbf{P})_{ij} = \langle \mathcal{D}(g)b_i(\mathbf{x}), b_j(\mathbf{x}) \rangle \qquad (17)$$

of the coefficient vector

$$\mathbf{c} := (c_1, c_2, ..., c_M) \qquad (18)$$

onto the interpolation function vector

$$\mathbf{a}(g) := (a_1(g), a_2(g), ..., a_N(g)) \ . \qquad (19)$$

As already pointed out by Michaelis and Sommer (Michaelis and Sommer, 1995), the basis functions have not to be the transformed versions of the filter kernel as assumed in other approaches (Freeman and Adelson, 1991; Simoncelli and Farid, 1996). It is sufficient that the synthesized function is steerable. If it is nonetheless desired to design basis functions which are transformed versions $h_g(\mathbf{x}) := \mathcal{D}(g)h(\mathbf{x})$ of the filter kernel $h(\mathbf{x})$ a basis change is sufficient

$$h(\mathbf{x}) = \sum_{j=1}^{M} a_j(g) b_j(\mathbf{x}) = \sum_{j=1}^{M} \tilde{a}_j h_{g_j}(\mathbf{x}) \ . \tag{20}$$

The relation between $\{a_j(g)\}$ and $\{\tilde{a}_j(g)\}$ can be found by a projection of both sides of equation (20) on $b_m(\mathbf{x})$

$$\begin{aligned} a_m(g) &= \left\langle \sum_{j=1}^{M} a_j(g) h_{g_j}(\mathbf{x}), b_m(\mathbf{x}) \right\rangle \qquad (21) \\ &= \sum_{j=1}^{M} \tilde{a}_j(g) \underbrace{\langle h_{g_j}(\mathbf{x}), b_m(\mathbf{x}) \rangle}_{B_{jm}} \ . \end{aligned}$$

This can be written as a matrix/vector operation with $\tilde{\mathbf{a}}^T := (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)$ and $\mathbf{a}^T := (a_1, a_2, \dots, a_n)$

$$\mathbf{a} = \mathbf{B}\tilde{\mathbf{a}} \ . \tag{22}$$

The matrix $\mathbf{B}$ describing the basis change is invertible

$$\tilde{\mathbf{a}} = \mathbf{B}^{-1}\mathbf{a} \tag{23}$$

and the steerable basis can be designed as steered versions of the original filter kernel.

3.4 Relation to Recent Approaches

We present a steerable filter approach for computing the basis functions and interpolation functions for arbitrary Lie groups. Since two steerable filter approaches based on Lie group theory (Michaelis and Sommer, 1995; Teo and Hel-Or, 1996; Teo and Hel-Or, 1998) have already been developed, the purpose of this section is to examine their relation to our approach.

Freeman and Adelson (Freeman and Adelson, 1991) consider steerable filters with respect to the rotation group in 2D and 3D, respectively. For the 2D case they propose a Fourier basis (of the function space) times a rotational invariant function as well as a polynomial basis (of the function space) times a rotational invariant function as basis functions of the steerable filter. They realized that the minimum required set of basis functions depend on the kind of basis itself but their approach failed to explain the reason for it. Michaelis and Sommer (Michaelis and Sommer, 1995) answer this question based on Lie group theory: the basis of an irreducible group representation span an invariant subspace of minimum size. Since the Fourier basis is the basis for an irreducible representation of the rotation group $SO(2)$, the required number of basis function is less as for the polynomial basis. Our approach can be considered as an extension of the approach of Michaelis and Sommer from Abelian Lie groups to arbitrary Lie group transformation. Whereas the approach of Michaelis and Sommer construct the basis function from the generators of the group, our approach uses a Casimir operator. Since the generators of an Abelian Lie group commutate with each other, their linear combination constitute a Casimir operator and thus both methods become equal in this case. But our method also works for the case of general compact groups, since in this case, a self-adjoint Casimir operator with a discrete spectrum delivers finite dimensional invariant subspaces. For non-compact, non-Abelian groups we showed that polynomials serve always as basis for an invariant subspace. The approach of Teo and Hel-Or significantly differs from our approach in the way how the invariant subspaces are generated. The basis functions of the invariant subspace are constructed by applying all combinations of Lie group generators to the function that is to be made steerable. A certain sequence of generators, denoted as generator chain in case of Abelian Lie groups and generator trees in the case of non-Abelian Lie groups, is stopped if the resulting function is linearly dependent to the basis functions which have already been constructed. In the following, we will show that this approach may fail.

Let us consider the function $h(x, y) = \exp(-x^2)$ and the rotation in 2D as the group transformation. Applying the generator chain which is simply the successive application of the group generator $\mathcal{L} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ does not converge since $h(x, y)$ is not expandable by a finite number of basis functions of a representation of the rotation group. In our approach, $h(x, y)$ is first approximated by a finite number of basis functions each belonging to a finite dimensional invariant subspace. Such a filter is always steerable by construction.

4 GROUP INVARIANT QUADRATURE FILTERS

Quadrature filters have become an appropriate tool for computing the local phase and local energy of one dimensional signals. They are obtained by a bandpass filter and its Hilbert transformation. The bandpass filter is applied to reduce the original signal to a signal with small bandwidth which is necessary to obtain a reasonable interpretation of the local phase. The Hilbert transformation is applied to shift the phase

Table 3.1: Several examples of Lie groups, the corresponding operator(s), generator(s), Casimir operator(s) and basis functions. Terminology: $\mathbf{T}_N$: translation group in the N-dimensional Signal space; $SO(N)$: special orthogonal group; $\mathbf{U}_N$: uniform scaling group; $\mathbf{S}_N$: shear group.

Group	Operators	Generators	Casimir operator	basis functions
$\mathbf{T}_N$	$\mathcal{D}(\mathbf{a})h(\mathbf{x}) = h(\mathbf{x} - \mathbf{a})$	$\{\mathcal{L}_i = \frac{\partial}{\partial x_i}\}$	$\mathcal{C} = \sum_{i=1}^{N} \mathcal{L}_i^2$	$\{\exp\left(j\mathbf{n}^T\mathbf{x}\right)\}$
$SO(2)$	$\mathcal{D}(\alpha)h(r,\varphi) = h(r,\varphi - \alpha)$	$\mathcal{L} = \frac{\partial}{\partial\varphi}$	$\mathcal{C} = \mathcal{L}^2$	$\{f_k(r)\exp(jk\varphi)\}$
$SO(3)$	$\mathcal{R}h(\mathbf{x}) = h(\mathbf{R}^{-1}\mathbf{x})$	$\{\mathcal{L}_k = x_j\frac{\partial}{\partial x_i} - x_i\frac{\partial}{\partial x_j}\}$	$\mathcal{C} = \sum_{i=1}^{3} \mathcal{L}_i^2$	$\{f_k(r)Y_{\ell m}(\theta,\varphi)\}$
$\mathbf{U}_N$	$\mathcal{D}(\alpha)h(\mathbf{x}) = h(e^{-\alpha}\mathbf{x})$	$\{\mathcal{L}_i = x_i\frac{\partial}{\partial x_i}\}$	$\mathcal{C} = \sum_{i=1}^{N} \mathcal{L}_i^2$	$\{r^k\}$
$\mathbf{S}_N$	$\mathcal{D}(\mathbf{u})h(\mathbf{x},t) = h(\mathbf{x} - \mathbf{u}t, t)$	$\{\mathcal{L}_i = t\frac{\partial}{\partial x_i}\}$	$\mathcal{C} = \sum_{i=1}^{N} \mathcal{L}_i^2$	$\{f_k(t)\exp(jk\mathbf{x}/t)\}$

of the original signal by ninety degrees such that the squared sum of the output of the bandpass and its Hilbert transformation results in a phase invariant local energy. In order to apply this concept to image processing, large efforts have been made to generalize the Hilbert transform to 2D dimensional signals (Bülow and Sommer, 2001; Granlund and Knutsson, 1995). All of these approaches fail to be rotational invariant, but rotational invariance is an essential property of all feature detection methods. An appropriate 2D generalization of the analytic signal is the monogenic signal which is based on the vector-valued Riesz transformation (Felsberg and Sommer, 2001). The Riesz transformation is valid for all dimensions and reduces to the Hilbert transformation in the one dimensional case. A further generalization of 2D rotational invariant quadrature filters can be done by steerable filters which behave under certain conditions like quadrature filter pairs (Köthe, 2003). The monogenic signal is included in this approach. We will go a step further, using the theory of Lie groups and the steerable filter approach presented in the last section to develop a generalization of the rotational invariant quadrature filters to quadrature filters which are invariant to compact or Abelian Lie groups and is also valid for arbitrary signal dimensions. In particular, we are able to design rotational invariant quadrature filters in 3D. But also feature detection methods that are invariant with respect to other transformation groups are important as in the case of motion estimation. The signal in the space time volume is sheared and not steered by motion and thus filters for detecting this signal shall be designed invariant with respect to the shear transformation.

4.1 Properties of a General Quadrature Filter

We will first recall the main properties of a quadrature filter. The main idea of a quadrature filter is to apply two filters to a signal such that the sum of the square filter responses reflect the local energy of the signal. Also the local phase of the selected frequency band shall be determined by the two filter outputs. Furthermore, the local energy shall be group invariant, i.e. the filter outputs shall be invariant with respect to the deformation of the signal by the corresponding group. In order to achieve group invariance, we construct our quadrature filter from the basis of an unitary group representation. Groups with an unitary representation are compact groups and Abelian groups (Wigner, 1959). The even $\mathbf{h}_e$ and odd $\mathbf{h}_o$ components of the quadrature filter are constructed by a vector valued impulse response consisting of the basis functions of an unitary representation of dimension m_e and m_o, respectively.

$$\mathbf{h}_e = \begin{pmatrix} h_{e1}(\mathbf{x}) \\ h_{e2}(\mathbf{x}) \\ \vdots \\ h_{em_e}(\mathbf{x}) \end{pmatrix}, \quad \mathbf{h}_o = \begin{pmatrix} h_{o1}(\mathbf{x}) \\ h_{o2}(\mathbf{x}) \\ \vdots \\ h_{om_o}(\mathbf{x}) \end{pmatrix} . \quad (24)$$

In the following we show that all basis functions of an invariant subspace generated by a Casimir operator which is point symmetric, i.e. commutates with the mirror group that acts on the coordinate vector like $\mathbf{P}\mathbf{x} \to -\mathbf{x}$, own the same parity. Since the parity operator commutates with the Casimir operator, there exists simultaneous eigenfunctions. Applying $\mathcal{P}$ two times equals the identity operator and thus the eigenvalues of $\mathcal{P}$ are $\lambda = \pm 1$. Thus every basis function has a certain parity, i.e. is either point symmetric or point anti-symmetric

$$\mathcal{P}b_j(\mathbf{x}) = \pm b_j(\mathbf{x}) \ . \quad (25)$$

Let us consider one basis function $b_1(\mathbf{x})$ with positive parity, i.e. $\mathcal{P}b_1(\mathbf{x}) = b_1(\mathbf{x})$. All other basis functions of the same subspace can be generated by linear combinations of generators of $\mathcal{G}$, where k_j^i denote the coefficients of the linear combination and n the number of generators of $\mathcal{G}$

$$\sum_{j=1}^{n} k_j^i \mathcal{L}_j b_1(\mathbf{x}) = b_i(\mathbf{x}), \quad i = 1, 2, ..., m_e \ . \quad (26)$$

Applying the parity operator on both sides of equ.(26) and considering that $\mathcal{P}$ commutates with all generators yields

$$\begin{aligned}\mathcal{P}\sum_{j=1}^{n} k_j^i \mathcal{L}_j b_1(\mathbf{x}) &= \mathcal{P}b_i(\mathbf{x}) \qquad (27)\\ \Leftrightarrow \sum_{j=1}^{n} k_j^i \mathcal{L}_j b_1(\mathbf{x}) &= \mathcal{P}b_i(\mathbf{x})\\ \Rightarrow Pb_i(\mathbf{x}) &= b_i(\mathbf{x}) \quad \forall i \ .\end{aligned}$$

If we assume $\mathcal{P}b_1(\mathbf{x}) = -b_1(\mathbf{x})$ we obtain with the same deduction $\mathcal{P}b_i(\mathbf{x}) = -b_i(\mathbf{x})$ for all basis functions of this subspace. Thus, all basis functions belonging to the same subspace attain the same parity. The filter responses of $\mathbf{h}_e$ and $\mathbf{h}_o$ are denoted as the filter channels $\mathbf{c}_e = s(\mathbf{x}) * \mathbf{h}_e(\mathbf{x})$ and $\mathbf{c}_o = s(\mathbf{x}) * \mathbf{h}_o(\mathbf{x})$, respectively. The square of the filter response of each channel are denoted as even and odd energies. Due to the unitary representation, both energies are invariant under the corresponding group action

$$E_s = (\mathbf{D}(g)\mathbf{c}_s)^T (\mathbf{D}(g)\mathbf{c}_s) = \mathbf{c}_s^T \mathbf{c}_s \quad s \in \{e, o\} \ .$$

Note that the inner product is taken with respect to the invariant subspace, not with respect to the function space. The local energy of the signal is given by the sum of the even and odd energy. In the following we will examine the properties of the filter channels required to achieve a phase invariant local energy when applied to bandpass signals. In the ideal case, a simple [1] bandpass filtered signal consists of only one wave vector $\mathbf{k}_0$ and its Fourier transform[2] reads with the Dirac delta distribution $\delta(\mathbf{k})$

$$S(\mathbf{k}) = S_0\delta(\mathbf{k} - \mathbf{k}_0) + S_0\delta(\mathbf{k} + \mathbf{k}_0) \ . \qquad (28)$$

We start with examining the Fourier transform of the even and odd energies

$$E_s = \mathbf{c}_s^T \mathbf{c}_s = \sum_{j=1}^{m_s} (s(\mathbf{x}) * h_{sj}(\mathbf{x}))^2 \ . \qquad (29)$$

Applying the convolution theorem to E_s reads

$$\mathcal{F}\{E_s\}(\mathbf{k}) = \sum_{j=1}^{m_s} (S(\mathbf{k})H_{sj}(\mathbf{k})) * (S(\mathbf{k})H_{sj}(\mathbf{k})) \ .$$

Inserting the signal (28) in the equation above, computing the convolution and performing the inverse Fourier transformation reads

$$\begin{aligned}E_s(\mathbf{x}) &= S_0^2 \sum_{i=1}^{m_s} (H_{si}(\mathbf{k}_0))^2 e^{4\pi j \mathbf{k}_0^T \mathbf{x}} \qquad (30)\\ &+ S_0^2 \sum_{i=1}^{m_s} (H_{si}(-\mathbf{k}_0))^2 e^{-4\pi j \mathbf{k}_0^T \mathbf{x}}\\ &+ S_0^2 \sum_{i=1}^{m_s} H_{si}(\mathbf{k}_0) H_{si}(-\mathbf{k}_0)\\ &+ S_0^2 \sum_{i=1}^{m_s} H_{si}(-\mathbf{k}_0) H_{si}(\mathbf{k}_0) \ .\end{aligned}$$

Note that the first two terms are phase variant whereas the last two ones are not. In order to achieve phase invariant local energy, the first two terms have to cancel when adding the even and odd energy. This is exactly the case when all basis functions of one invariant subspace are either even or odd and the sum of squared Fourier transformed filter components are equal

$$\sum_{i=1}^{m_e} |H_{ei}(\mathbf{k}_0)|^2 = \sum_{k=1}^{m_o} |H_{ok}(\mathbf{k}_0)|^2 \ . \qquad (31)$$

All basis functions of one invariant subspace are either even (= their Fourier transforms are real and even), or odd (= their Fourier transforms are imaginary and odd). Thus, the Fourier transformed filter components become

$$\sum_{i=1}^{m_e} H_{ei}(\pm\mathbf{k}_0)^2 = \sum_{i=1}^{m_e} |H_{ei}(\mathbf{k}_0)|^2 \qquad (32)$$

in the even case and

$$\sum_{i=1}^{m_o} H_{oi}(\pm\mathbf{k}_0)^2 = -\sum_{i=1}^{m_o} |H_{oi}(\mathbf{k}_0)|^2 \qquad (33)$$

in the odd case. Since the inner product of the Fourier transform of both filter channels are equal, the first two terms cancel out resulting in a phase invariant local energy

$$E = 2S_0^2 \left(\sum_{j=1}^{s} |H_{nj}(\mathbf{k}_0)|^2 + \sum_{k=1}^{d} |H_{mk}(\mathbf{k}_0)|^2 \right) \ .$$

The local phase ϕ of an intrinsic one dimensional signal is given by

$$\tan(\phi) = \frac{\left[\sum_{i=1}^{m_o} (h_{oi}(\mathbf{x}) * s(\mathbf{x}))^2\right]^{\frac{1}{2}}}{\left[\sum_{i=1}^{m_e} (h_{ei}(\mathbf{x}) * s(\mathbf{x}))^2\right]^{\frac{1}{2}}} \ . \qquad (34)$$

In the next section an example of an group invariant quadrature filter is presented.

[1] simple signal: signal with intrinsic dimensionality one.

[2] Note that the Fourier transformed entities are labeled with capital letters.

4.2 An Example: 3D Rotational Invariant Quadrature Filters

We now apply the approach presented in the last section to the 3D rotational invariant quadrature filter. The even $\mathbf{h}_e$ and odd $\mathbf{h}_o$ vector valued impulse responses have to be the basis functions of an unitary representation of the rotation group $SO(3)$. A possible basis of an unitary invariant subspaces are the well known spherical harmonics times an arbitrary radial function $f_n(|\mathbf{x}|) \in L^2$

$$b_{n\ell m}(\mathbf{x}) = f_n(|\mathbf{x}|) Y_{\ell m}(\hat{\mathbf{x}}) \ . \tag{35}$$

The spherical harmonics are either even or odd, thus the even vector valued impulse responses can be constructed from all spherical harmonics of even order, the odd vector valued impulse response from all spherical harmonics of odd order. According to equ.(31), we have to show that the scalar product of the Fourier transformed vector valued impulse responses are equal. It is well known that a radial functions times a spherical harmonic is also spherical separable in the Fourier domain and vise versa. If we require, like in the 2D case (Köthe, 2003), the radial function $F_{n\ell m}(|\mathbf{k}_0|) = F(|\mathbf{k}_0|)$ in the Fourier domain to be the same for all transfer functions, the constraint equ.(31) becomes

$$\sum_{\ell}^{\text{odd}} \sum_{m=-\ell}^{\ell} |Y_{\ell m}(\hat{\mathbf{k}}_0)|^2 = \sum_{\ell}^{\text{even}} \sum_{m=-\ell}^{\ell} |Y_{\ell m}(\hat{\mathbf{k}}_0)|^2 \ . \tag{36}$$

Since the scalar product of the even as well as the odd spherical harmonics are rotational invariant the right and the left hand side of equ.(36) is constant. Therefore, the constraint equation can be always be fulfilled by an appropriate scaling of the spherical harmonics.

5 CONCLUSION

We have presented a theory for steerable filters and quadrature filters based on Lie group theory. Both approaches are most general with respect to the signal dimension as well as with respect to the transformation Lie group. For the steerable filter case, we provide for every quadratic integrable function (at least approximately) the method for constructing the basis functions for every Lie group transformation. For compact and Abelian groups we even showed that this is the minimum required number of basis functions. Furthermore, we generalized the 2D rotational invariant quadrature filter approach with respect to arbitrary dimension of the signal space and to Lie group transformation which own an unitary representation. It turned out that the group invariant quadrature filter is a special steerable filter. The future work will be the integration of the general quadrature filter approach into a tensor representation and its application to motion and orientation estimation in 3D.

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