PAIRWISE COMPARISONS, INCOMPARABILITY AND PARTIAL ORDERS

Ryszard Janicki*
Department of Computing and Software, McMaster University, Hamilton, Ontario, Canada L8S 4K1

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Abstract: A new approach to *Pairwise Comparisons based Ranking* is presented. An abstract model based on the concept of partial order instead of numerical scale is introduced and analysed. Many faults of traditional, numerical-scale based models are discussed. The importance of the concept of *equal importance* or *indifference* is discussed.

1 INTRODUCTION

The method of *Pairwise Comparisons* could be traced to Marquis de Condorcet 1785 paper (see (Arrow, 1951)), was explicitly mentioned and analysed by Fechner in 1860 (Fechner, 1860), made popular by Thurstone in 1927 (Thurstone, 1927), and was transformed into a kind of (semi) formal methodology by Saaty in 1977 (called *AHP*, Analytic Hierarchy Process, see (Dyer, 1990; French, 1986; Satty, 1977)). In principle it is based on the observation that while ranking ("weighting") the importance of *several* objects is frequently problematic, it is much easier when restricted to *two* objects. The problem is then reduced to constructing a global ranking ("weighting") from the set of partially ordered pairs. The name "Pairwise" is slightly misleading, since for every *triple* some consistency rules are required (Dyer, 1990; Koczkodaj, 1993; Satty, 1986). For the orderings of *a*, *b*, and *c* are made independently, it frequently occurs that the consistency rules are not satisfied. To deal with this problem various consistency concepts have been introduced and analysed.

At present Pairwise Comparisons are practically identified with the controversial Saaty's AHP. On one hand AHP has respected practical applications, on the other hand it is still considered by many (see (Dyer, 1990)) as a flawed procedure that produces arbitrary rankings.

The strange and contradictory examples of rankings obtained by AHP cited in the literature (see (Dyer, 1990)) follow from the following sources:

- When two objects are being compared, interval scales are mainly used; but later, when reciprocal matrices are constructed the values of interval scales are treated as the values of ratio scales. The concept of a scale was never formally defined or analysed.
- The domain of real numbers, $\mathcal{R}$, has *two* distinct interpretations, and these two interpretations are mixed up in the calculi provided by AHP. The first interpretation is standard, numbers describe quantative values that can be added, multiplied, etc. The second interpretation is that this is just a representation of a *total order*, and for instance 1.0 is "better" than 3.51 and "much better" than 13.6, but that's it! In this interpretation one must be very careful when any arithmetic operations are performed.
- Even though the input numbers are "rough" and imprecise the results are treated as if they were very precise; "incomparability" is practically disallowed.

In the author's opinion the problems mentioned above stem mainly from the following sources:

- The final outcome is expected to be totally ordered (i.e. for all a, b, either $a < b$ or $b > a$),

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- Numbers are used to calculate the final outcome.

A non-numerical solution was proposed and discussed in (Janicki and Koczkodaj, 1996), but it assumed that on the initial level of pairwise comparisons, the answer could be only "$a < b$" or "$a \approx b$", or "$a > b$", i.e. "*slightly in favour*" and "*very strongly in favour*" were indistinguishable. In this paper we provide the means to make such a distinction *without using numbers*.

The model presented below is an extension of the one from (Janicki and Koczkodaj, 1996) with some influence due the results of (Fishburn, 1985; Janicki and Koutny, 1993).

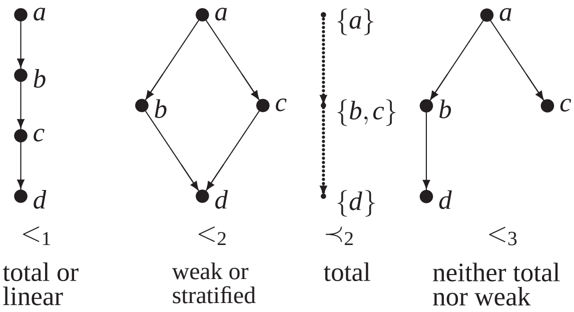

Figure 1: Various types of partial orders (represented as Hase diagrams). The total order $\prec_2$ represents the weak order $<_2$, The partial order $<_3$ is nether total nor weak.

2 TOTAL, WEAK AND PARTIAL ORDERS

Let X be a *finite* set. A relation $\lhd \subseteq X \times X$ is a (*sharp*) *partial order* if it is irreflexive and transitive, i.e. if $a \lhd b \Rightarrow \neg(b \lhd a)$ and $a \lhd b \lhd c \Rightarrow a \lhd c$, for all $a,b,c \in X$. A pair $(X, \lhd)$ is called a *partially ordered set*. We will often identify $(X, \lhd)$ with $\lhd$, when X is known.

We write $a \sim_\lhd b$ if $\neg(a \lhd b) \wedge \neg(b \lhd a)$, that is if a and b are either *distinct incompatible* (w.r.t. $\lhd$) or *identical* elements of X.

We also write

$$a \approx_\lhd b \iff \{x \mid x \sim_\lhd a\} = \{x \mid x \sim_\lhd b\}.$$

The relation $\approx_\lhd$ is an equivalence relation (i.e. it is reflexive, symmetric and transitive) and it is called *the equivalence with respect to* $\lhd$, since if $a \approx_\lhd b$, there is nothing in $\lhd$ that can distinquish between a and b (see (Fishburn, 1985) for details). We always have $a \approx_\lhd b \Rightarrow a \sim_\lhd b$, and one can show that (Fishburn, 1985):

$$a \approx_\lhd b \iff \begin{array}{l} \{x \mid a \lhd x\} = \{x \mid b \lhd x\} \\ \wedge\ \{x \mid x \lhd a\} = \{x \mid x \lhd b\} \end{array}$$

A partial order is (Fishburn, 1985)

- *total* or *linear*, if $\sim_\lhd$ is empty, i.e., for all $a,b \in X$. $a \lhd b \vee b \lhd a$,
- *weak* or *stratified*, if $a \sim_\lhd b \sim_\lhd c \Rightarrow a \sim_\lhd c$, i.e. if $\sim_\lhd$ is an equivalence relation,
- *interval*, if for all $a,b,c,d \in X$, $a \lhd c \wedge b \lhd d \Rightarrow a \lhd d \vee b \lhd c$,
- *semiorder*, if it is interval and for all $a,b,c,d \in X$, $a \lhd b \wedge b \lhd c \Rightarrow a \lhd d \vee d \lhd c$.

Evidently, every total order is weak, every weak order is a semiorder, and every semiorder is interval. We mention semiorders and interval orders to make the classification of our rankings (defined in Section 4) complete, as in this paper we will not use much of their rich theory. An interested reader is referred to (Fishburn, 1985; French, 1986).

Weak orders are often defined in an alternative way, namely (Fishburn, 1985),

- a partial order $(X, \lhd)$ is a weak order iff there exists a total order $(Y, \prec)$ and a mapping $\phi : X \to Y$ such that $\forall x,y \in X.\ x \lhd y \iff \phi(x) \prec \phi(y)$.

This definition is illustrated in Figure 1, let $\phi : \{a,b,c,d\} \to \{\{a\},\{b,c\},\{d\}\}$ and $\phi(a) = \{a\}$, $\phi(b) = \phi(c) = \{b,c\}$, $\phi(d) = \{d\}$. Note that for all $x,y \in \{a,b,c,d\}$ we have

$$x <_2 y \iff \phi(x) \prec_2 \phi(y).$$

Weak orders can easily be represented by *step-sequences*. For instance the weak order $<_2$ from Figure 1 is uniquely represented by the step-sequence $\{a\}\{b,c\}\{d\}$ (c.f. (Janicki and Koutny, 1993)).

Following (Fishburn, 1985), in this paper $a \lhd b$ is interpreted as "*a is less preferred than b*", and $a \approx_\lhd b$ is interpreted as "a and b are *indifferent*".

The preferable outcome of any ranking is a total order. For any total order $\lhd$, $\sim_\lhd$ is the empty relation and $\approx_\lhd$ is just the equality relation. A total order has two natural models, both deeply embedded in the human perception of reality, namely: *time* and *numbers*. Most people consider the causality relation and its time related version "earlier than" as total orders, even though their formal models are actually only interval orders (Janicki and Koutny, 1993).

Unfortunately in many cases it is not reasonable to insist that everything can or should be totally ordered. We may not have sufficient knowledge or such a perfect ranking may not even exist (Arrow, 1951). Quite often insisting on a totally ordered ranking results in an artificial and misleading "global index".

Weak (stratified) orders are a very natural generalization of total orders. They allow the modelling of some regular indifference, their interpretation is very simple and intuitive, and they are reluctantly accepted by decision makers. Although not as much as one might expect given the huge theory of such orders (see (Fishburn, 1985; French, 1986)). A non-numerical ranking technique proposed in (Janicki and Koczkodaj, 1996) produces a ranking that is weakly ordered.

If $\lhd$ is a weak order then $a \approx_{\lhd} b \iff a \sim_{\lhd} b$, so indifference means distinct incomparability or identity, and the relation $\lhd$ can be interpreted as a sequence of equivalence classes of $\sim_{\lhd}$. For the weak order $<_2$ from Figure 1, the equivalence classes of $\sim_{<_2}$ are $\{a\}$, $\{b,c\}$, and $\{d\}$, and $<_2$ can be interpreted as a sequence $\{a\}\{b,c\}\{d\}$.

There are, however, cases where insisting on weak orders may not be reasonable. Physiophysical measurements of perceptions of length, pitch, loudness, and so forth, provides other examples of qualitative comparisons that might be analysed from the perspective of semiorders and interval orders rather than the more precise but less realistic weak and total orders. The reader is referred to (Fishburn, 1985; Janicki and Koutny, 1993) for more details. In this paper we will only use total, weak, and general partial orders.

3 PARTIAL AND WEAK ORDER APPROXIMATIONS

Let X be a set of objects to be ranked. The problem is that X is believed to be partially or weakly ordered but the data acquisition process is so influenced by informational noise, imprecision, randomness, or expert ignorance that the collected data R is only some relation on X. We may say that R gives a fuzzy picture, and to focus it, we must do some pruning and/or extending. Without loss of generality we may assume that R is irreflexive, i.e. $(x,x) \notin R$. Suppose that R is not transitive. The "best" transitive approximation of R is its *transitive closure* $R^+ = \bigcup_{i=1}^{\infty} R^i$, where $R^{i+1} = R^i \circ R$ (c.f. (Fishburn, 1985)). Evidently $R \subseteq R^+$ and R^+ is transitive. The relation R^+ may not be irreflexive, but in such a case we can use the following classical result (which is due to E. Schröder, 1895, see (Janicki and Koczkodaj, 1996)) .

Lemma 1 *Let $Q \subseteq X \times X$ be a transitive relation. Define: $x <_Q y \iff xQy \wedge \neg yQx$. The relation $<_Q$ is a partial order.* □

Following (Janicki and Koczkodaj, 1996) we will call $<_{R^+}$, a *partial order approximation* of (*ranking relation*) R. If R is a partial order then $<_{R^+}$ equals R. The relation $<_{R^+}$ is usually not a weak order.

Let us assume that X is believed to be weakly ordered by a relation $\lhd$ but the discriminatory power of the data acquisition process, which seeks to uncover this order, is limited. The acquired data establishes only a partial order $\lhd$ which is a partial picture of the underlying order. We seek, however, an extension process which is expected to correctly identify the ordered pairs that are not part of the data.

Note that weak order extensions reflect the fact that if $x \approx_{\lhd} y$ than *all reasonable methods* for extending $\lhd$ will have x equivalent to y in the extension since there is nothing in the data that distinguishes between them (for details see (Fishburn, 1985)), which leads to the definition (Janicki and Koczkodaj, 1996) below (for both weak an total orders).

A weak (or total) order $\lhd_w \subseteq X \times X$ is a *proper weak (or total) order extension of* $\lhd$ if and only if :

$$(x \lhd y \Rightarrow x \lhd_w y) \text{ and } (x \approx_{\lhd} y \Rightarrow x \sim_{\lhd_w} y).$$

If X is finite then for every partial order $\lhd$ its proper weak extension always exists. If $\lhd$ is weak, than its only proper weak extension is $\lhd_w = \lhd$. If $\lhd$ if not weak, there are usually more than one such extensions. Various methods were proposed and discussed in (Fishburn, 1985). For our purposes, the best seem to be the method based on the concept of a *global score function*, which is defined as:

$$g_{\lhd}(x) = |\{z \mid z \lhd x\}| - |\{z \mid x \lhd z\}|.$$

Given the global score function $g_{\lhd}(x)$, we define the relation $\lhd_w^g \subseteq X \times X$ as

$$a \lhd_w^g b \iff g_{\lhd}(a) < g_{\lhd}(b).$$

Proposition 1 ((Fishburn, 1985)) *The relation $\lhd_w^g$ is a proper weak extension of a partial order $\lhd$.* □

Some other variations of $g_{\lhd}$ and their interpretations were analyzed in (Janicki and Koczkodaj, 1996). From Proposition 2 it follows that every finite partial order has a proper weak extension. The well known procedure "topological sorting", popular in scheduling problems, guarantees that every partial order has a total extension, but even finite partial orders may not have proper total extensions. Note that the total order $\lhd_t$ is a proper total extension of $\lhd$ if and only if the relation $\approx_{\lhd}$ equals the identity, i.e $a \approx_{\lhd} b \iff a = b$. For example no weak order has a proper total extension unless it is also already total. This indicates that while *expecting a final ordering to be weak may be reasonable, expecting a final total ordering is often unreasonable*. It may however happen, and often does, that a proper weak extension is a total order, which suggests that *we should stop seeking a priori total orderings since weak orders appear to be more natural models of preferences than total orders*.

4 THE MODEL

Let X be a finite set of objects to be "ranked", and let $\approx$, $\sqsubset$, $\subset$, $<$, and $\prec$ be a family of disjoint relations on X. The interpretation of these relations is the following (compare (Satty, 1986)), $a \approx b$: a and b are of *equal importance*, $a \sqsubset b$: *slightly in favour* of b, $a \subset b$: *in favour* of b, $a < b$: b is *strongly better*, $a \prec b$: b is *extremely better*. The lists $\sqsubset$, $\subset$, $<$, $\prec$ may be shorter or longer, but not empty and not much longer (due to limitations of the human mind(Satty, 1986)).

We define the relations $\widehat{\sqsubset}$, $\widehat{\subset}$, $\widehat{<}$, and $\widehat{\prec}$ as follows:

$$\widehat{\prec} = \prec \qquad \widehat{<} = \prec \cup <$$
$$\widehat{\subset} = \prec \cup < \cup \subset \qquad \widehat{\sqsubset} = \prec \cup < \cup \subset \cup \sqsubset$$

The relations $\widehat{\sqsubset}$, $\widehat{\subset}$, $\widehat{<}$, and $\widehat{\prec}$ are interpreted as *combined preferences*, i.e. $a\widehat{\sqsubset}b$: *at least slightly in favour* of b, $a\widehat{\subset}b$: *at least in favour* of b, $a\widehat{<}b$: *at least strongly in favour* of b, and $a\widehat{\prec}b$: *at least b is far superior than a*.

A relational structure $Rank = (X,\approx,\sqsubset,\subset,<,\prec)$ is called a *ranking* if the following axioms (*consistency* rules) are satisfied:

1. $\widehat{\sqsubset},\widehat{\subset},\widehat{<},\widehat{\prec}$ are partial orders,
2. $\approx = \sim_{\widehat{\sqsubset}}$, i.e. $\approx \cup \widehat{\sqsubset} \cup \widehat{\sqsubset}^{-1} = X \times X$,
3. $(a \approx b \wedge b \approx c) \;\Rightarrow\; (a \approx c \vee a \sqsubset c \vee c \sqsubset a)$,

4.1. $(a \approx b \wedge b \sqsubset c) \vee (a \sqsubset b \wedge b \approx c) \Rightarrow (a \sqsubset c \vee a \subset c)$,
4.2. $(a \approx b \wedge b \subset c) \vee (a \subset b \wedge b \approx c) \Rightarrow (a \subset c \vee a < c)$,
4.3. $(a \approx b \wedge b < c) \vee (a < b \wedge b \approx c) \Rightarrow (a < c \vee a \prec c)$,
5. $(a \approx b \wedge b \prec c) \vee (a \prec b \wedge b \approx c) \;\Rightarrow\; a \prec c$,
6.1. $(a \sqsubset b \wedge b \sqsubset c) \;\Rightarrow\; (a \sqsubset c \vee a \subset c)$,
6.2. $(a \subset b \wedge b \sqsubset c) \vee (a \sqsubset b \wedge b \subset c) \Rightarrow (a \subset c \vee a < c)$,
6.3. $(a < b \wedge b \sqsubset c) \vee (a \sqsubset b \wedge b < c) \Rightarrow (a < c \vee a \prec c)$,
7.1. $(a \sqsubset b \wedge b \prec c) \vee (a \prec b \wedge b \sqsubset c) \;\Rightarrow\; a \prec c$,
7.2. $(a \subset b \wedge b \prec c) \vee (a \prec b \wedge b \subset c) \;\Rightarrow\; a \prec c$,
7.3. $(a < b \wedge b \prec c) \vee (a \prec b \wedge b < c) \;\Rightarrow\; a \prec c$,
7.4. $(a \prec b \wedge b \prec c) \vee (a \prec b \wedge b \prec c) \;\Rightarrow\; a \prec c$.

The axioms $1-7.4$ follow from the interpretation of $\approx$ as equal importance and $\sqsubset$, $\subset$, $<$, $\prec$ as increasing preferences.

We will say that a ranking $(X,\approx,\sqsubset,\subset,<,\prec)$ is *totally* (*weakly, semi-, intervally*) *ordered* if the relation $\widehat{\sqsubset}$ is a total (weak, semi-, interval) order.

Due to the nature of stronger preferences it is unreasonable to expect any specific ordering of $\widehat{<}$, or $\widehat{\prec}$, however if such a specific (for example semiorder) ordering occurs, it may give some important information about the nature of hierarchy that is modelled by a given ranking.

Let $\widehat{\sqsubset}_w$ be a *proper weak extension* of $\widehat{\sqsubset}$ and let
$\sqsubset_w = \widehat{\sqsubset}_w \setminus \widehat{\subset}$,
$Rank_w = (X,\approx,\sqsubset_w,\subset,<,\prec)$.

Proposition 2 *The relational structure* $Rank_w = (X,\approx,\sqsubset_w,\subset,<,\prec)$ *is a weakly ordered ranking.* $\square$

We will call $Rank_w = (X,\approx,\sqsubset_w,\subset,<,\prec)$ a *weak order extension* of $Rank = (X,\approx,\sqsubset,\subset,<,\prec)$.

When transforming $Rank$ into $Rank_w$, we change only the weakest preference, so if $Rank = (X,\approx,\subset,<,\prec)$ then $Rank_w = (X,\approx,\subset_w,<,\prec)$, and if $Rank = (X,\approx,<,\prec)$ then $Rank_w = (X,\approx,<_w,\prec)$. We proceed similarly when the list of preferences is longer.

As mentioned above (see (French, 1986; Satty, 1986)) defining a proper ranking is problematic if $|X| > 2$, but it usually can be done if $|X| = 2$. Note that if $|X| = 2$ only one of the relations $\approx$, $\sqsubset$, $\subset$, $<$, $\prec$ is not empty.

A relational structure $(X,\approx,\sqsubset,\subset,<,\prec)$ is called a *pairwise comparisons pre-ranking*, if the following properties are satisfied:

1. the relations $\approx$, $\sqsubset$, $\subset$, $<$, $\prec$ are defined by *pairwise comparisons*,
2. $\approx$, $\sqsubset$, $\subset$, $<$, $\prec$ are disjoint and their union equals $X \times X$,
3. $\approx$ is interpreted as equal importance and $a \approx a$ for each $a \in X$,
4. $\sqsubset$, $\subset$, $<$, $\prec$ are interpreted as increasing preferences.

The pre-ranking is not usually a ranking. Our goal is to find such a, preferably weakly ordered, ranking that is "the best" approximation of a given pre-ranking. In the classical, "numerical" approach this is handled by the concept of *consistency* (Koczkodaj, 1993; Satty, 1986). The case $(X,\approx,<)$ was solved in (Janicki and Koczkodaj, 1996). The technique used in (Janicki and Koczkodaj, 1996) does not require any assumptions about the pre-ranking relations $\approx$ and $<$ (except $a \approx b \vee a < b \vee a <^{-1} b$, for all a,b). The technique can be extended to the general case, however, at the present stage, the algorithms are complex and lacking the elegance of the simpler case of $(X,\approx,<)$. Instead, we propose a more interactive approach.

First notice that it is very unlikely that a given pre-ranking is "random", i.e. it does not even resemble a ranking. When the process of classification of pairs is well designed, the outcome is a pre-ranking that is "almost" a ranking. Only some pairs violate the ranking axioms, the majority of pairs satisfy the ranking axioms. Checking those axioms is in principle a triad analysis, and the major violators are usually easy to detect.

After finding the pairs that violate more axioms than the other pairs (violations of axioms propagate, but "innocent" pairs violate much less frequently), we can either:

- Repeat the pairwise comparison process for those pairs and change the judgment. Or
- Introduce a "moderator" process which will change the relationship between those pairs, to satisfy the axioms.

In both cases, the changes are usually minor, like from $\subset$ to $\sqsubset$, etc.

The resulting ranking *Rank* is usually not weak, in most cases it is only semi- or intervally ordered, so, if we expect the outcome to be weak or total ranking, we need to compute $Rank_w$.

Computing $Rank_w$ is an extension process which is expected to identify correctly the ordered pairs that are not part of the data. The order identification power of weak extension procedures is substantial and *vastly underestimated*. If the ranked set of objects is, by its nature, expected to be totally ordered, the weak extension can detect it, even if the pairwise comparison process is not very precise, and often results in "indifference" (see example from Table 3 in the next section). It is a *serious error* to attempt to find a *total extension* without going through a weak extension process (see comments at the end of the previous chapter). In general, *admitting incomparability on the level of pairwise comparisons is better than insisting on an order at any cost*. The latter approach leads to an arbitrary and often incorrect total ordering.

5 AN EXAMPLE

The following experiment has been conducted. A blindfolded person compared the weights of the eight different stones, named A, B, C, D, E, F, G, H. The person put one stone in their left hand and another in their right, and then decided which of the relations $\approx$, $\sqsubset$, $\subset$, $<$, or $\prec$ held. The experiment was repeated for the same set of stones by various people; and then again for different stones and different number of stones. The results were very similar, and Table 1 presents the results of one such an experiment.

The pre-ranking $(X,\approx,\sqsubset,\subset,<,\prec)$, where $X = \{A,B,C,D,E,F,G,H\}$, described by Table 1 is *not* a ranking. However, a simple verification of axioms

Table 1: The first pre-ranking. It is not a ranking, the gray cells indicate problematic relations.

	A	B	C	D	E	F	G	H
A	$\approx$	$\subset$	$\approx$	$\sqsubset$	$\supset$	$\approx$	$<$	$\sqsupset$
B	$\supset$	$\approx$	$>$	$\approx$	$\succ$	$\sqsupset$	$\approx$	$<$
C	$\approx$	$<$	$\approx$	$\subset$	$\sqsupset$	$\sqsubset$	$<$	$\sqsubset$
D	$\sqsupset$	$\approx$	$\supset$	$\approx$	$\succ$	$\approx$	$\sqsubset$	$>$
E	$\subset$	$\prec$	$\sqsubset$	$\prec$	$\approx$	$<$	$\prec$	$\approx$
F	$\approx$	$\sqsubset$	$\sqsupset$	$\approx$	$>$	$\approx$	$\subset$	$\supset$
G	$>$	$\approx$	$>$	$\sqsupset$	$\succ$	$\supset$	$\approx$	$\succ$
H	$>$	$\prec$	$\sqsupset$	$<$	$\approx$	$\subset$	$\prec$	$\approx$

Table 2: The second (revised) pre-ranking which is a ranking. The grey cells were revised.

	A	B	C	D	E	F	G	H
A	$\approx$	$\subset$	$\approx$	$\sqsubset$	$\supset$	$\approx$	$<$	$\sqsupset$
B	$\supset$	$\approx$	$>$	$\approx$	$\succ$	$\sqsupset$	$\approx$	$<$
C	$\approx$	$<$	$\approx$	$\subset$	$\sqsupset$	$\sqsubset$	$<$	$\approx$
D	$\sqsupset$	$\approx$	$\supset$	$\approx$	$>$	$\approx$	$\sqsubset$	$>$
E	$\subset$	$\prec$	$\sqsubset$	$<$	$\approx$	$<$	$\prec$	$\approx$
F	$\approx$	$\sqsubset$	$\sqsupset$	$\approx$	$>$	$\approx$	$\subset$	$\supset$
G	$>$	$\approx$	$>$	$\sqsupset$	$\succ$	$\supset$	$\approx$	$\succ$
H	$>$	$\prec$	$\approx$	$<$	$\approx$	$\subset$	$\prec$	$\approx$

shows that the only violations are caused by relations between C and H, and D and E (grey cells in Table 1). The person was asked to repeat the comparison of pairs (C,H), and (D,E). This time he produced a slightly different answers. The new pre-ranking presented by Table 2, is now a ranking. However, it is *not* a weakly ordered ranking. The partial orders $\widehat{\sqsubset}$, $\widehat{\subset}$, $\widehat{<}$, and $\widehat{\prec}$ are presented in Figure 2. The relations $\widehat{\sqsubset}$, $\widehat{\subset}$, $\widehat{<}$ are semiorders while $\widehat{\prec}$ is an interval order, i.e. the ranking is *semiordered*. This was expected due to the nature of this type of experiments (c.f. (Fishburn, 1985)).

The stones were weighed and their weights created an increasing total order E,H,C,A,F,D,B,G. Note that this is the *same* order as *weak* extensions $\widehat{\sqsubset}_w$, $\widehat{\subset}_w$ and $\widehat{<}_w$. The fact that $\widehat{<}_w$ correctly describes the real ordering is very interesting since it means that the very rough ranking described in Table 3, where only high preferences were recorded, all weak ones were discarded and coded as "equal importance", was sufficient to produce a correct total ordering. It also shows the order identification power of the weak order extension procedure. Note also that if in the pre-ranking from Table 1, which is not a ranking, we will use only high preferences and replace weak by indifference (i.e. we replace $\sqsubset$ and $\subset$ with $\approx$, and leave only $<$ and $\prec$), the new pre-ranking will be a ranking. It will differ slightly from the one of Table 3 ($E \prec D$ in

Table 3: The third pre-ranking, where only $\approx$, $<$ and $\prec$ were recorded. It is a ranking. It can be obtained from Table 2 by replacing $\sqsubset$ and $\subset$ with $\approx$.

	A	B	C	D	E	F	G	H
A	$\approx$	$\approx$	$\approx$	$\approx$	$\approx$	$\approx$	$<$	$\approx$
B	$\approx$	$\approx$	$>$	$\approx$	$\succ$	$\approx$	$\approx$	$<$
C	$\approx$	$<$	$\approx$	$\approx$	$\approx$	$\approx$	$<$	$\approx$
D	$\approx$	$\approx$	$\approx$	$\approx$	$>$	$\approx$	$\approx$	$>$
E	$\approx$	$\prec$	$\approx$	$<$	$\approx$	$<$	$\prec$	$\approx$
F	$\approx$	$\approx$	$\approx$	$\approx$	$>$	$\approx$	$\approx$	$\approx$
G	$>$	$\approx$	$>$	$\approx$	$\succ$	$\approx$	$\approx$	$\succ$
H	$>$	$\prec$	$\approx$	$<$	$\approx$	$\approx$	$\prec$	$\approx$

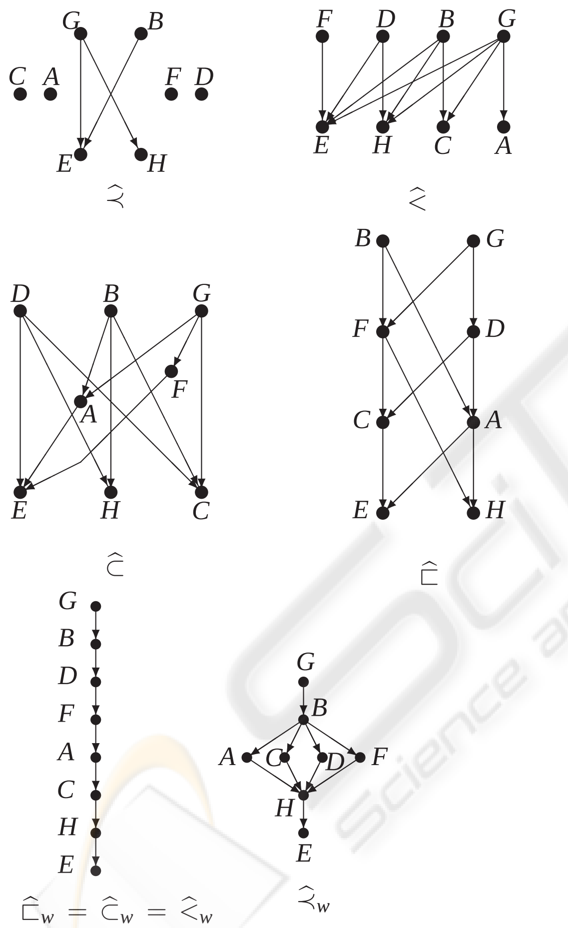

Figure 2: Partial orders $\widehat{\sqsubset}$, $\widehat{\subset}$, $\widehat{<}$, and $\widehat{\prec}$ defined by the ranking from Table 2, and their proper weak order extensions $\widehat{\sqsubset}_w$, $\widehat{\subset}_w$, $\widehat{<}_w$, and $\widehat{\prec}_w$ (created using global score function). The orders $\widehat{\sqsubset}$, $\widehat{\subset}$, $\widehat{<}$ are semiorders, $\widehat{\prec}$ is an interval order, $\widehat{\sqsubset}_w$, $\widehat{\subset}_w$, $\widehat{<}_w$ are the same *total.* order, and $\widehat{\prec}_w$ is a weak order.

Table 1 and $E < D$ in Table 3), but the order $\widehat{<}$ will be the same in both cases! This suggests that if an original pairwise comparison pre-ranking is not a ranking, a "moderator" might replace lower preferences by indifference, and check if the outcome is a ranking. If it is, a weak extension may be constructed and on its bases the initial pre-ranking might be modified. Alternately, a problematic decision could be re-judged.

6 FINAL COMMENTS

Apparently, all of the popular techniques based on pairwise comparison principles suffer from many serious problems and may lead to strange and contradictive results. These problems follow mainly from their treatment of imprecision, knowledge incompleteness and indifference (incomparability) (Dyer, 1990; Janicki and Koczkodaj, 1996).

The method proposed in this paper does not use numbers at all, it is entirely based on the concept of *partial orders*. It emphasizes and advocates using *incomparability* and *weak orderings*, as opposed to insistence on the comparability of all objects and a final total ordering. The order identification power of the weak order extension procedure is discussed.

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