ON ANALYZING SYMMETRY OF OBJECTS USING ELASTIC DEFORMATIONS

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Abstract: We introduce a framework for analyzing symmetry of 2D and 3D objects using elastic deformations of their boundaries. The basic idea is to define spaces of elastic shapes and to compute shortest (geodesic) paths between the objects and their reflections using a Riemannian structure. Elastic matching, based on optimal (nonlinear) re-parameterizations of curves, provides a better registration of points across shapes, as compared to the previously-used linear registrations. A crucial step of orientation alignment, akin to finding planes of symmetry, is performed as a search for shortest geodesic paths. This framework is fully automatic and provides: a measure of asymmetry, the nearest symmetric shape, the optimal deformation to make an object symmetric, and the plane of symmetry for a given object.

1 INTRODUCTION

Symmetry is an important feature of an object and in symmetry analysis one asks questions of the type: Is a given object symmetric? What is the level (quantification) of asymmetry in an object? What is the nearest symmetric object for a given asymmetric object and how far is it? How to minimally deform an object to make it symmetric? What are the planes(s) of symmetry of a given symmetric object? And so on. Such an analysis plays an important role in several applications, including object design, manufacturing, packaging, segmentation (Simari et al., 2006), view selection (Thrun and Wegbreit, 2005), model reduction (Mitra et al., 2006), medical diagnosis, and surgical planning. Reflection symmetry occurs in many biological objects, and is an important part of how we as humans perceive them. Symmetry analysis can also play an important role in medical diagnostics, for example, when the level of asymmetry in an organ relates to its health. The procedures for quantification of asymmetry, estimation of symmetry hyperplane, and symmetrization of objects are also gaining attention in 3D graphics, object recognition, indexing and retrieval.

In this paper we will consider both 2D and 3D objects, and we will restrict to the shapes of their boundaries for symmetry analysis. For the 2D case, we will study a variety of objects from public databases, but for the 3D case we will focus on shapes of facial surfaces. Symmetry of faces has large relevance in multiple contexts. For example, in orthodontics, see for example (Tomaka, 2005), that symmetry analysis can help plan surgical interventions in the craniofacial area and to monitor their long-term effects. The detection of facial landmarks can benefit from a symmetry analysis. An important component of cosmetic facial surgeries is to enhance facial symmetries, long considered a factor in improving appearances. A surgeon can be guided by the optimal deformation of a face that will make it symmetric.

By definition, a solid or a surface is reflection-symmetric if its reflection, with respect to a certain plane, is identical to it. Consequently, classifying an object as symmetric or not, or measuring the level of its asymmetry, can be reduced to the task of computing differences in shapes between two objects: the original one and its mirror reflection. Several recent efforts in shape analysis have focused on symmetry detection of 2D and 3D shapes (Mitra et al., 2006), (Martinet et al., 2006), (Kazhdan et al., 2004). The general framework used is as follows: Let β be an ob-

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ject (curve, surface, etc) in an appropriate $\mathbb{R}^n$ and R be a plane of reflection in $\mathbb{R}^n$. R will be denoted by the vector $v \in \mathbb{R}^n$ perpendicular to R. Assuming that β is centered in that coordinate system, define a measure of asymmetry as:

$$\rho(\beta) = \underset{v \in \mathbb{R}^n}{\operatorname{argmin}} \|\beta - H(v)\beta\|^2, \; H(v) = (I - 2\frac{vv^T}{v^T v}), \quad (1)$$

where $\|\cdot\|$ is the two norm integrated over the points in the object. $H(v)$ is the Householder reflection operator which rotates any vector into its reflection in a plane orthogonal to v. In case $\rho(\beta)$ is zero, the object is said to be symmetric and the corresponding v provides the the plane of symmetry. Zabrodsky et al. (Zabrodsky et al., 1995) suggested a slightly different formulation where they find the nearest symmetric object to the given object. That is, define

$$SD(\beta) = \underset{v \in \mathbb{R}^n, s.t. \;\; H(v)\alpha = \alpha}{\operatorname{argmin}} \|\beta - \alpha\|^2 . \quad (2)$$

This idea has been called the symmetry distance (Zabrodsky et al., 1995). Mitra et al. (Mitra et al., 2007) formulate the search for symmetrization deformation in a similar way, but based on points sampled from the original model. Sun et al. (Sun and Sherrah, 1997) proposed a method to detect symmetry based on the Extended Gaussian Image (EGI).

Since symmetry analysis is intimately tied to quantification of differences in shapes of objects and their reflections, one should look more carefully at how shape quantification is being performed. It is a common trend in papers on symmetry to use Euclidean norms between points sets to form cost functions. Additionally, the authors have invariably used a linear registration of points, between the original object and its reflection, to evaluate these norms. In contrast, the literature in shape analysis of curves suggests a larger variety of metrics and nonlinear registrations in measuring shapes (Michor and Mumford, 2006). In particular, the use of elastic deformations to compare and analyze shapes is gaining popularity. Here, the curves are allowed to optimally stretch/shrink and bend to match one another during comparisons. Mathematically, this is accomplished by applying all possible re-parameterizations, including nonlinear registrations, on curves to find the optimal registration. In this paper, we utilize the framework of Joshi et al. (Joshi et al., 2007a), on elastic shape analysis of curves, for performing symmetry analysis of 2D shapes. To extend this idea to symmetry analysis of surfaces, we use the approach of Samir et al. (Samir et al., 2006) where a facial surface is represented as a collection of level curves, and faces can be elastically compared by comparing the corresponding curves.

The rest of this paper is organized as follows. We present the general framework in Section 2, particularize it for 2D shapes in Section 3 and for surfaces in Section 4.

2 GENERAL FRAMEWORK

We advocate the use of geometric approaches in symmetry analysis. In particular, we suggest the use of elastic shape analysis of curves and surfaces to help quantify differences between objects and their reflections. A geometric approach for shape analysis involves: (i) defining a space of shapes using their mathematical representations, (ii) imposing a Riemannian structure on it, and (iii) numerically computing geodesic paths between arbitrary shapes. Care is taken to remove symmetry-preserving transformations from the representation using algebraic equivalences.

More precisely, one starts with a space, say $\mathcal{C}$, of mathematical representations of objects, e.g. closed curves, and studies its differential geometry to identify tangent spaces $T\mathcal{C}$. Then, choosing a Riemannian metric – a positive-definite, bilinear, symmetric form on tangent spaces – one can define lengths of paths on $\mathcal{C}$. Given any two objects, i.e. two elements of $\mathcal{C}$, one can use a numerical approach to find a shortest geodesic path between them. Let d_c denote the length of this geodesic.

Symmetry of a curve or a surface is invariant to its translation, scaling, rotation, and re-parametrization. Scaling and translation are usually accounted for in defining $\mathcal{C}$, but the other two are handled explicitly as follows. One defines the action of the rotation group $SO(n)$ and the re-parametrization group Γ on $\mathcal{C}$, and defines the orbits of objects under these actions as equivalence classes. In other words, for a $q \in C$, if $[q]$ is the set of all variations of q obtained by rotating and re-parameterizing it, then $[q]$ is defined to be an equivalence class. The set of all such equivalence classes is the quotient space $\mathcal{S} = \mathcal{C}/(SO(n) \times \Gamma)$. The distance between any two elements of $\mathcal{S}$, say $[q_1]$ and $[q_2]$, is the length of the shortest geodesic in $\mathcal{C}$ between elements of those two sets:

$$\begin{aligned} d_s([q_1],[q_2]) &= \inf_{p_1 \in [q_1], p_2 \in [q_2]} d_c(p_1, p_2) \\ &= \inf_{p_2 \in [q_2]} d_c(p_1, p_2) \; . \quad (3) \end{aligned}$$

The last equality assumes that $SO(n)$ and Γ act on $\mathcal{C}$ as isometries. The distance d_s is invariant to rotation, translation, scaling, and re-parametrization of the objects. How can this distance be used to measure the symmetry of an object? The answer comes from the following result.

Theorem 1. 1) 2D Case. *Assuming a simple, closed curve β is bounded, and $\tilde{\beta}$ is an arbitrary reflection of β, then the distance d_s between β and $\tilde{\beta}$ is zero if and only if β is symmetric.*
2) 3D Case. *For a two-dimensional surface S in $\mathbb{R}^3$, if the distance d_s between S and any of its reflection $\tilde{S}$ is zero, then there exists a rigid motion taking S to $\tilde{S}$. In fact, this rigid motion is a composition of a reflection in some plane, and a rotation around an axis perpendicular to that plane.*

We note that this theorem actually holds for any bounded point set in $\mathbb{R}^2$ or $\mathbb{R}^3$, although in this paper we are concerned only with curves and surfaces.

This motivates the use of elastic distance d_s as a measure of asymmetry in an object. If an object is perfectly symmetric, the distance d_s, between itself and its arbitrary reflection, will be zero. If it is not symmetric then we can use d_s as a measure of asymmetry in that object. In the next two sections, we particularize this idea to analyze symmetries of 2D and 3D shapes.

3 SYMMETRY ANALYSIS OF 2D CURVES

We start this section by summarizing the geometric shape analysis of planar closed curves. Consider a closed curve as a mapping β from $\mathbb{S}^1$ to $\mathbb{R}^2$. To analyze its shape, we will represent this curve by a function $q : \mathbb{S}^1 \to \mathbb{R}^2$, where $q(t) = \frac{\dot{\beta}(t)}{\sqrt{||\dot{\beta}(t)||}}$. Here, $s \in \mathbb{S}^1$, where $\|\cdot\|$ is the usual two norm in $\mathbb{R}^2$. Given a q function, we can reconstruct the original curve up to a translation. We narrow our study to closed curves of length one by defining:

$$\mathcal{C} = \{q| \int \|q(t)\| dt = 1, \int q(t)\ ||q(t)|| dt = 0\} . \quad (4)$$

The first constraint forces the curves to be of length one and the second constraint ensures their closure. Shapes of curves are compared using geodesic paths on $\mathcal{C}$ which, in turn, requires a Riemannian structure. To impose a Riemannian metric on $\mathcal{C}$, define the inner product: for any $u, v \in T_q(\mathcal{C})$, $\langle u, v\rangle = \int_0^{2\pi} \langle u(t), v(t)\rangle\ dt$. For computing geodesic paths on $\mathcal{C}$, there are a variety of numerical approaches available. In this paper we use a path-straightening approach, where the given pair of shapes is connected by an initial arbitrary path in $\mathcal{C}$ and that path is iteratively "straightened" until it becomes a geodesic. For details of implementation, please refer to the paper by Joshi et al. (Joshi et al., 2007a). Let $d_c(q_1, q_2)$ be the length of geodesic connecting q_1 and q_2 in $\mathcal{C}$.

Since the symmetry of β is considered invariant to its rotation or re-parametrization, our measure of asymmetry should also be invariant to these transformations. (Note that translation and scaling of a curve have already been removed when curves are represented as elements of $\mathcal{C}$.) The rotation of a curve is represented by a 2×2 matrix, an element of $SO(2)$, while a re-parametrization is a diffeomorphism $\gamma : \mathbb{S}^1 \to \mathbb{S}^1$, an element of Γ, the space of all such diffeomorphisms. Define the actions of $SO(2)$ and Γ on $\mathcal{C}$ as follows:

$$\begin{aligned} SO(2) \quad &\times \quad \mathcal{C} \to \mathcal{C}, \quad (O, q) = \{Oq(t)|t \in [0,1]\} \\ \Gamma \quad &\times \quad \mathcal{C} \to \mathcal{C}, \quad (\gamma, q) = \{\sqrt{\dot{\gamma}} q(\gamma(t))|t \in [0,1]\} \end{aligned}$$

It can be shown that these two group actions commute and, hence, we can define the quotient space with respect to their direct product according to $\mathcal{S} = \mathcal{C}/(SO(2)\times\Gamma)$. The orbit of a shape, represented by q, is given by:

$$[q] = \{\sqrt{\dot{\gamma}} O q(\gamma(t)) | O \in SO(2), \gamma \in \Gamma\} ,$$

and this denotes the equivalence class of all rotations and re-parameterizations of q. The set of all such equivalence classes is $\mathcal{S}$.

To compute distances between any two elements of $\mathcal{S}$, one has to find the shortest path between the two corresponding orbits. This length of this shortest path is:

$$d_s = \inf_{O \in SO(2), \gamma \in \Gamma} d_c(q_1, \sqrt{\dot{\gamma}} O q_2(\gamma)) . \quad (5)$$

This minimization requires search over all rotations and re-parameterizations of q_2 so that it best matches q_1. Note that if q_2 is a reflection of q_1, then the optimization over $SO(2)$ is similar to the one in Eqn. 1, except that the Euclidean norm is replaced by the geodesic distance d_c. The other difference from that equation is the optimization over γ, which allows for nonlinear registration between the two shapes. How to solve this optimization problem? Joshi et al. (Joshi et al., 2007b) describe a technique that uses gradients to search over Γ but uses an exhaustive search over $SO(2)$ to minimize the cost function. We refer the reader to that paper for details.

To analyze the level of asymmetry of a closed curve β, we obtain a reflection $\tilde{\beta} = H(v)\beta$, where $H(v) = (I - 2\frac{vv^T}{v^T v})$, for any $v \in \mathbb{R}^2$. Denote the representation of β in $\mathcal{C}$ by q and that of $\tilde{\beta}$ by $\tilde{q}$. Let $\psi : [0,1] \mapsto \mathcal{S}$ be the geodesic path between q and $\tilde{q}$

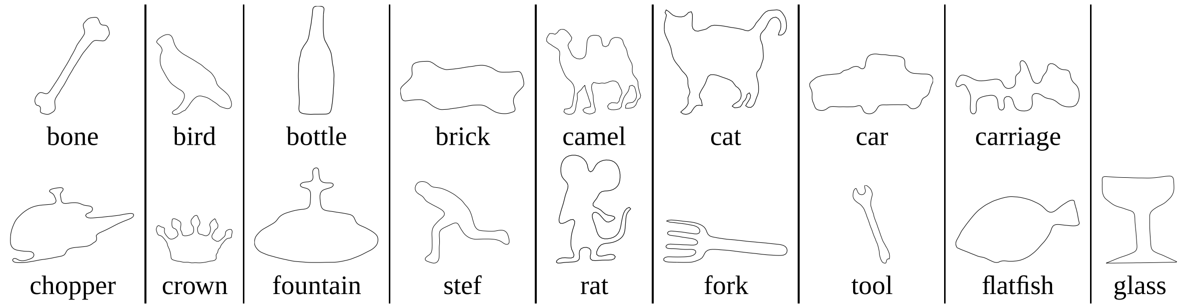

Figure 1: Seventeen 2D shapes used in experiments on symmetry.

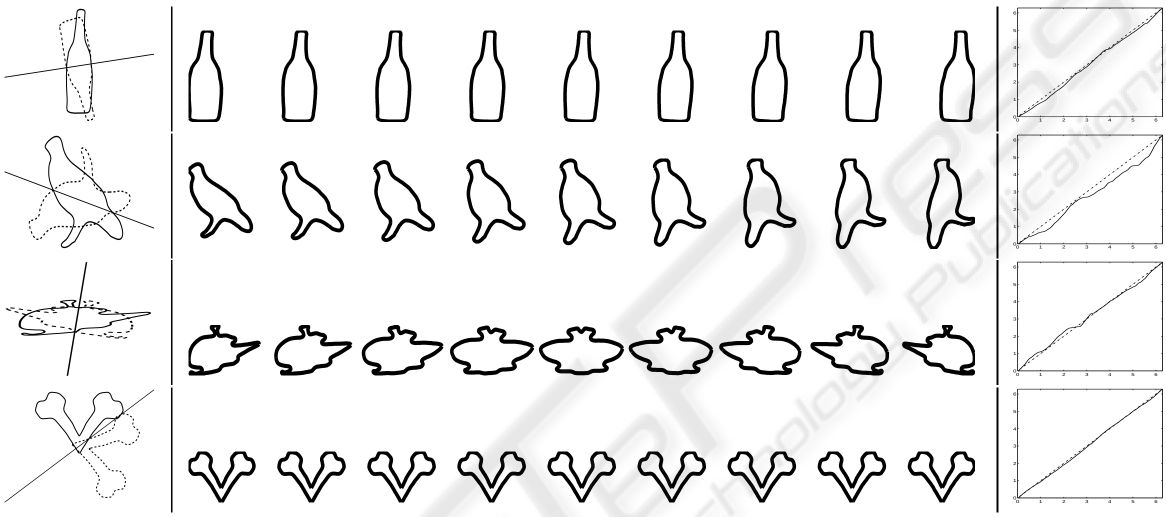

Figure 2: In each row, the left panel shows β and $\tilde{\beta}$, the middle panel shows geodesic ψ_t between them, and the last panel shows the optimal γ for their matching.

constructed using optimization techniques presented in (Joshi et al., 2007b); we have $\psi(0) = q$ and $\psi(1) = \tilde{q}$. ψ provides important information about the symmetry of β:

1. **Measure of Asymmetry.** Define the length of the path ψ as a measure of asymmetry of β:
$$\rho(\beta) = d_s(q, H(v)q)) \text{ , for any } v \in \mathbb{R}^n \text{ .} \quad (6)$$
2. **Nearest Symmetric Shape.** The halfway point along the geodesic, i.e. $\psi(0.5)$, is perfectly symmetric. Amongst all perfectly symmetric shapes, it is the nearest to q in $\mathcal{S}$.
3. **Deformation for Symmetrization.** The velocity vector $\dot{\psi}(0)$ provide a deformation (vector) field on β than transforms β into the nearest symmetric shape.

Next we present some experimental results on measuring asymmetry on some 2D shapes taken from one of the Kimia databases. We have used seventeen curves from different shape classes; these curves are shown in Figure 1. Based on their appearances, one can see that some curves show strong symmetries while others look far from symmetric.

Our approach is to take a curve β, select an arbitrary reflector in $\mathbb{R}^2$, and form a new curve $\tilde{\beta}$ by applying that reflector on β. Shown in the left column of Figure 2 are four examples of such random reflections. The last shape is an artificially constructed shape with perfect symmetry, just to test the algorithm. Then, we compute geodesic paths in $\mathcal{S}$ between the shapes of q and $\tilde{q}$, the representatives of β and $\tilde{\beta}$ in $\mathcal{C}$. Four examples of these paths are shown in the middle column of Figure 2. The lengths of these paths, $\rho(\beta)$, provide the level of asymmetry of these shapes. For the seventeen shapes shown in Figure 1, the values of $\rho(\beta)$ are shown in Figure 3. The bottle and the fork are the most symmetric objects while the fountain, the flatfish, the glass, and the tool are quite close. On the other extreme, the cat and the chopper are the most asymmetric shapes. One can say that the cat is almost ten times as asymmetric as the bottle. To put these numbers in perspective, this measure for the artificially constructed shape in the top right of Figure 2 is found to be 0.0814. This value can be

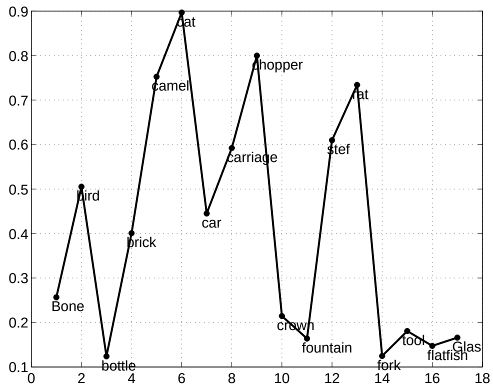

Figure 3: The values of $\rho(\beta)$ for the 17 shapes shown in Figure 1.

treated as a numerical error in measuring symmetry of perfectly symmetric objects. The last column shows the optimal γs that resulted from optimal alignment of curves with their reflections (Eqn. 5). For a perfectly symmetric shape γ is identity, while for largely asymmetric shapes optimal γs are quite nonlinear.

4 SYMMETRY ANALYSIS OF SURFACES

To analyze the symmetry of a surface is much more complicated due to the corresponding difficulty in analyzing shapes of surfaces. The space of parameterizations of a surface is much larger than that of a curve, and this hinders an analysis of symmetry in a way that is invariant to parametrization. (Recall from previous sections that invariance to parametrization requires solving an optimal re-parametrization of one object, in order to best match the other.) One solution is to restrict to a family of parameterizations and perform shape analysis over that space. Although this can not be done for all surfaces, it is natural for certain surfaces such as the facial surfaces as described next.

Using the approach of Samir et al., we can represent a facial surface S as an indexed collection of facial curves, as shown in Figure 4. Each facial curve, denoted by c_λ, is obtained as a level set of the (surface) distance function from the tip of the nose; it is a closed curve in $\mathbb{R}^3$. The treatment of symmetries of closed curves in $\mathbb{R}^3$ is similar to that of planar curves described in the previous section. As earlier, let d_s denote the geodesic distance between closed curves in $\mathbb{R}^3$, when computed on the shape space $\mathcal{S} = \mathcal{C}/(SO(3) \times \Gamma)$, where $\mathcal{C}$ is same as Eqn. 4 except this time it is for curves in $\mathbb{R}^3$. A surface S is

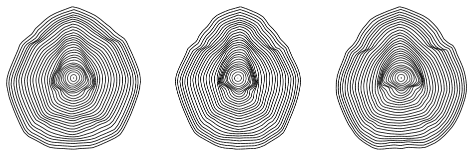

Figure 4: Representation of facial surfaces as indexed collection of closed curves in $\mathbb{R}^3$.

represented as a collection $\cup_\lambda c_\lambda$ and the elastic distance between any two facial surfaces is given by: $d_s(S_1, S_2) = \sum_\lambda d_s(\lambda)$, where

$$d_s(\lambda) = \inf_{O \in SO(3), \gamma \in \Gamma} d_c(q^1_\lambda, \sqrt{\dot{\gamma}} O q^2_\lambda(\gamma)) \ . \qquad (7)$$

Here q^1_λ and q^2_λ are q representations of the curves c^1_λ and c^2_λ, respectively. According to this equation, for each pair of curves in S_1 and S_2, c^1_λ and c^2_λ, we obtain an optimal rotation and re-parametrization of the second curve. To put together geodesic paths between full facial surfaces, we need a single rotational alignment between them, not individually for each curve as we have now. Thus we compute an average rotation:

$$\hat{O} = \text{average}\{O_\lambda\} \ ,$$

using a standard approach, and apply $\hat{O}$ to S_2 to align it with S_1. This global rotation, along with optimal re-parameterizations for each λ, provides an optimal alignment between individual facial curves and results in shortest geodesic paths between them. Combining these geodesic paths, for all λs, one obtains geodesic paths between the original facial surfaces.

We apply this idea for symmetry analysis of facial surfaces. We take a surface S and form its reflection in an arbitrary plane. Then, we extract facial curves out of each surface and use these curves, as described above, to form a geodesic path between the facial surfaces. Shown in Figure 5 is an example of a geodesic path between a face, distorted by a smile, and its reflection. For illustration, we show the geodesic using both the rendered surfaces (top) and the facial curves (bottom). In this case, the measure of asymmetry is $d_s(S, \tilde{S}) = 0.0210$.

In Figure 6, we present some additional examples of symmetrizing facial surfaces. From top to bottom, the measure of asymmetry in these faces is 0.0217, 0.0147, 0.0156, and 0.0195.

5 SUMMARY

We have presented a framework for analyzing symmetries in 2D and 3D objects. This framework is based on elastic deformations of objects in a fashion that is invariant to rigid transformations and global

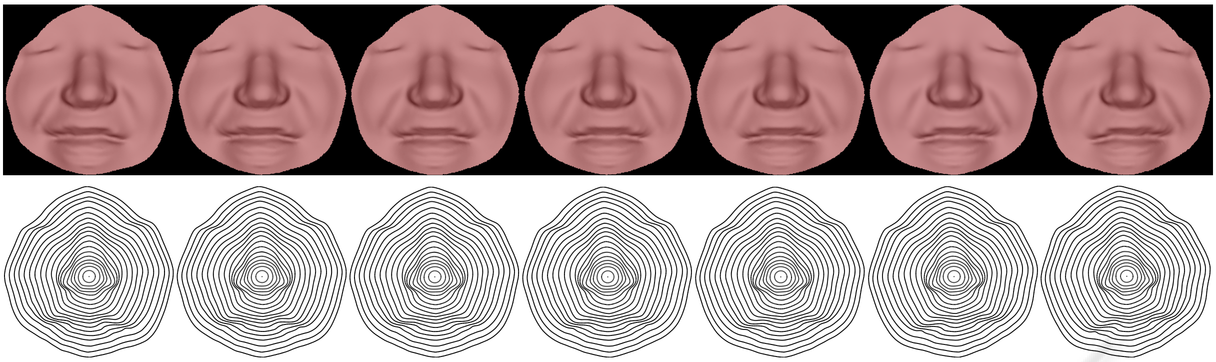

Figure 5: The geodesic path between a face and its reflection.

Figure 6: Examples of symmetry analysis of faces: geodesic between faces and their optimally aligned reflections.

scaling. The use of nonlinear registration techniques help improve the quantification of differences between shapes of objects and their reflections. This framework provides a measure of asymmetry, the nearest symmetric object, and a deformation field for symmetrizing an object.

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