ON THE STATE–SPACE REALIZATION OF VECTOR AUTOREGRESSIVE STRUCTURES
An Assessment

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Abstract: This study explores the interconnection between vector autoregressive (VAR) structures and state–space models and results in a compact framework for the representation of multivariate time–series, as well as the estimation of structural information. The corresponding methodology that is developed, applies the fact that every VAR process of order *n* may be described by an equivalent (non–unique) VAR model of first order, which is identical to a state–space realization. The latter uncovers many ”hidden” information of the initial model, it is more easy to manipulate and maintains significant second moments' information that can be reflected back to the original structure with no effort. The performance of the proposed framework is validated using vector time–series signatures from a structural system with two degrees of freedom, which retains a pair of closely spaced vibration modes and has been reported in the relevant literature.

1 INTRODUCTION

The analysis of vector time–series, generally referring to the determination of the dynamics that govern the performance of a system under unobservable excitations, has been a subject of constant development for more than two decades, as part of the broader system identification framework. Relative applications are extended from econometrics (Clements and Henry, 1998; Lütkepohl, 2005), to dynamics (Ljung, 1999; Koulocheris et al., 2008), vibration (Papakos and Fassois, 2003), modal analysis (Huang, 2001) and fault diagnosis (Dertimanis, 2006).

The study of vector time–series can be assessed from a variety of viewpoints, with respect to the application of interest. These include simulation, prediction and extraction of structural information. Yet, while in the first two areas the interrelation of the corresponding time–series structures, such as the VAR one (or the VARX and the VARMAX, under the availability of input information), to equivalent state–space models has been studied extensively (Hannan, 1976; Brockwell and Davis, 2002; Lütkepohl, 2005), not much have been done in the third (Lardies, 2008), from where it appears that state–space realizations may provide significant advantages, regarding structural estimation, with respect to other approaches (He and Fu, 2001).

This paper attempts to provide a unified framework for the representation of vector time–series, by means of VAR structures and their corresponding state–space realizations. Based on the fact that every VAR structure of order *n* (referred to from now on as VAR(n) structure) can be expressed as an equivalent (and non–unique) VAR(1) one, a corresponding state–space model is developed. This specific model qualifies, over other possible realizations, for having a transition matrix that coincides with the VAR(n) polynomial matrix. It turns out that the spectrum of this transition matrix has all the structural information about the system that generates the time–series ”hidden” in its spectrum. Consequently, by taking advantage standard results of matrix algebra, closed form expressions for the Green function and the covariance matrix are derived. The latter, unlike other estimation schemes, such as the Burg and the forward–backward methods (Brockwell and Davis, 2002), is by definition closely related to the energy distribution of the vector time–series. Thus, the corresponding expression that is assessed, quantifies the impact of each specific structural mode in the total energy of the system, a technique that has been recorded in the literature as

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dispersion analysis (Lee and Fassois, 1993).

The paper is organized as follows: in Sec. 2 the VAR(n) structure is presented and the reduction to the state–space realization is performed. Section 3 illustrates the properties of the state–space model, including the development of closed form representations for the Green function and the covariance matrix, and how these are reflected to the original VAR(n) structure. Section 4 contains the least–squares estimation of the state equation and Sec. 5 the validation of the estimated model, as well as the extraction of the structural information that is "hidden" in the transition matrix. Section 6 displays an application of the proposed framework to a simulated vibrating system that has been already used in the past (Lee and Fassois, 1993; Fassois and Lee, 1993) and in Sec. 7 the method is concluded and some remarks for further research are outlined.

2 THE VAR(n) STRUCTURE

2.1 The Model

Let $\mathbf{Y}[t] = \begin{bmatrix} y_1[t] & y_2[t] & \dots & y_s[t] \end{bmatrix}^T$ denote a s–dimensional vector time–series of zero mean random variables[1]. Under the stationarity assumption (Box et al., 2008), $\mathbf{Y}[t]$ can be described by a finite order VAR model of the following form:

$$\mathbf{Y}[t] + \mathbf{A}_1 \cdot \mathbf{Y}[t-1] + \dots + \mathbf{A}_n \cdot \mathbf{Y}[t-n] = \mathbf{Z}[t] \tag{1}$$

In the above equation n is the order of the VAR process, $\mathbf{A}_j$ designate the $[s \times s]$ AR matrices and $\mathbf{Z}[t]$ describes a vector white noise process with zero mean,

$$\boldsymbol{\mu}_{\mathbf{Z}} \equiv E\left\{\mathbf{Z}[t]\right\} = \mathbf{0} \tag{2}$$

and covariance function,

$$\boldsymbol{\Gamma}_{\mathbf{Z}}[h] \equiv E\left\{\mathbf{Z}[t+h] \cdot \mathbf{Z}^T[t]\right\} = \begin{cases} \boldsymbol{\Sigma}_{\mathbf{Z}} & h = 0 \\ \mathbf{0} & h \neq 0 \end{cases} \tag{3}$$

where $\boldsymbol{\Sigma}$ is a non–singular (and generally non–diagonal) matrix.

Taking advantage of the backshift operator q, defined such that $q^{-k} \cdot \mathbf{Y}[t] = \mathbf{Y}[t-k]$, the VAR(n) structure can be compactly written as,

$$\mathbf{A}(q) \cdot \mathbf{Y}[t] = \mathbf{Z}[t] \tag{4}$$

where $\mathbf{A}(q)$ is the $[s \times s]$ AR polynomial matrix:

$$\mathbf{A}(q) = \mathbf{I}_s + \mathbf{A}_1 \cdot q^{-1} + \dots + \mathbf{A}_n \cdot q^{-n} \tag{5}$$

[1] Throughout the paper, quantities in the brackets shall notate discrete–time units (or time lags, in the case of covariance functions) and hats shall notate estimators / estimates. $E\{\cdot\}$ shall notate expectation.

2.2 Reduction to State–space

Any VAR(n) process of Eq. 1 can be transformed to an equivalent VAR(1) structure (Lütkepohl, 2005). Define the $[n \cdot s \times 1]$ vectors,

$$\boldsymbol{\Xi}[t] = \begin{bmatrix} \mathbf{Y}[t-n+1] \\ \mathbf{Y}[t-n+2] \\ \vdots \\ \mathbf{Y}[t-1] \\ \mathbf{Y}[t] \end{bmatrix} \quad \boldsymbol{N}[t] = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{Z}[t] \end{bmatrix} \tag{6}$$

and the $[n \cdot s \times n \cdot s]$ and $[s \times n \cdot s]$ matrices,

$$\mathbf{F} = \begin{bmatrix} \mathbf{O}_s & \mathbf{I}_s & \dots & \mathbf{O}_s \\ \mathbf{O}_s & \mathbf{O}_s & \dots & \mathbf{O}_s \\ \dots & \dots & \ddots & \vdots \\ \mathbf{O}_s & \mathbf{O}_s & \dots & \mathbf{I}_s \\ -\mathbf{A}_n & -\mathbf{A}_{n-1} & \dots & -\mathbf{A}_1 \end{bmatrix} \tag{7}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{O}_s & \mathbf{O}_s & \dots & \mathbf{I}_s \end{bmatrix} \tag{8}$$

respectively, Eq. 1 can take the following form:

$$\boldsymbol{\Xi}[t] = \mathbf{F} \cdot \boldsymbol{\Xi}[t-1] + \boldsymbol{N}[t] \tag{9}$$

$$\boldsymbol{Y}[t] = \mathbf{C} \cdot \boldsymbol{\Xi}[t] \tag{10}$$

Equations 9–10 illustrate the state–space realization of the VAR(n) structure of Eq. 1. Naturally, the state–space model consists of a state equation (Eq. 9), in which $\mathbf{F}$ is the state transition matrix, and an observation equation (Eq. 10) that relates the original s–variate time–series $\mathbf{Y}[t]$ to the state vector, $\boldsymbol{\Xi}[t]$, by means of the output matrix $\mathbf{C}$. Obviously, the state equation can be viewed as a VAR(1) model, in which $\boldsymbol{\Xi}[t]$ is a well–defined stationary stochastic process and $\mathbf{N}[t]$ has properties similar to that of $\mathbf{Z}[t]$, as it will become clear at the following.

It must be noted that the state–space realization of Eq. 1 is not unique (Lardies, 2008). In fact they exist infinitely many pairs $\{\mathbf{F}, \mathbf{C}\}$ that can describe $\mathbf{Y}[t]$ in terms of Eqs. 9–10, since any transformation of the state vector by a non–singular $[n \cdot s \times n \cdot s]$ matrix $\mathbf{T}$ leads to new state equation, in which the transition matrix $\mathbf{T} \cdot \mathbf{F} \cdot \mathbf{T}^{-1}$ is similar to $\mathbf{F}$ and preserves its eigenvalues (Meyer, 2000). Yet, Eq. 1 has a very important property: the transition matrix $\mathbf{F}$, as defined in Eq. 7, is the block companion matrix of the polynomial matrix $\mathbf{A}(q)$ described by Eq. 5 and includes all the structural information of interest, regarding the process that generates the s–variate time–series $\mathbf{Y}[t]$.

3 PROPERTIES OF THE STATE–SPACE REALIZATION

3.1 Noise

From Eqs. 6, 8, it holds that,

$$\mathbf{N}[t] = \mathbf{C}^T \cdot \mathbf{Z}[t] \tag{11}$$

so the mean value and the covariance matrix of $\mathbf{N}[t]$ are given by,

$$\boldsymbol{\mu}_{\mathbf{N}} \equiv E\left\{\mathbf{N}[t]\right\} = \mathbf{C}^T \cdot E\left\{\mathbf{Z}[t]\right\} = \mathbf{0} \tag{12}$$

and,

$$\begin{aligned}\boldsymbol{\Gamma}_{\mathbf{N}}[h] &\equiv E\left\{\mathbf{N}[t+h]\cdot\mathbf{N}^T[t]\right\} \\ &= E\left\{\mathbf{C}^T\cdot\mathbf{Z}[t+h]\cdot\mathbf{Z}^T[t]\cdot\mathbf{C}\right\} \\ &= \mathbf{C}^T\cdot E\left\{\mathbf{Z}[t+h]\cdot\mathbf{Z}^T[t]\right\}\cdot\mathbf{C} \\ &= \mathbf{C}^T\cdot\boldsymbol{\Gamma}_{\mathbf{Z}}[h]\cdot\mathbf{C}\end{aligned} \tag{13}$$

leading to,

$$\boldsymbol{\Gamma}_{\mathbf{N}}[h] = \begin{cases}\boldsymbol{\Sigma}_{\mathbf{N}} & h = 0 \\ \mathbf{0} & h \neq 0\end{cases} \tag{14}$$

where $\boldsymbol{\Sigma}_{\mathbf{N}} = \mathbf{C}^T\cdot\boldsymbol{\Sigma}_{\mathbf{Z}}\cdot\mathbf{C}$.

3.2 State Vector

Since the state transition equation reflects the properties of an observed dynamic system, the output of which is the available s–variate time–series $\mathbf{Y}[t]$, it is desirable to obtain corresponding mathematical expressions that assess and quantify the relative information. Conventional time–series analysis usually is led to infinite, or recursive expressions for the representation / calculation of valuable quantities, such as the weighting function (referred to also as *Green function*, *process generating function*, or *transfer function*) and the covariance matrix. The analysis that follows leads to closed form representations, which reveal the spectral characteristics of the transition matrix $\mathbf{F}$.

3.2.1 The Green Function

Starting from the VAR(1) state equation,

$$\boldsymbol{\Xi}[t] = \mathbf{F}\cdot\boldsymbol{\Xi}[t-1] + \boldsymbol{N}[t] \tag{15}$$

it can be written as an infinite vector moving average,

$$\boldsymbol{\Xi}[t] = \sum_{k=0}^{\infty}\mathbf{F}^k\cdot\boldsymbol{N}[t-k] \tag{16}$$

which is a multivariate generalization of Wold's theorem (Box et al., 2008). Without loss of generality, assuming that $\mathbf{F}$ has a complete set of eigenvalues $\{\lambda_1,\lambda_2,...,\lambda_{n\cdot s}\}$, it can be expressed as,

$$\mathbf{F} = \sum_{j=1}^{n\cdot s}\mathbf{G}_j\cdot\lambda_j \tag{17}$$

where $\mathbf{G}_k$ are the spectral projectors of $\mathbf{F}$ (refer to the Appendix for a brief introduction to the spectral properties of square matrices). The substitution of Eq. 17 to Eq. 16, using the fact that $\mathbf{G}_j^k = \mathbf{G}_j$ and $\mathbf{G}_i\cdot\mathbf{G}_j = 0,\ \ i \neq j$, yields,

$$\begin{aligned}\boldsymbol{\Xi}[t] &= \sum_{k=0}^{\infty}\Big[\sum_{j=1}^{n\cdot s}\mathbf{G}_j\cdot\lambda_j\Big]^k\cdot\boldsymbol{N}[t-k] \\ &= \sum_{k=0}^{\infty}\sum_{j=1}^{n\cdot s}\mathbf{G}_j\cdot\lambda_j^k\cdot\boldsymbol{N}[t-k] \\ &= \sum_{k=0}^{\infty}\mathbf{H}_{\boldsymbol{\Xi}}[k]\cdot\boldsymbol{N}[t-k]\end{aligned} \tag{18}$$

so that the coefficients of the weighting (Green) function can be expressed in a closed form as,

$$\mathbf{H}_{\boldsymbol{\Xi}}[k] \equiv \mathbf{F}^k = \sum_{j=1}^{n\cdot s}\mathbf{G}_j\cdot\lambda_j^k \tag{19}$$

in terms of the spectrum of $\mathbf{F}$. Notice that by definition (Eq. 16), $\mathbf{H}_{\boldsymbol{\Xi}}[k]$ can be viewed as the impulse response of the state difference equation, which generally has a decaying performance, characterized by a mixture of damped exponentials and cosines, as for example in vibrating systems, where the eigenvalues λ_k often appear in complex conjugate pairs. Furthermore it holds that (see Appendix):

$$\mathbf{H}_{\boldsymbol{\Xi}}[0] = \sum_{j=1}^{n\cdot s}\mathbf{G}_j = \mathbf{I} \tag{20}$$

3.2.2 The Covariance Matrix

The covariance matrix related to the Wold decomposition of the state equation is (Brockwell and Davis, 2002):

$$\boldsymbol{\Gamma}_{\boldsymbol{\Xi}}[h] = \sum_{k=0}^{\infty}\mathbf{F}^{k+h}\cdot\boldsymbol{\Sigma}_{\mathbf{N}}\cdot\left[\mathbf{F}^k\right]^T \tag{21}$$

Using Eq. 17, the following apply:

$$\begin{aligned}
\boldsymbol{\Gamma}_{\boldsymbol{\Xi}}[h] &= \\
&= \sum_{k=0}^{\infty}\left\{\Big[\sum_{j=1}^{n\cdot s}\mathbf{G}_j\cdot\lambda_j\Big]^{k+h}\cdot\boldsymbol{\Sigma}_{\mathbf{N}}\cdot\Big\{\Big[\sum_{m=1}^{n\cdot s}\mathbf{G}_m\cdot\lambda_m\Big]^{k}\Big\}^{T}\right\} \\
&= \sum_{k=0}^{\infty}\left\{\sum_{j=1}^{n\cdot s}\mathbf{G}_j\cdot\lambda_j^{k+h}\cdot\boldsymbol{\Sigma}_{\mathbf{N}}\cdot\sum_{m=1}^{n\cdot s}\mathbf{G}_m^{T}\cdot\lambda_m^{k}\right\} \\
&= \sum_{j}\sum_{m}\mathbf{G}_j\cdot\boldsymbol{\Sigma}_{\mathbf{N}}\cdot\mathbf{G}_m^{T}\cdot\lambda_j^{h}\cdot\sum_{k=0}^{\infty}\lambda_j^{k}\cdot\lambda_m^{k} \\
&= \sum_{j}\sum_{m}\mathbf{G}_j\cdot\boldsymbol{\Sigma}_{\mathbf{N}}\cdot\mathbf{G}_m^{T}\cdot\lambda_j^{h}\cdot\frac{1}{1-\lambda_j\cdot\lambda_m} \\
&= \sum_{j=1}^{n\cdot s}\mathbf{G}_j\cdot\boldsymbol{\Sigma}_{\mathbf{N}}\cdot\sum_{m=1}^{n\cdot s}\frac{\mathbf{G}_m^{T}\cdot}{1-\lambda_j\cdot\lambda_m}\cdot\lambda_j^{h} \qquad (22)
\end{aligned}$$

Setting,

$$\mathbf{D}_j = \mathbf{G}_j\cdot\boldsymbol{\Sigma}_{\mathbf{N}}\cdot\sum_{m=1}^{n\cdot s}\frac{\mathbf{G}_m^{T}\cdot}{1-\lambda_j\cdot\lambda_m} \qquad (23)$$

the covariance matrix can be expressed as:

$$\boldsymbol{\Gamma}_{\boldsymbol{\Xi}}[h] = \sum_{j=1}^{n\cdot s}\mathbf{D}_j\cdot\lambda_j^{h} \qquad (24)$$

Equation 24 has some important features. First, as become directly evident, it has the same form as the Green function. Second, it describes the covariance matrix in terms of the spectral properties of the transition matrix (plus the noise covariance), which, as already mentioned, contains all the information about the dynamics that produce the state vector and, thus, $\mathbf{Y}[t]$. This fact leads to a third crucial feature: for $h=0$, Eq. 24 yields:

$$\boldsymbol{\Gamma}_{\boldsymbol{\Xi}}[0] = \mathbf{D}_1 + \mathbf{D}_2 + \cdots + \mathbf{D}_{n\cdot s} \qquad (25)$$

Recalling that $\boldsymbol{\Gamma}_{\boldsymbol{\Xi}}[0]$ can be treated as the multivariate equivalent of the variance (in fact its diagonal elements are the variances of each entry of $\boldsymbol{\Xi}[t]$), Eq. 25 can be used as a direct measure of the significance that every eigenvalue has, in the *total energy* of the vector time–series. This leads to the notion of *dispersion analysis*, originated in the work of (Lee and Fassois, 1993), in which the estimated modal characteristics of a vibrating system are qualified against some predefined thresholds. In the next section, a more practical version of Eq. 25 is presented, with respect to the estimation problem.

In the case that the correlation matrix is of interest, it can be calculated from (Box et al., 2008),

$$\mathbf{R}_{\boldsymbol{\Xi}}[h] = \mathbf{V}_{\boldsymbol{\Xi}}^{-1/2}\cdot\boldsymbol{\Gamma}_{\boldsymbol{\Xi}}[h]\cdot\mathbf{V}_{\boldsymbol{\Xi}}^{-1/2} \qquad (26)$$

where $\mathbf{V}_{\boldsymbol{\Xi}}$ is a diagonal matrix that contains the autocorrelations at zero lag:

$$\mathbf{V}_{\boldsymbol{\Xi}}^{-1/2} = diag\{\gamma_{11}^{-1/2}[0],\ldots,\gamma_{n\cdot s}^{-1/2}[0]\} \qquad (27)$$

3.3 Output Time–series

The previous analysis explored the advantages of the state equation and led to closed form representations for the coefficients of the Green function and the covariance matrix, which are exclusively depend on the spectrum of the transition matrix $\mathbf{F}$. Naturally, there exist strong connections between these quantities and the corresponding ones of the s–variate time–series $\mathbf{Y}[t]$. The link is just the output equation of the state–space model. The substitution of Eq. 16 to Eq. 10, under the result of Eq. 19, yields,

$$\begin{aligned}
\mathbf{Y}[t] = \mathbf{C}\cdot\boldsymbol{\Xi}[t] &= \mathbf{C}\cdot\sum_{k=1}^{\infty}\mathbf{F}^{k}\cdot\boldsymbol{N}[t-k] \\
&= \sum_{k=0}^{\infty}\mathbf{C}\cdot\mathbf{H}_{\boldsymbol{\Xi}}[k]\cdot\mathbf{C}^{T}\cdot\boldsymbol{Z}[t-k] \\
&= \sum_{k=0}^{\infty}\mathbf{H}_{\mathbf{Y}}[k]\cdot\boldsymbol{Z}[t-k] \qquad (28)
\end{aligned}$$

where:

$$\begin{aligned}
\mathbf{H}_{\mathbf{Y}}[k] &= \mathbf{C}\cdot\mathbf{H}_{\boldsymbol{\Xi}}[k]\cdot\mathbf{C}^{T} \\
&= \sum_{j=1}^{n\cdot s}\mathbf{C}\cdot\mathbf{G}_j\cdot\mathbf{C}^{T}\cdot\lambda_j^{k} \\
&= \sum_{j=1}^{n\cdot s}\boldsymbol{\Omega}_j\cdot\lambda_j^{k} \qquad (29)
\end{aligned}$$

The covariance matrix of $\mathbf{Y}[t]$ is:

$$\begin{aligned}
\boldsymbol{\Gamma}_{\mathbf{Y}}[h] &\equiv E\Big\{\mathbf{Y}[t+h]\cdot\mathbf{Y}^{T}[t]\Big\} = \\
&= \sum_{k=0}^{\infty}\mathbf{H}_{\mathbf{Y}}[k+h]\cdot\boldsymbol{\Sigma}_{\mathbf{Z}}\cdot\mathbf{H}_{\mathbf{Y}}^{T}[k] \qquad (30)
\end{aligned}$$

Recalling that,

$$\boldsymbol{\Sigma}_{\mathbf{N}} = \mathbf{C}^{T}\cdot\boldsymbol{\Sigma}_{\mathbf{Z}}\cdot\mathbf{C} \qquad (31)$$

and

$$\mathbf{H}_{\mathbf{Y}}[k] = \mathbf{C}\cdot\mathbf{H}_{\boldsymbol{\Xi}}[k]\cdot\mathbf{C}^{T} \qquad (32)$$

the following apply:

$$\begin{aligned}
\boldsymbol{\Gamma}_{\mathbf{Y}}[h] &= \sum_{k=0}^{\infty}\mathbf{C}\cdot\mathbf{H}_{\boldsymbol{\Xi}}[k+h]\cdot\mathbf{C}^{T}\cdot\boldsymbol{\Sigma}_{\mathbf{Z}}\cdot\mathbf{C}\cdot\mathbf{H}_{\boldsymbol{\Xi}}^{T}[k]\cdot\mathbf{C}^{T} \\
&= \mathbf{C}\cdot\left\{\sum_{k=0}^{\infty}\mathbf{H}_{\boldsymbol{\Xi}}[k+h]\cdot\boldsymbol{\Sigma}_{\mathbf{N}}\cdot\mathbf{H}_{\boldsymbol{\Xi}}^{T}[k]\right\}\cdot\mathbf{C}^{T} \\
&= \mathbf{C}\cdot\boldsymbol{\Gamma}_{\boldsymbol{\Xi}}[h]\cdot\mathbf{C}^{T} \qquad (33)
\end{aligned}$$

Thus, from Eq. 24,

$$\boldsymbol{\Gamma}_{\mathbf{Y}}[h] = \sum_{j=1}^{n\cdot s}\mathbf{Q}_j\cdot\lambda_j^{h} \qquad (34)$$

where,

$$\mathbf{Q}_j = \mathbf{C}\cdot\mathbf{D}_j\cdot\mathbf{C}^T \tag{35}$$

while if the correlation matrix $\mathbf{R}_\mathbf{Y}[h]$ is needed, corresponding versions of Eqs. 26– 27 apply to Eq. 34 as well.

Equations 29 and 34 show how the properties of the s–variate time–series $\mathbf{Y}[t]$ are related to the transition matrix. It is important to observe that the above analysis is strictly depended on the spectrum of $\mathbf{F}$. Indeed, when the VAR(n) structure described by Eq. 1 is available, all the information about the dynamics of the system that produces $\mathbf{Y}[t]$ can be assessed by the eigenvalue problem of $\mathbf{F}$, the state transition matrix of the state–space realization, which is identical to the block companion matrix of the polynomial matrix $\mathbf{A}(q)$. Of course, no VAR structure exists a–priori for an available data set and it rather has to be estimated. This is the topic of the next Section.

4 ESTIMATION

The estimation of VAR(n) structures pertains to the identification of the polynomial matrix order and coefficients, as well as the covariance matrix of the vector noise sequence, given observations of a s–variate times–series $\mathbf{Y}[t],\ t = 1,...,N$, that has been sampled at a period T_s. To this, the state–space realization may again be utilized, noting that, regardless the selected order n of the original VAR structure, the state equation retains the first order VAR form. In addition, Eq. 15 can be written as a linear regression,

$$\mathbf{\Xi}[t] = \mathbf{\Phi}[t]\cdot\boldsymbol{f} + \mathbf{N}[t] \tag{36}$$

with,

$$\mathbf{\Phi}[t] = -\mathbf{\Xi}^T[t-1]\otimes\mathbf{I}_{n\cdot s}\ \ [n\cdot s\times n\cdot s^2] \tag{37}$$

$$\boldsymbol{f} = vec\{\mathbf{F}\}\ \ [n\cdot s^2\times 1] \tag{38}$$

where $\otimes$ denotes Kronecker's product and $vec\{\cdot\}$ the vector that is produced by stacking the columns of the relative matrix, one underneath the other. Introducing,

$$\mathbf{\Xi} = \left[\mathbf{\Xi}^T[1]\ \ \ldots\ \ \mathbf{\Xi}^T[N]\right]^T\ \ [N\cdot n\cdot s\times 1] \tag{39}$$

$$\mathbf{\Phi} = \left[\mathbf{\Phi}[1]\ \ \ldots\ \ \mathbf{\Phi}[N]\right]^T\ \ [N\cdot n\cdot s\times n\cdot s^2] \tag{40}$$

$$\mathbf{N} = \left[\mathbf{N}^T[1]\ \ \ldots\ \ \mathbf{N}^T[N]\right]^T\ \ [N\cdot n\cdot s\times 1] \tag{41}$$

the minimization of the quadratic norm,

$$V(\boldsymbol{f}) = \frac{1}{2}\cdot\mathbf{N}^T\cdot\mathbf{\Lambda}\cdot\mathbf{N} \tag{42}$$

where $\mathbf{N} = \mathbf{\Xi} - \mathbf{\Phi}\cdot\boldsymbol{f}$ and $\mathbf{\Lambda}$ any arbitrary weighting matrix (the covariance of the residual vector $\mathbf{N}$ is presently utilized, calculated as $\mathbf{I}_N\otimes\widehat{\mathbf{\Sigma}}_\mathbf{N}^{-1}$), leads to the well-known normal equations for the least–squares estimation of $\boldsymbol{f}$,

$$\mathbf{\Phi}^T\cdot\mathbf{\Phi}\cdot\widehat{\boldsymbol{f}} = \mathbf{\Phi}^T\cdot\mathbf{\Xi} \tag{43}$$

whereas the covariance matrix associated with the estimate of Eq. 43 is:

$$\mathbf{P} = \left[\mathbf{\Phi}^T\cdot\mathbf{\Lambda}\cdot\mathbf{\Phi}\right]^{-1} \tag{44}$$

The diagonal entries of $\mathbf{P}$ are the variances of the parameter vector $\boldsymbol{f}$. Thus, assuming normality (provided that $N \gg n\cdot s^2$), the 95% confidence limits are derived from $\hat{f}_j \pm 1.96\cdot\sigma_j$ for $j = 1,\ldots,n\cdot s^2$. Note that if the zero value is contained in this interval, the relative parameter can be regarded as zero.

Having the state equation estimated, the transition to the original VAR(n) structure is designated by the matrix $\mathbf{C}$ of the state–space realization's output equation. To this, the transformation methods that were implied in Sec. 3 are applied.

5 VALIDATION

The vector time–series fitting strategy consists of finding an appropriate estimate of the order n, as well as of exploring the properties of the innovations, $\mathbf{Z}[t]$. Both may be qualified via minimization of the Bayesian Information Criterion (*BIC*), defined as,

$$BIC = ln\ det\ |\widehat{\mathbf{\Sigma}}_\mathbf{Z}| + n\cdot s^2\frac{ln\ N}{N} \tag{45}$$

while the innovations can be further tested for whiteness, using standard hypothesis tests. See (Papakos and Fassois, 2003) for details.

Once the final model has been available, complete structural information can be assessed in terms of the estimated transition matrix. Towards this, the spectrum of $\widehat{\mathbf{F}}$ is calculated, namely the eigenvalues and the eigenvectors, while using Eq. 34, the relative importance of each structural mode, within the total energy of the system is evaluated. With respect to the discussion that took place in Sec. 3.2.2, regarding the notion of the dispersion analysis, setting $h = 0$ in Eq. 34 yields:

$$\mathbf{\Gamma}_\mathbf{Y}[0] = \mathbf{Q}_1 + \mathbf{Q}_2 + \cdots + \mathbf{Q}_{n\cdot s} \tag{46}$$

Let γ_{ij} be the $[i,j]$ element of $\mathbf{\Gamma}_\mathbf{Y}[0]$. Then,

$$\gamma_{ij} = q_{1\,ij} + q_{2\,ij} + \cdots + q_{n\cdot s\,ij} \tag{47}$$

and since the eigenvalues of the transition matrix usually come as a mixture of real and complex conjugate

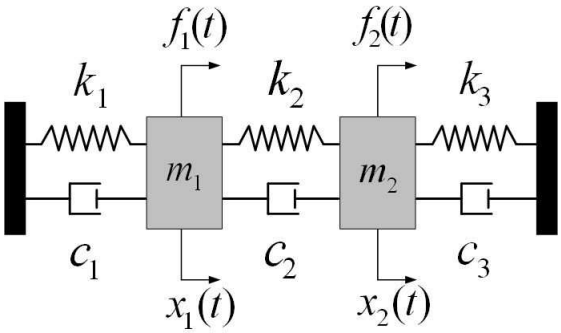

Figure 1: A structural system with two degrees of freedom: $m_1 = m_2 = 4.5\ kg$, $c_1 = 45\ Ns/m$, $c_2 = 35\ Ns/m$, $c_3 = 15\ Ns/m$, $k_1 = k_3 = 17500\ N/m$, $k_2 = 100\ N/m$

numbers, the structural dispersions within the content of the $[i, j]$ covariance estimate are defined as:
Real mode:

$$\delta_{ij\,k} = q_{ij\,k} \tag{48}$$

Complex mode:

$$\delta_{ij\,k} = q_{ij\,k} + q^*_{ij\,k} \tag{49}$$

Thus, the relative importance of each dispersion in the $[i, j]$ covariance estimate is:

$$\Delta_{ij\,k} = \frac{\delta_{ij\,k}}{\gamma_{ij}} \times 100\% \tag{50}$$

This procedure allows the determination of the contribution of the k^{th} identified mode in every element of the covariance matrix, by building corresponding $\mathbf{\Delta}_k$, which store the relative normalized dispersions $\Delta_{ij\,k}$.

6 EXPERIMENTAL VALIDATION

The method's performance was examined through the structural identification problem of a vibrating system with two degrees of freedom, presented in Fig. 1. The system is characterized by a pair of closely spaced modes, as indicated in Tab. and the vector time–series used for the identification tasks was the vibration displacement of the masses. The statistical consistency of the method was investigated via Monte Carlo analysis that consisted of 20 data records of vibration displacement time–series (with each such record having 1000 samples, see Fig. 2 for a single realization and Fig. 3 for its covariance matrix), obtained with different white excitations and noise–corrupted at 5% noise to signal (N/S) ratio. Regarding the simulation, the continuous system was discretized using the impulse–invariant transformation, at a sampling period $T_s = 0.025\ s$.

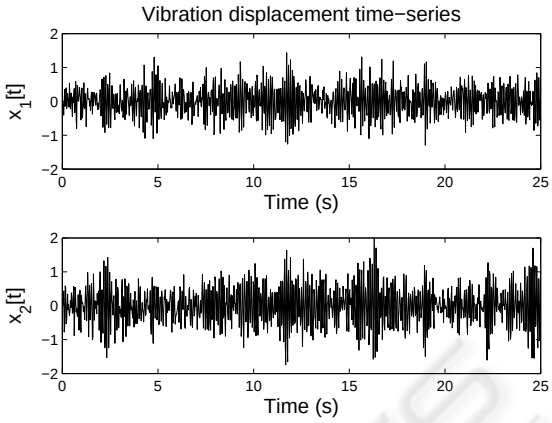

Figure 2: A realization of the noise corrupted (at 5% N/S ratio) vibration displacement time–series.

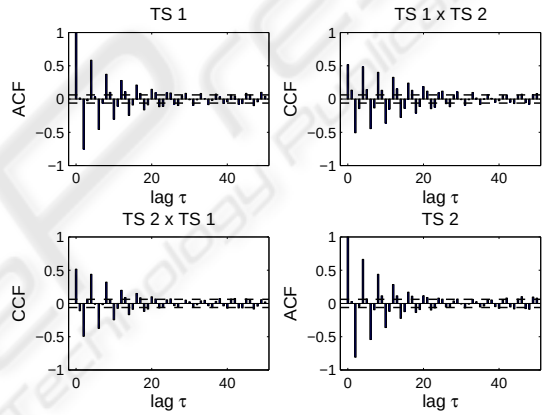

Figure 3: The correlation matrix of the series in Fig. 2 for 50 lags. TS1: $x_1[t]$, TS2: $x_2[t]$, ACF: autocorrelation, CCF: cross–correlation.

Following the estimation procedure described in Sec. 5, a VAR(2) structure was found adequate to describe system dynamics. Table 1 illustrates the estimates of the natural frequencies and the damping ratios (in fact the corresponding mean values and the standard deviations of the Monte Carlo simulation), together with the theoretical ones, from where it is clear that the method performed satisfactory and identified the relative quantities, even in the absence of the input excitations. Table 1 further displays the percentage dispersion matrices for each mode of vibration, showing that the second mode is a heavier contributor in the total energy of the system. This result coincides with the previous assessment of the specific simulated system, reported in (Fassois and Lee, 1993).

For further validation of the results, Figs. 4–5 display the theoretical correlation matrix of the estimated model for the vector time–series of Fig. 1 and the sample correlation matrix of the innovations, for the

Table 1: Theoretical / identified natural frequencies (Hz) and damping ratios and dispersions of the identified VAR(n) structure.

	Theoretical	Identified	Dispersion matrices (%) $1^{st}Mode$	$2^{nd}Mode$
w_n	9.9793	9.9903 ± 0.1446		
	9.9274	9.9307 ± 0.0636	$\begin{bmatrix} 37.68 \pm 5.06 & -32.95 \pm 8.82 \\ -37.01 \pm 10.18 & 9.58 \pm 2.73 \end{bmatrix}$	$\begin{bmatrix} 62.32 \pm 5.06 & 132.95 \pm 8.82 \\ 137.01 \pm 10.18 & 90.42 \pm 2.73 \end{bmatrix}$
ζ	0.1826	0.1848 ± 0.0174		
	0.0480	0.0477 ± 0.0073		

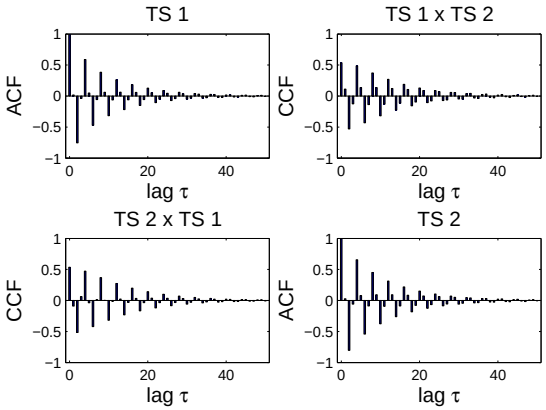

Figure 4: Theoretical correlation matrix: estimated model for the series in Fig. 2 (50 lags). Notation is the same as in Fig. 3.

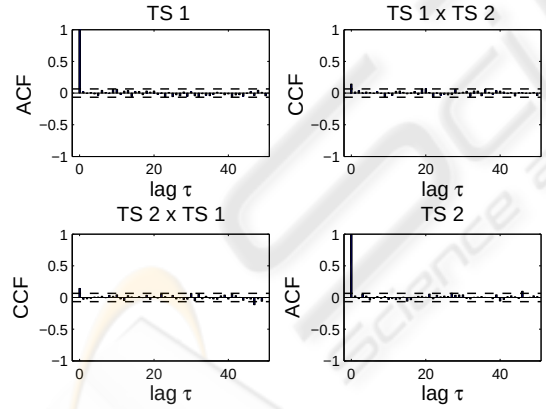

Figure 5: Sample correlation matrix: innovations of the estimated model for the series in Fig. 2 (50 lags). Notation is the same as in Fig. 3.

same model. The estimated theoretical correlation is very accurate, it follows its sample counterpart and exhibits a damped sinusoidal behavior, as a result of the identified complex conjugate eigenvalues of the transition matrix. In addition, the sample correlations of the innovations satisfy the whiteness hypothesis test, at a 95% level of significance, since they are kept within the $1.96/\sqrt{N}$ thresholds (Fig. 5, dash lines).

7 CONCLUSIONS

A novel method for the representation of vector time–series, by means of VAR(n) structures, was presented in this paper. Focusing on the estimation of structural information, the method takes advantage of the fact that every VAR(n) structure can be turn into a VAR(1) counterpart and is led to a state–space realization, whose transition matrix coincides with the block companion matrix of the VAR polynomial. Consequently, it is shown how important quantities of the original VAR(n) structure, such as the Green function and the covariance matrix, can be qualified and assessed only in terms of the spectrum of the transition matrix. This fact provides the user with the ability to accurately evaluate the significance of every structural mode in the total vector time–series energy (a technique referred to as dispersion analysis). Of the advantages of the method is the avoidance of iterative iteration schemes and the estimation of a unique structure for a given data set.

The encouraging results (reduced data acquisition, statistical consistency, accurate structural identification, no overdetermination, unique estimate) suggest the further research into this field. Extension of the method to vector time–series with structural indices governed by multiple eigenvalues, probably by means of Jordan canonical forms, as well as the investigation of VARMA models, ensues straightly. Of main interest is also the application of the method under the availability of input excitation and the expansion of its framework to non–stationary vector time–series, to closed–loop operations, as well as to fault diagnosis schemes.

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APPENDIX: MATRIX SPECTRUM

Every $[n \times n]$ matrix $\mathbf{A}$ with spectrum,

$$\sigma(\mathbf{A}) = \{\lambda_1, \lambda_2, ..., \lambda_k\}, \;\; k \leq n \tag{51}$$

has the following properties (Meyer, 2000):

- It is similar to a diagonal matrix.
- It retains a complete linearly independent set of eigenvectors.
- Every λ_j is semi–simple.

Any such matrix can be written as,

$$\mathbf{A} = \lambda_1 \cdot \mathbf{G}_1 + \lambda_2 \cdot \mathbf{G}_2 + ... + \lambda_k \cdot \mathbf{G}_k \tag{52}$$

where the $\mathbf{G}_j$'s are the, so called, spectral projectors, for which the following properties hold:

- $\mathbf{G}_1 + \mathbf{G}_2 + ... + \mathbf{G}_k = \mathbf{I}$
- $\mathbf{G}_i \cdot \mathbf{G}_j = 0, \;\; i \neq j$
- $\mathbf{G}_i^m = \mathbf{G}_i$

There are various ways to calculate the spectral projectors. Among them, the one that is presently utilized uses only the matrix $\mathbf{A}$ and its eigenvalues λ_j to compute $\mathbf{G}_j$:

$$\mathbf{G}_j = \frac{\prod\limits_{\substack{i=1 \\ i \neq j}}^{k} \left(\mathbf{A} - \lambda_i \cdot \mathbf{I} \right)}{\prod\limits_{\substack{i=1 \\ i \neq j}}^{k} \left(\lambda_j - \lambda_i \right)} \tag{53}$$