OPTIMAL SPARSE CONTROLLER STRUCTURE WITH MINIMUM ROUNDOFF NOISE GAIN

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Keywords: Roundoff noise gain, Sparse controller structure, Optimization, Direct-form II transposed (DFIIt) structure.

Abstract: This paper investigates the roundoff noise effect in the digital controller on the closed-loop output for a discrete-time feedback control system. Based on a polynomial parametrization approach, a sparse controller structure is derived. The performance of the proposed structure is analyzed by deriving the corresponding expression of closed-loop roundoff noise gain and the problem of finding optimized sparse structures is solved. A numerical example is presented to illustrate the design procedure and the performance of the proposed structure compared with those of some existing well-known structures.

1 INTRODUCTION

Finite word length (FWL) effects have been a well studied field in the design of digital filers for more than three decades (Mullis and Roberts, 1976), (Hwang, 1977), (Roberts and Mullis, 1987), (Gevers and Li, 1993). However, they have received less attention in the area of digital control. Nowadays, many researchers have recognized the importance of the numerical problems caused by FWL effects in digital controller implementation. The optimal FWL controller structure design (Fialho and Georgiou, 1994), (Li, 1998), (Wu et al., 2001), (Yu and Ko, 2003) has been considered as one of the most effective methods to minimize the effects of FWL errors on the performance of closed-loop control systems. The basic idea behind this approach is that for a given digital controller, there exist different structures which have different numerical properties, and the optimal structure problem is to identify those structures that optimize a certain FWL performance criterion.

Generally speaking, there are two types of FWL errors in the digital controller. The first one is the perturbation of the controller parameters implemented with FWL, and the second one is the rounding errors that occur in arithmetic operations, which are usually measured with the so-called roundoff noise gain. The effects of roundoff noise have been well studied in digital signal processing, particularly in digital filter implementation (Wong and Ng, 2000), (Wong and Ng, 2001). However, it was not until the late 1980s that the problem of optimal controller realizations minimizing the roundoff noise gain was addressed. The roundoff noise gain was derived for a control system with a state-estimate feedback controller and the corresponding optimal realization problem was solved in (Li and Gevers, 1990), while the roundoff error effect on the linear quadratic regulation (LQG) performance was investigated in (Williamson and Kadiman, 1989) and the optimal solution was obtained by Liu *et al* (Liu et al., 1992). The problem of finding the optimum roundoff noise structures of digital controllers in a sampled-data system has been investigated in (Li et al., 2002).

It has been noted that the optimal controller realizations obtained with the above design methods are usually fully parametrized, which increase the complexity for real-time implementations. From a practical point of view, it is desired that the actually implemented controller have a nice performance against the FWL effects as well as a sparse structure that possesses many trivial parameters[1] which produce no FWL errors. As far as we know, a few results have been published on the sparseness issue for the controller structure design (Li, 1998), (Wu et al., 2003), however, it is noted that in these approaches, sophisticated numerical algorithms were utilized and the positions of trivial parameters were not predictable. In (Hao et al., 2006), we proposed two sparse structures

[1]By trivial parameters we mean those that are 0 and ± 1, other parameters are, therefore, referred to as nontrivial parameters.

Hao J., Wee T., Karatzas L. and Lee Y.
OPTIMAL SPARSE CONTROLLER STRUCTURE WITH MINIMUM ROUNDOFF NOISE GAIN.
DOI: 10.5220/0002170900130019
In *Proceedings of the 6th International Conference on Informatics in Control, Automation and Robotics (ICINCO 2009)*, page
ISBN: 978-989-674-001-6

for digital controllers, which have some degrees of freedom that can be used to enhance the closed-loop stability robustness against the FWL effects.

In this paper, a new sparse controller structure is derived by adopting the polynomial parametrization approach in (Hao et al., 2006) and using the l_2-scaling scheme. This structure can be considered as a l_2-scaled generalized DFIIt (direct-form II transposed) structure. The expression of the roundoff noise gain is derived for a closed-loop feedback control system, in which the digital controller is implemented with the proposed structure. The problem of finding optimized sparse structures is solved by minimizing the corresponding closed-loop roundoff noise gain. A numerical example is given to illustrate the design procedure, which shows that the proposed structure beats the traditional DFIIt structures greatly in terms of roundoff noise performance, and furthermore, outperforms the fully parametrized optimal realization (Li et al., 2002) in terms of both roundoff noise gain and computation efficiency.

2 A SPARSE CONTROLLER STRUCTURE

Consider a discrete-time feedback control system depicted in Fig. 1, where $P_d(z)$ is the discrete-time plant and $C_d(z)$ is a well-designed digital controller. The controller can be represented by its transfer function which is parametrized with $\{\xi_k, \zeta_k\}$ in the shift operator z:

$$C_d(z) = \frac{\sum_{k=0}^{K} \zeta_k z^{K-k}}{z^K + \sum_{k=1}^{K} \xi_k z^{K-k}}. \tag{1}$$

This controller can be implemented with many different structures, such as the direct forms or the following state-space equations:

$$\begin{cases} x(n+1) &= Ax(n) + Bu(n) \\ y(n) &= Cx(n) + du(n) \end{cases} \tag{2}$$

where $x(n) \in \mathcal{R}^{K\times 1}$ is the state variable vector and $u(n)$, $y(n)$ are the input and output of the controller $C_d(z)$, respectively, while $r(n)$ is the input signal of the closed-loop system. $R \triangleq (A,B,C,d)$ is called a realization of $C_d(z)$ with $A \in \mathcal{R}^{K\times K}, B \in \mathcal{R}^{K\times 1}, C \in \mathcal{R}^{1\times K}$ and $d \in \mathcal{R}$, satisfying

$$C_d(z) = d + C(zI - A)^{-1}B.$$

Denote S_C as the set of all the realizations: $S_C \triangleq \{(A,B,C,d) : C_d(z) = d + C(zI-A)^{-1}B\}$. Let $R_0 \triangleq (A_0,B_0,C_0,d) \in S_C$ be an initial realization. It can be shown that S_C is characterized by

$$A = T^{-1}A_0T, \;\; B = T^{-1}B_0, \;\; C = C_0T \tag{3}$$

where $T \in \mathcal{R}^{K\times K}$ is any nonsingular matrix. Such a matrix T is usually called a similarity transformation. Once an initial realization R_0 is given, different controller realizations correspond to different similarity transformations T.

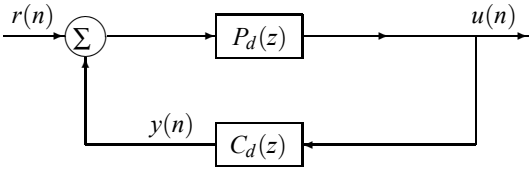

Figure 1: A discrete-time feedback control system.

2.1 A Generalized DFIIt Structure

Based on the approach in (Hao et al., 2006), we define

$$\rho_k(z) \triangleq \frac{z-\gamma_k}{\Delta_k}, \; k = 1,2,...,K, \tag{4}$$

where $\{\gamma_k\}$ and $\{\Delta_k > 0\}$ are two sets of constants to be discussed later. Let

$$\begin{aligned} p_k(z) &\triangleq \prod_{m=k+1}^{K} \rho_m(z), \;\; \forall k \in \{0,1,\cdots,K-1\}, \\ p_K(z) &\triangleq 1. \end{aligned} \tag{5}$$

It can be shown that (1) can be rewritten as

$$C_d(z) = \frac{\beta_0 p_0(z) + \beta_1 p_1(z) + \ldots + \beta_K p_K(z)}{p_0(z) + \alpha_1 p_1(z) + \ldots + \alpha_K p_K(z)}, \tag{6}$$

where

$$\begin{aligned} \bar{\alpha} &\triangleq [1 \quad \alpha_1 \quad \cdots \quad \alpha_K]^{\mathcal{T}} \\ &= \kappa \bar{T}_p^{-\mathcal{T}} [1 \quad \xi_1 \quad \cdots \quad \xi_K]^{\mathcal{T}} \\ \bar{\beta} &\triangleq [\beta_0 \quad \beta_1 \quad \cdots \quad \beta_K]^{\mathcal{T}} \\ &= \kappa \bar{T}_p^{-\mathcal{T}} [\zeta_0 \quad \zeta_1 \quad \cdots \quad \zeta_K]^{\mathcal{T}} \end{aligned}$$

with $\kappa = \prod_{k=1}^{K} \Delta_k^{-1}$ such that $\bar{\alpha}(1) = 1$ and $\bar{T}_p$ an upper triangular matrix whose kth row is formed with the coefficients of $p_{k-1}(z)$ defined above. Equation (6) implies that the controller transfer function $C_d(z)$ is reparametrized with $\{\alpha_k\}$ and $\{\beta_k\}$ in the new set of polynomial operators $\{p_k(z)\}$.

It follows from (5) and (6) that the output of the controller can be computed with the following equations

$$\begin{aligned} y(n) &= \beta_0 u(n) + w_1(n) \\ w_k(n) &= \rho_k^{-1}[\beta_k u(n) - \alpha_k y(n) + w_{k+1}(n)] \\ w_K(n) &= \rho_K^{-1}[\beta_K u(n) - \alpha_K y(n)] \end{aligned} \tag{7}$$

where $w_k(n)$ is the output of $\rho_k^{-1}(z)$ and can be computed with the structure depicted in Fig. 2. Fig. 3 shows the corresponding structure to (7). For convenience, a structure defined by Fig.s 2 and 3 is called a generalized DFIIt structure, denoted as ρDFIIt. This structure possesses $\{\alpha_k,\ \beta_k,\ \Delta_k\}$ and a set of free parameters $\{\gamma_k\}$. For a given digital controller $C_d(z)$, there exists a class of such structures, depending on the space within which $\{\gamma_k\}$ take values. Clearly, when $\gamma_k = 0,\ \Delta_k = 1,\ \forall k$, Fig. 3 is the conventional direct-form II transposed (DFIIt) structure.

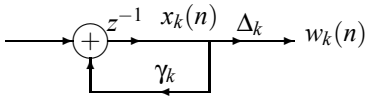

Figure 2: A realization of $\rho_k^{-1}(z)$ defined in (4).

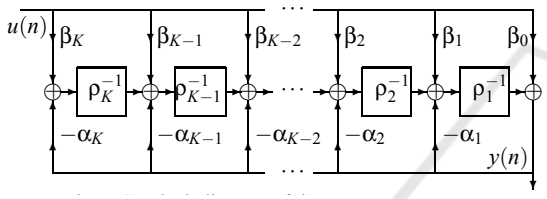

Figure 3: Block diagram of the ρDFIIt structure.

With $\{x_k(n)\}$ indicated in Fig. 2 as the state variables and $x(n)$ denoting the state vector, one can find the equivalent state-space realization, denoted as $(A_\rho, B_\rho, C_\rho, \beta_0)$, of the proposed ρDFIIt structure:

$$C_d(z) = \beta_0 + C_\rho (zI - A_\rho)^{-1} B_\rho \tag{8}$$

with $B_\rho = \bar{V}_\beta - \beta_0 \bar{V}_\alpha$, where $\bar{V}_x \triangleq [x_1\ \cdots\ x_k\ \cdots\ x_K]^{\mathcal{T}}$ for $x = \alpha, \beta$, $C_\rho = [\Delta_1\ 0\ \cdots\ 0\ 0]$, and

$$A_\rho \triangleq \begin{bmatrix} a_{11} & \Delta_2 & 0 & \cdots & 0 & 0 \\ a_{21} & \gamma_2 & \Delta_3 & \cdots & 0 & 0 \\ & & & \vdots & & \\ a_{(K-1)1} & 0 & 0 & \cdots & \gamma_{K-1} & \Delta_K \\ a_{K1} & 0 & 0 & \cdots & 0 & \gamma_K \end{bmatrix}$$

with $a_{11} = \gamma_1 - \Delta_1\alpha_1$ and $a_{k1} = -\Delta_1\alpha_k,\ k \in \{2,3,\cdots,K\}$.

2.2 Scaling Scheme

It is well known that in an implementation system, all the signals should be sustained within a certain dynamic range in order to avoid overflow. Under the assumption that the input $r(n)$ and the output $u(n)$ of the closed-loop system are properly pre-scaled, the only signals which may have overflow are the elements of the controller state vector $x(n)$, which, therefore, have to be scaled.

There exist different scaling schemes for preventing variables from overflow. The popularly used ones are the l_2- and l_∞-scalings. In what follows, we will concentrate on the l_2-scaling scheme. The l_2-scaling means that each element of the controller state vector $x(n)$ should have a unit variance when the input $r(n)$ is a white noise with a unit variance. This can be achieved if

$$\bar{\mathcal{K}}(l,l) = 1,\ \ l = N+1, N+2, ..., N+K \tag{9}$$

where $\bar{\mathcal{K}}$ is the controllability Gramian of the closed-loop system of order $N+K$. Assuming that $P_d(z)$ is strictly proper and has a realization $(A_z, B_z, C_z, 0)$, let $(A_{cl}, B_{cl}, C_{cl}, 0)$ be the closed-loop realization, where

$$\begin{aligned} A_{cl} &= \begin{bmatrix} A_z + dB_zC_z & B_zC \\ BC_z & A \end{bmatrix} \\ B_{cl} &= \begin{bmatrix} B_z \\ \mathbf{0} \end{bmatrix} \\ C_{cl} &= [C_z \quad \mathbf{0}] \end{aligned} \tag{10}$$

with $\mathbf{0}$ denoting the zero vector of appropriate dimension. Then $\bar{\mathcal{K}}$ is given by

$$\bar{\mathcal{K}} = \sum_{k=0}^{+\infty} A_{cl}^k B_{cl} B_{cl}^{\mathcal{T}} (A_{cl}^{\mathcal{T}})^k \tag{11}$$

satisfying

$$\bar{\mathcal{K}} = A_{cl}\bar{\mathcal{K}}A_{cl}^{\mathcal{T}} + B_{cl}B_{cl}^{\mathcal{T}}.$$

Let (A_{cl}, B_{cl}, C_{cl}) and $(A_{cl}^0, B_{cl}^0, C_{cl}^0)$ be two realizations of the closed-loop system with A_{cl}, B_{cl} and C_{cl} defined in (10), corresponding to the two digital controller realizations $R \triangleq (A,B,C,d)$ and $R_0 \triangleq (A_0, B_0, C_0, d)$ which are related with (3), respectively. It can be shown that

$$\begin{aligned} A_{cl} &= \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & T \end{bmatrix}^{-1} A_{cl}^0 \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & T \end{bmatrix} \\ B_{cl} &= \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & T \end{bmatrix}^{-1} B_{cl}^0 \\ C_{cl} &= C_{cl}^0 \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & T \end{bmatrix}. \end{aligned} \tag{12}$$

It then follows from (12) that

$$\bar{\mathcal{K}} = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & T \end{bmatrix}^{-1} \bar{\mathcal{K}}_0 \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & T \end{bmatrix}^{-\mathcal{T}}$$

where $\bar{\mathcal{K}}^0$ is the closed-loop controllability Gramian corresponding to R_0. Let

$$\bar{\mathcal{K}} \triangleq \begin{bmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} \\ \mathcal{K}_{21} & \mathcal{K} \end{bmatrix},\ \ \bar{\mathcal{K}}^0 \triangleq \begin{bmatrix} \mathcal{K}_{11}^0 & \mathcal{K}_{12}^0 \\ \mathcal{K}_{21}^0 & \mathcal{K}_0 \end{bmatrix} \tag{13}$$

have the same partition as $\begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & T \end{bmatrix}$, then

$$\mathcal{K} = T^{-1}\mathcal{K}_0 T^{-\mathcal{T}} \tag{14}$$

where $\mathcal{K}_0$ is a positive-definite matrix independent of T.

It is easy to see from above equations that the l_2-scaling constraint (9) can be satisfied if the diagonal elements of $\mathcal{K}$ are all equal to one, that is

$$\mathcal{K}(k,k) = 1, \forall k. \tag{15}$$

When the ρDFIIt structure is used to implement a digital controller, it has to be l_2-scaled in order to prevent the signals in the controller from overflow, which can be achieved by choosing $\{\Delta_k\}$ properly. It is interesting to note that

$$p_k(z) = [\prod_{l=k+1}^{K} \Delta_l^{-1}]\bar{p_k}(z),\ \forall k \tag{16}$$

where all $\bar{p_k}(z)$ are obtained using (5) with $\Delta_k = 1,\ \forall k$.

Let $(A_\rho^0, B_\rho^0, C_\rho^0, \beta_0)$ be the equivalent state-space realization corresponding to $\Delta_k = 1,\ \forall k$. With (16), it can be shown that

$$A_\rho = T_{sc}A_\rho^0 T_{sc}^{-1},\ B_\rho = T_{sc}B_\rho^0,\ C_\rho = C_\rho^0 T_{sc}^{-1}$$

where T_{sc}^{-1} is a diagonal scaling similarity transformation, and

$$T_{sc} = diag(d_1, d_2, \cdots, d_K),\ \ d_k = \prod_{l=1}^{k} \Delta_l^{-1},\ \forall k.$$

Denote $\bar{\mathcal{K}_\rho}$ and $\bar{\mathcal{K}_\rho^0}$ as the closed-loop controllability Gramians, corresponding to the controller realizations $(A_\rho, B_\rho, C_\rho, \beta_0)$ and $(A_\rho^0, B_\rho^0, C_\rho^0, \beta_0)$, respectively. Let $\mathcal{K}_\rho$ be the sub-matrix of $\bar{\mathcal{K}_\rho}$ with the partition defined in (13), then (14) becomes $\mathcal{K}_\rho = T_{sc}\mathcal{K}_\rho^0 T_{sc}^{\mathcal{T}}$ with $\mathcal{K}_\rho^0$ the corresponding sub-matrix of $\bar{\mathcal{K}_\rho^0}$. It is easy to see that the l_2-scaling can be achieved if $\mathcal{K}_\rho(k,k) = 1, \forall k$, or equivalently,

$$d_k^2 \mathcal{K}_\rho^0(k,k) = 1,\ \ k = 1,2,...,K$$

which leads to

$$\Delta_1 = \sqrt{\mathcal{K}_\rho^0(1,1)},\ \Delta_k = \sqrt{\frac{\mathcal{K}_\rho^0(k,k)}{\mathcal{K}_\rho^0(k-1,k-1)}}, \tag{17}$$

$$k = 2,3,...,K.$$

In the sequel, all the structures under discussion, including the ρDFIIt structure, are assumed to have been l_2-scaled. Here we should note that the l_2-scaled ρDFIIt structure to be analyzed in this paper is different from the structure in (Hao et al., 2006) where $\{\Delta_k\}$ are free parameters used for maximizing the stability robustness measure.

3 PERFORMANCE ANALYSIS AND OPTIMIZED STRUCTURE

In this section, we will analyze the performance of the ρDFIIt structure in terms of closed-loop roundoff noise gain. The problem of finding the optimized structure will then be formulated and solved.

One notes that for a given digital controller $C_d(z)$, there exists a class of l_2-scaled ρDFIIt structures, which are determined by a space, denoted as S_γ, from which the free parameters $\{\gamma_k\}$ take values. It is easy to see that $\{\gamma_k\}$ are the parameters to be implemented directly in the structure. Since we are confined to fixed-point implementation for which the FWL effects are more serious, it is desired that γ_k be absolutely not bigger than one and of FWL format. For a fixed-point implementation of B_p bits, define

$$S_{FWL} \triangleq \{-1,1\} \cup \{\pm \sum_{l=1}^{B_p} b_l 2^{-l},\ b_l = 0,1,\ \forall l\} \tag{18}$$

which is a discrete space, containing $2^{B_p+1}+1$ elements. Therefore, one can choose $S_\gamma \subset S_{FWL}$, which means that all γ_k are of exact B_γ-bit format with $B_\gamma \leq B_p$.

3.1 Closed-loop Roundoff Noise Gain

In practice, a designed digital controller has to be implemented with finite precision and a rounding operation has to be applied if less-than-double precision fixed-point arithmetic is utilized. Assuming rounding occurs after multiplication (RAM), a variable, say x, computed with a multiplication, has to be replaced by its quantized version, denoted as $Q[x]$, in the ideal computation model. The difference $Q[x] - x$ is the corresponding roundoff noise, which is usually modelled as a white noise sequence and statistically independent of those produced by other sources.

Let μ be a parameter in a controller structure and $Q[\mu s(n)]$ the quantized version of the product $\mu s(n)$. The roundoff noise due to the parameter μ can be defined as

$$\psi(\mu)\varepsilon_\mu(n) \triangleq Q[\mu s(n)] - \mu s(n)$$

where $\psi(\mu) = 1$ if μ is nontrivial, otherwise, $\psi(\mu) = 0$. In fact, the function $\psi(\mu)$ is used for indicating the fact that μ produces no roundoff noise when it is trivial. Denote $\Delta u(n)$ as the corresponding output deviation of the closed-loop system to $\psi(\mu)\varepsilon_\mu(n)$ and $F(z)$ as the transfer function between $\psi(\mu)\varepsilon_\mu(n)$ and $\Delta u(n)$. It is well known (see, e.g., (Gevers and Li, 1993)) that $\Delta u(n)$ is a stationary process and the variance $E[(\Delta u(n))^2] = \psi(\mu)\|F(z)\|_2^2 E[\varepsilon_\mu^2(n)]$. Then the

roundoff noise gain for μ is defined as

$$G_\mu \triangleq \frac{E[(\Delta u(n))^2]}{E[\varepsilon_\mu^2(n)]} = \psi(\mu)\|F(z)\|_2^2 \qquad (19)$$

where $||.||_2$ is the L_2-norm:

$$\begin{aligned} \| F(z) \|_2 &\triangleq \left\{ \frac{1}{2\pi}\int_0^{2\pi} \sum_{i=1}^{l}\sum_{k=1}^{m} |f_{ik}(e^{j\omega})|^2 d\omega \right\}^{1/2} \\ &= \left\{ tr[\frac{1}{j2\pi}\oint_{|z|=1} F(z)F^{\mathcal{H}}(z)z^{-1}dz] \right\}^{1/2} \end{aligned} \qquad (20)$$

with $F(z) = \{f_{ik}(z)\} \in \mathcal{R}^{l\times m}$, and $\mathcal{H}$, $tr[.]$ denoting the conjugate-transpose and trace operators, respectively. Let $F(z) = D + L(zI-\Phi)^{-1}J$. It can be shown that

$$\|F(z)\|_2^2 = tr[DD^T + LW_cL^T] = tr[D^TD + J^TW_oJ] \qquad (21)$$

where W_c, W_o are the controllability and observability Gramians of the realization (Φ, J, L, D), respectively.

Consider a digital controller implemented with a ρDFIIt structure. We note that the parameters in a ρDFIIt structure are $\{\alpha_k\}, \{\beta_k\}$, $\{\Delta_k\}$, and $\{\gamma_k\}$. It follows from (6) that

$$y(n) = \beta_0 u(n) + \sum_{l=1}^{K}[\frac{p_l(z)}{p_0(z)}\beta_l u(n) - \frac{p_l(z)}{p_0(z)}\alpha_l y(n)]. \qquad (22)$$

Let us first look at the effect of roundoff noise $\psi(\beta_0)\varepsilon_{\beta_0}(n)$ due to β_0 on the closed-loop output. Let $u^*(n)$ and $y^*(n)$ be the corresponding output of the closed-loop system and the controller, respectively. Clearly, they obey (22) with $\beta_0 u^*(n)$ replaced by $\beta_0 u^*(n) + \psi(\beta_0)\varepsilon_{\beta_0}(n)$. Denote $\Delta y(n) \triangleq y^*(n) - y(n)$. Then one can show that

$$\begin{aligned} \Delta y(n) &= [\beta_0 \Delta u(n) + \psi(\beta_0)\varepsilon_{\beta_0}(n)] \\ &+ \sum_{l=1}^{K}\frac{p_l(z)}{p_0(z)}\beta_l \Delta u(n) - \sum_{l=1}^{K}\frac{p_l(z)}{p_0(z)}\alpha_l \Delta y(n) \end{aligned} \qquad (23)$$

where $\Delta u(n) \triangleq u^*(n) - u(n)$, satisfying

$$\Delta u(n) = P_d(z)\Delta y(n). \qquad (24)$$

Let $H_{cl}(z)$ be the transfer function of the closed-loop system, which is given by

$$H_{cl}(z) = \frac{P_d(z)}{1 - P_d(z)C_d(z)}$$

where $P_d(z)$ is the transfer function of plant and $C_d(z)$ the polynomial parametrized controller transfer function given by (6). It is easy to see that

$$H_{cl}(z) = D_{cl} + C_{cl}(zI - A_{cl})^{-1}B_{cl} \qquad (25)$$

with $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$ the realization of closed-loop system. It then follows from (23) and (24) that

$$\Delta u(n) = S_0(z)\psi(\beta_0)\varepsilon_{\beta_0}(n)$$

where $S_0(z)$ is the transfer function between $\psi(\beta_0)\varepsilon_{\beta_0}(n)$ and $\Delta u(n)$, which is given by

$$S_0(z) = H_{cl}(z)V_0(z)$$

with

$$V_0(z) \triangleq \frac{p_0(z)}{p_0(z) + \sum_{l=1}^{K}\alpha_l p_l(z)}.$$

Comparing $V_0(z)$ with (6), it follows from (8) that

$$\begin{aligned} V_0(z) &= [\beta_0 + C_\rho(zI - A_\rho)^{-1}B_\rho]_{|_{\beta_0=1, \bar{V}_\beta=\mathbf{0}}} \\ &= 1 - C_\rho(zI - A_\rho)^{-1}\bar{V}_\alpha. \end{aligned}$$

One observes that $S_0(z)$ is of the form $S_0(z) = [D_2 + C_2(zI_2 - A_2)^{-1}B_2][D_1 + C_1(zI_1 - A_1)^{-1}B_1]$, where $A_1 = A_\rho, B_1 = -\bar{V}_\alpha, C_1 = C_\rho, D_1 = 1$, $A_2 = A_{cl}, B_2 = B_{cl}, C_2 = C_{cl}, D_2 = D_{cl}$, and $I_k, k = 1,2$ denotes the identity matrix of a proper dimension. It is easy to verify that

$$S_0(z) \triangleq \tilde{D} + \tilde{C}(z\tilde{I} - \tilde{A})^{-1}\tilde{B}$$

where

$$\begin{aligned} \tilde{D} &= D_2D_1, \ \tilde{C} = [D_2C_1 \quad C_2] \\ \tilde{I} &= \begin{bmatrix} I_1 & \mathbf{0} \\ \mathbf{0} & I_2 \end{bmatrix} \\ \tilde{A} &= \begin{bmatrix} A_1 & \mathbf{0} \\ B_2C_1 & A_2 \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}. \end{aligned}$$

According to (19) and (21), the roundoff noise gain due to parameter β_0 is given by

$$\begin{aligned} G_{\beta_0} &= \psi(\beta_0)||S_0(z)||_2^2 = \psi(\beta_0)tr(\tilde{D}^T\tilde{D} + \tilde{B}^T\tilde{W}\tilde{B}) \\ &\triangleq \psi(\beta_0)G_0 \end{aligned}$$

where $\tilde{W}$ is the observability Gramian of the realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$.

Using the same procedure, one can analyze the roundoff noise gain due to the parameter β_k. Let $\psi(\beta_k)\varepsilon_{\beta_k}(n)$ be the corresponding roundoff noise. It can be shown that the transfer function from $\psi(\beta_k)\varepsilon_{\beta_k}(n)$ to $\Delta u(n)$, denoted as $S_k(z)$, is

$$S_k(z) = H_{cl}(z)V_k(z)$$

with $H_{cl}(z)$ given by (25) and

$$V_k(z) = \frac{p_k(z)}{p_0(z) + \sum_{l=1}^{K}\alpha_l p_l(z)} = C_\rho(zI - A_\rho)^{-1}e_k$$

for $k = 1, 2, \cdots, K$, where e_k is the kth elementary vector whose elements are all zero except the kth one which is 1. Therefore,

$$G_{\beta_k} = \psi(\beta_k)||S_k(z)||_2^2 \triangleq \psi(\beta_k)G_k, \ \forall k$$

with

$$G_k = tr(\tilde{D}_k^T \tilde{D}_k + \tilde{B}_k^T \tilde{W}_k \tilde{B}_k)$$

where $(\tilde{A}_k, \tilde{B}_k, \tilde{C}_k, \tilde{D}_k)$ is the realization of $S_k(z)$ and $\tilde{W}_k$ is the corresponding observability Gramian.

Comparing the positions of α_k, γ_k and Δ_{k+1} with that of β_k in Fig. 3, one can see easily that

$$G_{\alpha_k} = \psi(\alpha_k) G_k,\ G_{\gamma_k} = \psi(\gamma_k) G_k,\ G_{\Delta_k} = \psi(\Delta_k) G_{k-1}$$

for $k = 1, 2, \cdots, K$.

Therefore, the total closed-loop roundoff noise gain of the ρDFIIt structure is

$$G_\rho \triangleq \sum_{k=1}^{K} [G_{\alpha_k} + G_{\gamma_k} + G_{\Delta_k}] + \sum_{k=0}^{K} G_{\beta_k} \triangleq \sum_{k=0}^{K} \upsilon_k G_k \quad (26)$$

where the coefficients υ_k can be specified easily with the expressions, obtained above, of roundoff noise gain for all the parameters.

3.2 Structure Optimization

For a given digital controller $C_d(z)$ and any given free parameters $\{\gamma_k\}$, one can obtain the l_2-scaled ρDFIIt structure with the procedure presented in Section II. The roundoff noise gain G_ρ can then be evaluated with (26). Since different sets of $\{\gamma_k\}$ yield different ρDFIIt structures and hence lead to different roundoff noise gain G_ρ, an interesting problem is to minimize G_ρ with respect to these free parameters, which leads to the following optimal ρDFIIt structure problem:

$$\min_{\gamma_k \in S_\gamma} G_\rho. \quad (27)$$

It seems impossible to obtain analytical solutions to the problem (27) due to the high nonlinearity of G_ρ in $\{\gamma_k\}$. However, noting that S_γ is of finite number of elements, the problem can be well solved using the exhaustive searching method.

4 A DESIGN EXAMPLE

In this section, we illustrate our design procedure and the performance of the proposed structure with a numerical example, in which $S_\gamma = \{\pm 1, \pm(2^{-1} + 2^{-2}), \pm 2^{-1}, \pm 2^{-2}, 0\}$. The elements in the set S_γ are of exact 3-bit fixed-point format (including one bit for the sign). Using more bits or floating-point formats will lead to a further improved performance, which can also confirm the effectiveness of our design procedure.

Consider a discrete-time control system, where the digital plant $P_d(z) = 10^{-1} \times \frac{0.0181z^4+0.0033z^3-0.1628z^2+0.0111z+0.0163}{z^5-3.7174z^4+5.7458z^3-4.6673z^2+2.0336z-0.3953}$ and controller $C_d(z) = 0.0577 + \frac{0.2258z^5-0.6588z^4+0.8195z^3-0.5320z^2+0.1814z-0.0234}{z^6-3.6172z^5+5.9513z^4-5.6335z^3+3.2509z^2-1.0895z+0.1690}$.

The corresponding poles of the closed-loop system are $\{0.4523 \pm j0.5315, 0.4837 \pm j0.4556, 0.6055 \pm j0.4108, 0.7814 \pm j0.3099, 0.8886 \pm j0.3326, 0.9113\}$.

Applying exhaustive searching to (27), one gets the optimal ρDFIIt structure, denoted as ρDFIIt, for which $\gamma_1 = 1,\ \gamma_5 = 0.5,\ \gamma_k = 0.75,\ k \in \{2,3,4,6\}$. For comparison, an optimal fully parametrized state-space realization, denoted by R_f, is obtained using the procedure in (Li et al., 2002). zDFIIt and δDFIIt are the traditional DFIIt structures in the shift- and δ-operators, corresponding to $\gamma_k = 0,\ \forall k$ and $\gamma_k = 1,\ \forall k$, respectively.

Table 1: Comparison of Different Structures.

	zDFIIt	δDFIIt	R_f	ρDFIIt
G	1.5191×10^4	7.1763	4.9919	1.0085
N_p	19	19	49	24

The comparative results of different structures are presented in Table I, where G is the roundoff noise gain and N_p is the number of nontrivial parameters in each structure.

From this example, one can see that zDFIIt yields a very large roundoff noise gain, though it has only 19 parameters to implement, while δDFIIt has a much better performance. The fully parametrized optimal realization R_f yields a further better performance, however, all the 49 parameters in R_f are nontrivial. It is interesting to see that ρDFIIt beats R_f in terms of the roundoff noise performance. Moreover, ρDFIIt is very sparse and has only 24 nontrivial parameters, which is less than half of those in R_f.

5 CONCLUSIONS

In this paper, we have addressed the optimal controller structure problem in a discrete-time control system with roundoff noise consideration. Our major contribution is twofold. Firstly, a sparse controller structure, which is a l_2-scaled generalized DFIIt structure, has been derived. Secondly, the performance of the proposed structure has been analyzed by deriving the corresponding expression of closed-loop roundoff noise gain and the problem of finding optimized sparse structures has been solved. Finally, a numerical example has been given, which shows that the proposed structure can achieve much better performance than some well-known structures and particularly, outperforms the traditional optimal fully

parametrized realization greatly in terms of reducing roundoff noise and implementation complexity. This optimal controller design strategy with high precision arithmetic can be utilized to develop suitable control systems for robotic platforms performing complex movements, where efficiency, accuracy and fast speed are essential.

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