# CIRCULAR PROCESSING OF THE HUE VARIABLE
## *A Particular Trait of Colour Image Processing*

Alfredo Restrepo Palacios, Carlos Rodríguez and Camilo Vejarano
*Laboratorio de Señales, Dpt. Ing. Eléctrica y Electrónica*
*Universidad de los Andes, Bogotá, Colombia*

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Abstract: Novel tools for colour image processing are presented. Unlike many magnitudes dealt with in engineering, the hue variable of a colour image is circular and requires a special treatment. Special techniques have been advanced in statistics for the analysis of data from angular variables; likewise in image processing for the processing of the hue variable. We give a definition of the median and of the range of angular data and apply their running versions on images to smooth them and to detect hue edges. We also give definitions of hue morphology; one based on the topological concept of lifting and on grey level morphology; another definition is wholly given in a circular context.

# 1 INTRODUCTION

We consider a specific aspect of colour image processing, namely the processing of the H variable of the HVS colour system. The H variable is an angular variable, i.e. one that lives in the circle which is the one dimensional sphere $S^1$, that we interpret as the one-point compactification of the interval $(-\pi, \pi]$. $S^1$ is orientable; we assume that the orientation is positive when it is counterclockwise; in this sense, colours are *positively sequenced* as in red, orange, yellow, citrine, green, cyan, blue violet, red etc.; see Figure 1. Unless otherwise stated, in the examples, we assume V = S = 0.8 (constant value and saturation). The tools given here can be combined with other tools that process the V and S variables.

The elements of $S^1$ cannot be linearly ordered in any way compatible with its topology; thus, a circular version of most statistics already defined for linearly ordered data is usually not obvious. A typical problem for the definition of *location* statistics is that for certain uniformly distributed samples it is best to leave them undefined; e.g. the average of a sample of equally spaced angles. Also, the problem of multiplicity is more ubiquitous than in the case of a linearly ordered range space (e.g., in the linearly ordered case, the median of an even sized sample). We consider four sample statistics: the circular average, a circular median, a circular range and the circular concentration, and their corresponding running versions.

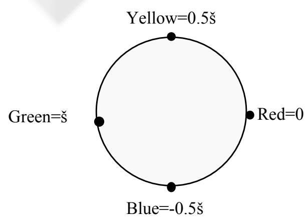


Figure 1: The space $S^1$ of hues.

A 2D image (i.e. a 2D signal) $h:A \rightarrow B$ is a function whose domain set A is two dimensional; the image is *discrete* if A is countable, it is *digital* if its range set B is finite. In the case of discrete images, we use as domain set a product of *integer intervals*. An integer interval /n, m/ is the set of the integers less than or equal to m and larger than or equal to n. We concentrate here on images whose range set is $S^1$; we call them *hue images* and picture them as computer images in the HSV colour system with S and V constant. The elements of the domain set are called *pixels* and the elements of the range set are the possible *values* of the pixels.

Several proposals for doing hue mathematical morphology have been advanced e.g. (Peters II,

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1997), (Comer and Delp, 1999), (Hanbury and Serra, 2001), (Hanbury and Serra, 2001), (Vejarano, 2002) and others. We give one based on the uniqueness of the lifting of a *map* (a map is a continuous function) with domain a *simply connected space* to the *universal cover* of the range space (Christenson and Voxman, 1998); another, based on intrinsic aspects of the circular relation of angular data.

## 2 STATISTICS OF LOCATION AND DISPERSION FOR DIRECTIONAL DATA

Most of the time we assume angles (i.e. hues) to be either numbers in the interval $(-\pi, \pi]$ or numbers in the interval $[0, 2\pi)$; the choice by default is $(-\pi, \pi]$. Except in the case of *liftings*, we do not use multiple code representations (equivalent mod-$2\pi$) for the same angle.

It is convenient to use the complex number $e^{j\phi}$ as a representation the angle $\phi$. Assuming the function arctan to have range $(-\pi/2, \pi/2)$; the angle arg(z) of a nonzero complex number (arg(z) is not defined if z = 0+j0) is defined as:

$$\arg(z)=\arctan(\mathrm{Im}(z)/\mathrm{Re}(z)) \text{ if } \mathrm{Re}(z)>0$$
$$\arg(z)=-\pi+\arctan(\mathrm{Im}(z)/\mathrm{Re}(z)) \text{ if } \mathrm{Re}(z)<0$$
$$\arg(z)=\pi/2 \text{ if } \mathrm{Re}(z)=0 \text{ and } \mathrm{Im}(z)>0$$
$$\arg(z)=-\pi/2 \text{ if } \mathrm{Re}(z)=0 \text{ and } \mathrm{Im}(z)<0$$

Let $[\phi_1, \ldots \phi_N]$ be a sample of N angles; unlike a *set,* in a sample there may be repeated data. Let $[z_1, \ldots z_N]$, with $\phi_i=\arg(z_i)$, be the N-tuple of the corresponding complex numbers on the unit circle. We respectively call

$$Z := \sum_{i=1}^{N} z_i \quad \text{and } \Phi = \Big(\sum_{i=1}^{N} \phi_i\Big) \text{ mod-}2\pi$$

the *complex sum* and the *angular sum* of the angles. Clearly, $\arg(Z) \neq \Phi$. For example, consider $3\pi/4$ and $5\pi/4$.

### 2.1 Sample Mean or Average

We give the standard definition of the average of a sample of angles; see e.g. (Nicolaidis and Pitas, 1998) and (www.higp.hawaii.edu, 2006).

Let $[\phi_1, \ldots \phi_N]$ be a *sample* of N angles; if their complex sum Z is 0+j0, we leave the *average of the sample* undefined, otherwise, the *average of the sample* is set to be the angle arg(Z) of the complex sum.

For the texture images at the top in Figure 2, after the application of the running average (we use windows of size 3x3) the images at the bottom result. When the average of a window is undefined, the central pixel was left unchanged. The textures are based on the Von Mises distribution, given by:

$$f(\phi) = (1/2\pi I_0(\kappa)) \exp(\kappa \cos(\phi-\alpha))$$

where $I_0$ is a Bessel function that ensures that the pdf integrates to 1, the parameter $\kappa$ has to do with the variance and $\alpha$ is the mean of the distribution.

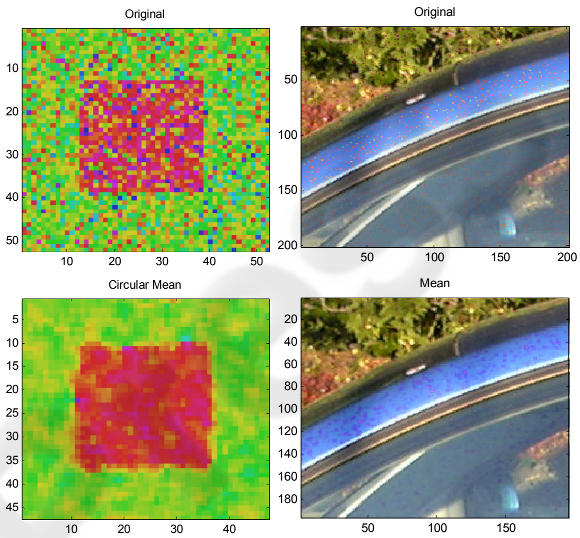


Figure 2: A Von Mises' texture images and a noisy image (top) and the result after applying a moving average (below).

### 2.2 Sample Median

As above, let $[\phi_1, \ldots \phi_N]$ be a sample of angular data. Let $d_{ij}$ be the *distances* between pairs (a pair is a set of cardinality two) of *consecutive* angles $\phi_i$ and $\phi_j$, given by $d_{ij} = T(|\phi_i - \phi_j|)$, where:

$$T: [0, 2\pi) \rightarrow [0, \pi]$$

has the graph indicated in Figure 3.
To find out which angles are consecutive, order the angles in the sample, in their domain of representation $(-\pi, \pi]$, get the pairs of consecutive angles in this ordering, and add an extra pair given by the largest and the smallest angles in this interval.

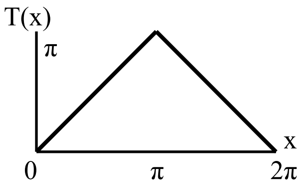


Figure 3: The function T, used to define a metric on $\mathbf{S}^1$.

If each component of the sample has the same value, let the median be this common value. If the sample is not constant, the set of distances $\{d_{ij}\}$ has a positive (i.e. larger than zero) maximum; if this positive maximum is achieved for a unique pair of angles $\phi_i$ and $\phi_j$, subtract the 0-sphere $\{\phi_i, \phi_j\}$ from the 1-sphere $S^1$; two connected components result (Jordan theorem in dimension 1;) of these two components, one contains the data; subtract the other connected component, which we call the *gap;* the resulting arc contains the data and and may now be linearly ordered (see Figure 4); take the median of the data in this arc, if the sample has even size, take the (angular) average of the two central data.

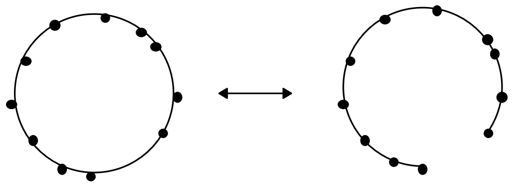

Figure 4: The data determine a gap on $\mathbf{S}^1$, which is taken off.

If the maximum of the distances between consecutive angles is achieved more than once, the definitions must be refined; see Figs. 5a. and 5b. For one thing, the sample has *several gaps of unique length* and it still may have a median. Proceed in two steps to find it. Initially, for each of the multiple, equally maximally sized gaps, obtain a median as above. The resulting set of medians may further have a median or it may not; it does not if this set of preliminary medians determines again a unique distance between consecutive points; in such a case the sample is said to be *uniformly spaced* and to have no median (nor a mean.) Otherwise, compute the median of this set of preliminary medians, as above, and call it *the median* of the sample.

A rule of thumb to check things is to slightly separate repeated data. The definition given of the sample median of angular data gives a unique answer in cases when the median defined in (Nicolaidis and Pitas, 1998) does not; see Figure 5c.

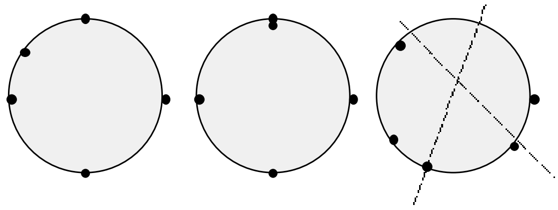


Figure 5: In a, there are three maximal gaps; in b, four. In each case, they are of the same length.

In Figure 6, we show the result of applying a 3x3 hue median filter to the texture images at the top of Figure 2.

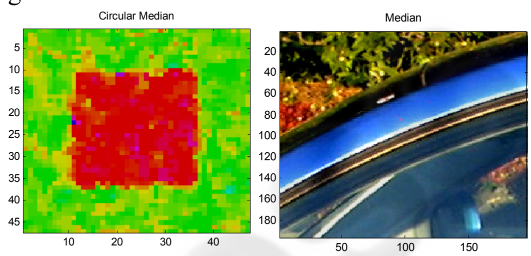


Figure 6: The result of applying the median filter to the images at the top in Figure 2.

## 2.3 Max and Min

Both the average and the median of a sample of hues are hues and so are the max and the min as defined below; nevertheless, the length of the gap, the concentration and the range are angles but no hues: they are differences of hues. Two different pairs of hues may have the same difference and a sample may have a range but no median.

Let $[\phi_1, \ldots \phi_N]$ be a sample of angular data. If the sample is uniformly spaced or if there are a multiple gaps, the max and the minimum of the sample are left undefined; otherwise, for a unique gap, the *maximum* (max) and *minimum* (min) of the sample are defined as follows: take the gap off as above and, in the remaining arc, let the point most ccw (counterclockwise) be the *maximum* of the sample and the point most cw (clockwise) be the *minimum.* For example, for the sample [orange, red, yellow], red is the min and yellow is the max.

## 2.4 Concentration and Range

Let $[\phi_1, \ldots \phi_N]$ be a sample of angular data; let the *concentration C of the sample* be given by the magnitude of the complex sum Z divided by N,

$$C := (1/N)|Z|$$

C ranges between 0 and 1, and clearly is a measure of the concentration of the data. If the complex sum of the sample is 0, the sample has no average but it has (zero) concentration. The name given here to this statistic is not standard (*mean resultant length,* in (www.higp.hawaii.edu, 2006)).
For a constant sample, the range is set to be $2\pi$; otherwise, proceed as in Section 2.2 and define the *range* $\rho$ of the sample as:

$$\rho = 2\pi - \max\{d_{ij}\} = 2\pi - \text{length(gap)}$$

$\rho$ clearly is a measure of the dispersion of the data.
The moving range and moving concentration are used to get maps of hue edges, from the image at the top, left, in Figure 7.

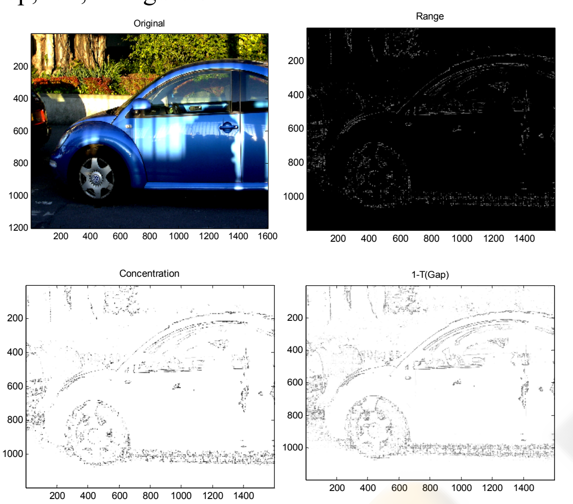


Figure 7: Hue edge maps.

# 3 HUE IMAGES

A *continuous 2D image* (as opposed to a *discrete image* as defined in Section I) is a function with domain set a product I×J of real intervals The *continuity* referred to here is of the domain; thus, a continuous image may be a discontinuous function. If the range set is $S^1$, the values of the function are *hues.*

The existence of certain mathematical tools for handling continuous functions makes the corresponding images important from a theoretical viewpoint. For example, the lifting of a continuous function to a simply connected cover of the range set, is unique. Thus, even though the discontinuities of a function carry important information, it may be convenient to restrict attention to continuous, and even differentiable, functions. After all, the rates of change can be arbitrarily large. In fact, the largest possible jump of a hue image has value $\pi$ and it corresponds to a change from a hue to its *opposing* hue, which may correspond as well to its complementary colour. (Two colours are said to be *complementary* if their additive mixture produces an achromatic colour).

But, perhaps more to the point, in digital image processing, one considers discrete images. Moreover, in practice, discrete images are digital (as defined in Section I); nevertheless, we disregard here the discrete nature of the range set. The range set of the hue component of an image, in all cases will be assumed to be $S^1$.

Since the basic definitions of mathematical morphology are given in terms of the operations of taking the maxima and minima of sets, it is assumed that the range set has at least the structure of a linearly ordered set (and the domain set, that of a lattice). This is the case of grey level images, for example, but not the case of hue images. We explore below two approaches for performing morphology on angle-valued functions; one is based on the topological concept of the lift of a function while the other works in the natural ambient space of the graph of the function; both are based on grey level morphology. Initially, we consider briefly the geometry of the ambient space of the graph of the hue function.

## 3.1 The Graph of a Function from a 2D Interval to the Circle

Consider 1D continuous hue images. The graph of a function $f{:}I \to S^1$ lives on $I \times S^1$ which is a cylinder and can be also thought of an annulus (the annulus is homeomorphic but not isometric); if the function is continuous, the graph is connected. See Figure 8.

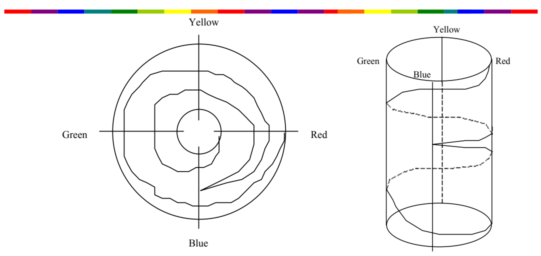


Figure 8: Two pictures of the graph of the function $f{:}I \to S^1$, corresponding to the coloured line, shown discontinuous above.

The graph of a function $f : I \times J \to S^1$ lives on $I \times J \times S^1$ which can be thought of a solid cylinder; if the function is continuous, the graph is connected.

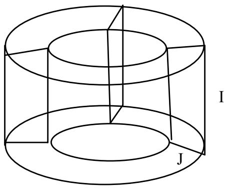


Figure 9: The product $I^2\times S^1$.

For each $s \in S^1$ the intersection $K_s$ of the graph of f with the plane $\{(x, s) : x \in I\times J\} \subset I\times J\times S^1$ has as connected components points, arcs, Jordan curves and unions of these. We may give a partial order to the set of the simple closed curves in

$$T_s := K_s \ast \partial(I\times J)$$
$$= \{(x, s) : x \in I\times J,\ f(x) = s\} \ast \partial(I\times J)$$

A simple closed curve $C_1$ bounds a region on its *interior* (i.e. the region *farthest* from $\partial(I\times J)$); if another Jordan curve $C_2$ in $T_s$ lies in this region, we write $C_2 < C_1$. This is a partial order.

## 3.2 Lifting Angular Maps

$R^1$ is the *universal covering space* of $S^1$ (Christenson and Voxman, 1998), as projection map we have $p(x) := [x]_{2\pi}$ where $[x]_{2\pi}$ is the only number in $[0, 2\pi)$ equivalent to x, mod-$2\pi$. See Figure 10.

$$\begin{tikzcd} & R^1 \arrow[d, "p"] \\ I\times J \arrow[ur, "F"] \arrow[r, "f"] & S^1 \end{tikzcd}$$

Figure 10: $R^1$ is the universal cover of $S^1$.

Now to lift a continuous function $f{:}I\times J\to S^1$ is to find a continuous function $F{:}I\times J\to R^1$ such that $p(F(x)) = f(x)$. If we specify $F(0, 0) = f(0, 0)$, F is unique.

The max minus the min of the values taken by a lift F of a hue function f is said to be the *degree* of f. It is rare that in an image of a common scene, the degree of the hue be larger than $2\pi$; it implies more than one set of colours as in a rainbow, with the same ordering of colours along some path.

To lift f is, in a sense, to unfold its graph. The smallest curves in $T_0$, according to the partial ordering defined in Section 3.1, are the starting point for an algorithm that finds the lift F. Find these smallest simple closed curves and then define F on the regions bounded by these components. Initially, on these smallest regions set F:=f. Then, depending on whether F is positive or negative on a region, for the next region, define F as $f+2\pi$ or $f-2\pi$. On a region of *order* n, add or subtract $2n\pi$.

# 4 MATHEMATICAL MORPHOLOGY FOR A CONTINUOUS ANGULAR VALUED FUNCTION

The (unique) lifting $F{:}I\times J\to R^1$ of a hue map $f{:}I\times J\to S^1$, has as range set $R^1$, which is linearly ordered. We define MOP(f) as p(mop(F)); where mop is a standard grey level morphological operator (Heijmans, 1994), (Serra, 1998) and MOP is the corresponding hue, morphological operator being defined.

For example, the hue image in Figure 11 corresponds to the function with graph (plotted in a flat 3D ambient space) as shown in Figure 12; the lifting of this function is also shown in Figure 12; applying ("grey level") dilation/erosion and opening/closing, with the structuring element in Figure 13, and projecting back on $S^1$ with the projection $p = (\mathrm{mod}\ 2\pi)$, we get the image shown in Figure 11.

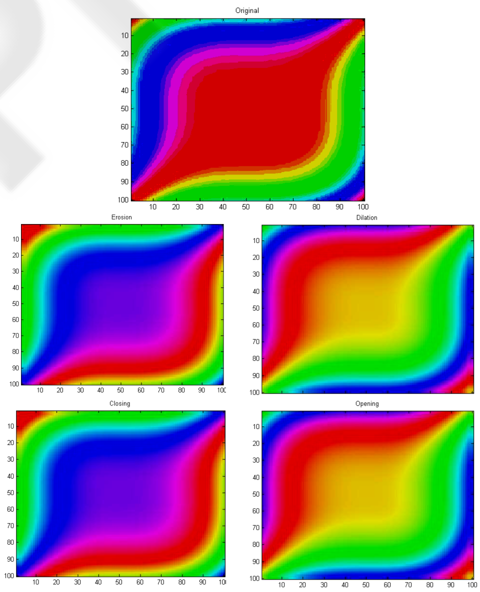


Figure 11: Hue morphology based on lifting.

As we see, for the given structuring element, in the erosion, reds become violet, in general there is a cw shift of hues which, for the given image, gives an impression of migration of hues in the (opposite) ccw direction (violets migrating toward red regions.)

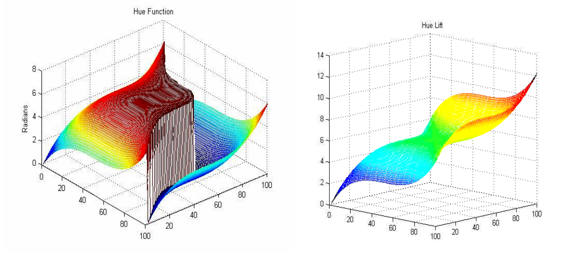


Figure 12: Hue function and lifted version (false colours).

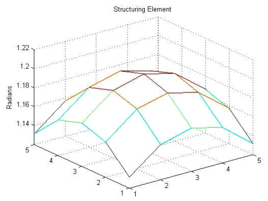


Figure 13: Structuring element.

## 4.1 Algorithms for Lifting Discrete Images

For a function $f{:}N{\times}M{\rightarrow}S^1$, where N and M are discrete intervals, several algorithms can be proposed to obtain functions $F{:}N{\times}M{\rightarrow}R^1$ such that $p(F(x)) = f(x)$; however, the difference between two such F´s (we chose to call them "lifts") is not necessarily constant. One kind of algorithm is based on the idea of sweeping the domain set N×M another kind is based on the idea of interpolating the discrete image and obtain a continuous image.

### 4.1.1 Lifting Along Paths

Consider initially a coloured line, that is, a function with domain set an integer interval /0, N-1/. We start at either one of the extrema of the interval, say at 0. Initially, we set F(0) = f(0), to compute the remaining values of F, we proceed as follows. Assume F has been defined up to pixel i; next, let $F_{i+1} = F_i + \Delta(\delta_i)$ where $\delta_i := f_{i+1} - f_i$ and $\Delta$ is defined as $\Delta(\delta) = \delta$, if $|\delta| < \pi$; $\Delta(\delta) = 2\pi + \delta$, if $\delta < -\pi$; and $\Delta(\delta) = \delta - 2\pi$ if $\delta > \pi$. Clearly, f = F (mod-$2\pi$). The algorithm does not give a unique *lift*. The lift obtained starting from the right may be different. Nevertheless, the two lifts are *equivalent*, i. e. their difference is a constant congruent with 0 (mod-$2\pi$). Then, the operators of grey-level morphology can be applied, before projecting back on S1, See Figure 14. Now consider 2D, discrete, hue images f: /1, N/ x /1, M/ $\rightarrow S^1$; a lift F of f is such that F(i, j) = f(i, j) mod-$2\pi$. Sweeping the domain set along a simple path gives $\Delta$s between neighbours that depend on the path chosen to do the lift.

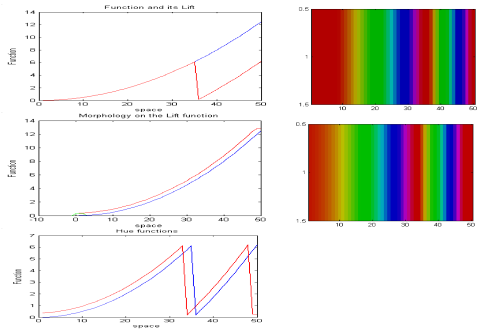


Figure 14: A 1D hue function and its lift, a structuring element and the resulting function, the original function and the projected processed function. The original 1D image and the resulting one on the right.

For example, consider the image and the two sweeping paths below

$$\begin{array}{ccc} 0 & 1.5\pi & \pi \\ 0.5\pi & 0 & 0.5\pi \\ \pi & 1.5\pi & 0 \end{array}$$

the corresponding resulting lifts are

$$\begin{array}{ccc} 2\pi & 1.5\pi & 1\pi \\ 2.5\pi & 0 & 0.5\pi \\ 3\pi & 3.5\pi & 4\pi \end{array} \qquad \begin{array}{ccc} 0 & -0.5\pi & \pi \\ 0.5\pi & 0 & 0.5\pi \\ \pi & -0.5\pi & 0 \end{array}$$

### 4.1.2 Interpolating Hue Images

The (e.g. linear) interpolation of a function defined on a rectangular grid ZxZ to a function defined on ZxZ is not uniquely defined; consider 4 neighbour pixels as in the corners of the square, notice that there are several, possibly conflicting, ways of interpolating along the diagonals. From an image with a very good resolution, that won't loose much from a process of decimation as it is redefined on a (non regular) hexagonal grid, discard the pixels (i, j) for which i+j is odd, as shown in Figure 15.

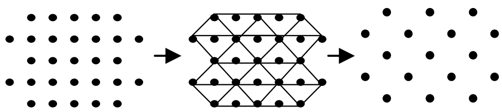

Figure 15: The decimation of a rectangular grid that give a hexagonal grid.

On each triangle linear interpolation is used. Once again, there are multiple possible linear interpolations, some of them of the same cost. By cost we mean the spent arc length on the circle. Consider first interpolation along a line. Here there are only two possible options: for x and y are angles in $[0, 2\pi)$, assuming $x \leq y$, there are two choices:

$(1-\alpha)x+\alpha y$ and $[(1-\alpha)(x+2\pi)+\alpha y]_{2\pi}$; if $|x-y| = T(|x-y|)$ then the first choice is less expensive and if $|x-y| > T(|x-y|)$ then the second choice is less expensive. The parameter $\alpha$ varies between 0 and 1.

Let x, y and z be three pixels which are the vertices of a triangle and let u=f(x), v=f(y) and w=f(z) be the corresponding hues and assume that, as numbers in $[0, 2\pi)$, $u \le v \le w$. There are three possible linear interpolations for the colours of the points in the triangle; for each point in the triangle let $\lambda_1$, $\lambda_2$, $\lambda_3$ be the (unique) barycentric coordinates of the point; the interpolations are given by $[\lambda_1(u+2\pi)+\lambda_2 v+\lambda_3 w]_{2\pi}$, $[\lambda_1 u+\lambda_2 v+\lambda_3 w]_{2\pi}$ and $[\lambda_1 u+\lambda_2 v+\lambda_3(w-2\pi)]_{2\pi}$. The choice depends on the cost of each interpolation and on the choices made for neighbour triangles.

Once a continuous function is obtained, the image has a unique lift.

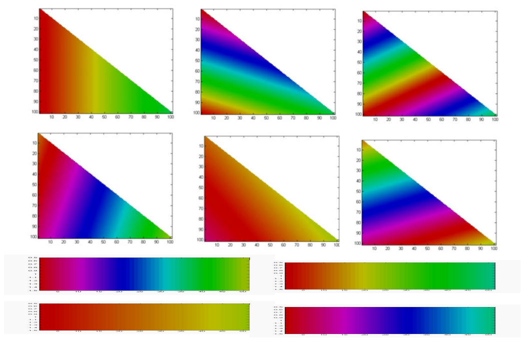

Figure 16: Interpolations of triangles given the vertices red red green and violet-orange-green; and of lines given extremes red-citrine and red-cyan.

## 4.2 Circular Mathematical Morphology

An alternate way of applying morphological operators to the hue component of colour images is to interpret the max and min operators and the addition operation in the context of a circular variable, as in Section 2.3. We translate *addition* as *counter clockwise rotation*, i.e. as addition of real numbers followed by the operation of taking modulo $2\pi$. Then, apply the standard definitions of grey level image morphology. Occasionally, the data in, say a 3x3 window, have multiple gaps and there is not a max nor a min, and a special treatment must be given to the pixel at hand. For this sort of sample, we choose to leave the corresponding pixels unchanged.

As can be observed in Figure 17, in the erosion, the blue of the wall became cyan (a negative shift) while in the dilation, violet (a positive shift). The red arrow in a yellowish background grew larger in the erosion and thinner in the dilation. In Figure 18, two Von Mises textures are shown; due to the shape of the structuring element, the inner square grows and becomes bluer with an erosion; with a dilation, it shrinks and becomes more yellow; both textures lost contrast.

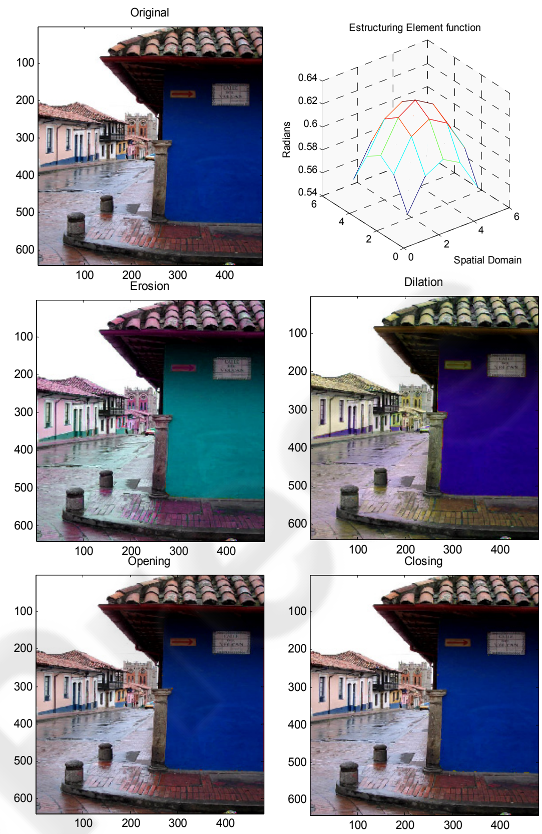


Figure 17: Circular morphological operations.

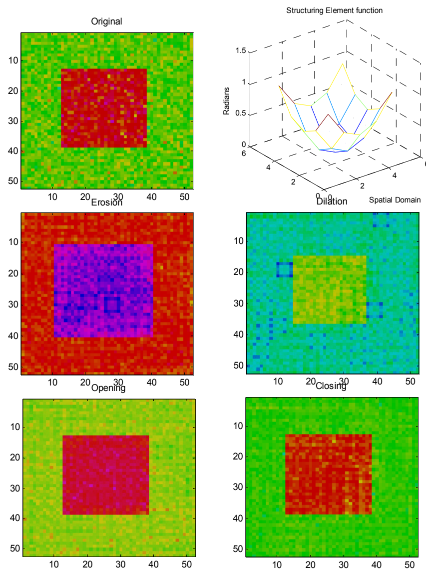


Figure 18: Circular morphological operations.

## 5 CONCLUSIONS

We have extended the definition and applied circular versions of statistics commonly used for linearly ordered data; in particular, the sample average, median, gap, min, max, concentration and range. We applied their running versions as smoothers and edge detectors of colour images. As expected, for the noises tried, the median filter works better visually than the average. We consider that the edge detector with best performance is the one given by 1-T(gap). As in the case of the phase of complex numbers, undeniable useful, in some cases these statistics must be left undefined. Two methods to apply morphological operators to angle valued signals are presented. These novel tools for colour image processing are likely to be useful.

Unlike previous versions of colour morphology, we consider only the hue component, leaving the components of saturation and value unaltered. Also, we respect the circular nature of the hue variable while taking advantage of grey level morphology. The processing of the hue component alone illustrates the effect of the tools which are particular due to the circular nature of the hue variable.

The cases of undefined statistics and morphological operators are more common, although on a, say 5x5 window, it is probably hard to find 25 hues uniformly distributed. We have chosen to leave the corresponding pixels unaltered but other choices are possible.

We have given algorithms for the interpolation of hue valued functions on triangular meshes as well as on 1D discrete domains. We found an unexpected lack of algorithms for the lifting of angle functions, this seems to be a fertile field of research.

The field of color image processing is important in computer vision tasks such as the detection of malaria in blood films (Ortiz et al., 2005) and also in tasks where the aesthetic quality of the processed image is important such as in commercial colour photography and in digital document restoration.

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