# ABOUT THE DECOMPOSITION OF RATIONAL SERIES IN NONCOMMUTATIVE VARIABLES INTO SIMPLE SERIES

Mikhail V. Foursov
*IRISA/Université de Rennes–1, Campus Universitaire de Beaulieu, 35042 Rennes Cedex, France*

Christiane Hespel
*IRISA/INSA de Rennes, 20, avenue des Buttes de Coësmes, 35043 Rennes Cedex, France*

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Abstract: Similarly to the partial fraction decomposition of rational fractions, we provide an approach to the decomposition of rational series in noncommutative variables into simpler series. This decomposition consists in splitting the representation of the rational series into simpler representations. Finally, the problem appears as a joint block–diagonalization of several matrices. We present then an application of this decomposition to the integration of dynamical systems.

## 1 INTRODUCTION

This article deals with the problem of splitting a rational formal power series into simple series. We present first well–known results on decomposition of rational series in a single variable and on reduced linear representations of a rational series in noncommutative variables.

Fliess showed that decomposition of rational formal power series can be done by joint block-diagonalization of several matrices. This is a difficult problem which was approached by numerous researchers such as Gantmacher, Jordan, Dunford and Jacobi.

The decomposition into simple series has many different applications in the dynamical system theory (such as subsystem independence, integration or stability) and in the automata theory, among others. We illustrate the application to the integration of dynamical systems.

## 2 PRELIMINARIES

In this paper, we consider a rational series $s$ with coefficients in the field $K = \mathbb{C}$. In some sections, $K$ can be taken as a semi–ring or as a commutative field.

### 2.1 Decomposition of Rational Series in a Single Variable into Simple Series

A rational series $s$ in a single variable can be rewritten as a rational fraction (Gantmacher, 1966).

**Theorem 2.1.** *Let $s = \sum_{j=0}^{\infty} s_j X^{j+1} \in K[[X]]$ be a formal power series with coefficients in a field $K$ of characteristic $0$. Then there are $2$ polynomials $P, Q \in K[X]$, such that*

$$deg(Q) < deg(P), \quad \frac{Q}{P} = \sum_{j=0}^{\infty} \frac{s_j}{X^{j+1}} \qquad (1)$$

*if and only if there is an integer $p \in \mathbb{N}$ such that the ranks of the Hankel matrices of orders $k$, $\forall k \geq p$, are all equal to $p$.*

*In this case there exist polynomials $P$ of degree $p$ and $Q$ of degree at most $p-1$. The minimal possible degree of $P$ is $p$, and the pair $(P,Q)$ is completely determined by these degree conditions and the condition that $P$ is monic. The polynomials $P$ and $Q$ are then prime.*

The proof of this theorem is based on the resolution of a system of linear equations obtained by identifying the coefficients of $X^l$. Let us remark that the finiteness condition on the rank of the Hankel matrix of $s$ expresses the recognizability of $s$, that is the rationality, for a single variable. This rational fraction can be easily split up into simple fractions of the form


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$s_i = \frac{a_i}{(1-\alpha_i X)^{r_i}}$ where $a_i, \alpha_i \in \mathbb{C}, r_i \in \mathbb{N}$, $s_i$ being expanded as a rational simple series.

**Remark.** A rational series can be considered as a weighted automaton (also known as automaton with multiplicity). The previous decomposition of $s$ as $s = \sum_{i \in I} s_i$ appears as a decomposition of the weighted automaton $\mathrm{A_s}$ of dimension $r$ into $\cup_{i \in I} \mathrm{A_{s_i}}$, where $\mathrm{A_{s_i}}$ are simple independent automata of dimension $r_i$ such that

$$\begin{cases} dim(\mathrm{A_{s_i}}) = & r_i \\ \sum_{i \in I} r_i = & r \end{cases} \quad (2)$$

## 2.2 Reduced Linear Representation of Rational Series in Noncommutative Variables

### 2.2.1 Series in Noncommutative Variables

These definitions and notations are from (Berstel and Reutenauer, 1988; Reutenauer, 1980; Salomaa and Soittola, 1978; Schützenberger, 1961). $K$ is a semi–ring.

**Definition 2.1.** *(Formal power series in noncommutative variables)*

1. *An alphabet $X$ is a nonempty finite set. Elements of $X$ are letters. The free monoid $X^*$ generated by the alphabet $X$ is the set of finite words $X_{i_1} \cdots X_{i_l}$, where $X_{i_j} \in X$, including the empty word denoted by 1. The set $X^*$ is a monoid with respect to concatenation.*
2. *A formal power series $s$ in noncommutative variables is a function*
$$s: \ X^* \to K \quad (3)$$
*The coefficient $s(w)$ of the word $w$ in the series $s$ is denoted by $\langle s|w \rangle$.*
3. *The set of formal power series $s$ over $X$ with coefficients in $K$ is denoted by $K\langle\langle X \rangle\rangle$. A structure of semi–ring is defined on $K\langle\langle X \rangle\rangle$ by the sum and the Cauchy product. Two external operations (left and right products) from $K$ to $K\langle\langle X \rangle\rangle$ are also defined. The set of polynomials is denoted by $K\langle X \rangle$.*

### 2.2.2 Rational Series in Noncommutative Variables

**Definition 2.2.** *(Rational formal power series in noncommutative variables)*

1. *The rational operations in $K\langle\langle X \rangle\rangle$ are the sum, the product, two external products as well as the Kleene star operation defined by $T^* = \sum_{n \geq 0} T^n$ for a proper series $T$ (i.e. such that $\langle T|1 \rangle = 0$).*
2. *A subset of $K\langle\langle X \rangle\rangle$ is rationally closed if it is closed under the rational operations. The smallest rationally–closed subset containing a subset $E \subseteq K\langle\langle X \rangle\rangle$ is called the rational closure of $E$.*
3. *A series $s$ is rational if $s$ is an element of the rational closure of $K\langle X \rangle$.*

### 2.2.3 Recognizable Series in Noncommutative Variables

We propose several equivalent definitions (Berstel and Reutenauer, 1988; Fliess, 1977; Fliess, 1974; Fliess, 1976; Jacob, 1980), $K$ being a commutative field.

**Definition 2.3.** *(Recognizable formal power series in noncommutative variables)*

1. *A series $s \in K\langle\langle X \rangle\rangle$ is recognizable if there exists an integer $N \geq 1$, a monoid morphism*
$$\mu : X^* \to K^{N*N} \quad (4)$$
*and 2 matrices $\lambda \in K^{1*N}$ and $\gamma \in K^{N*1}$ such that*
$$\forall w \in X^*, \ \langle s|w \rangle = \lambda \mu(w) \gamma. \quad (5)$$
2. *A series $s \in K\langle\langle X \rangle\rangle$ is recognizable if there exists an integer $N$, the rank of its Hankel matrix $H(s) = (\langle s|w_1.w_2 \rangle)_{w_1,w_2 \in X^*}$. The first row of $H(s)$ indexed by the word 1 describes $s$. The other rows are the remainders of $s$ by a word $w$. For instance, the row $L_{X_1}$ represents the right remainder of $s$ by $X_1$, denoted by $s \triangleright X_1$.*
3. *A series $s \in K\langle\langle X \rangle\rangle$ is recognizable if it is described by a finite weighted automaton obtained from its Hankel matrix remainders.*

**Definition 2.4.** *The triple $(\lambda, \mu, \gamma)$ is called a linear representation of $s$. The representation with minimal dimension is called the reduced linear representation.*

### 2.2.4 Theorem of Schützenberger

For a series in several noncommutative variables, the theorem of Schützenberger proves the equivalence between the notions of rationality and of recognizability (Schützenberger, 1961; Berstel and Reutenauer, 1988).

**Theorem 2.2.** *A formal series is recognizable if and only if it is rational.*

### 2.2.5 Finite Weighted Automaton Obtained from a Rational Series

This method is developed in (Hespel, 1998). It is based on the following theorem (Fliess, 1976; Jacob, 1980).

**Theorem 2.3.** *A formal series $s \in \mathbb{R}\langle\langle X\rangle\rangle$ is recognizable if and only if its rank $N$ is finite. Then it is recognized by a $\mathbb{R}$–matrix automaton $M = (N, \gamma, \lambda, \mu)$. Two sets of words $\{g_i\}_{1\le i\le N}$ and $\{d_j\}_{1\le j\le N}$, whose lengths are $< N$, can be determined so that the application $\chi$ from $X^*$ to $\mathbb{R}^{N\times N}$ defined by*

$$(\chi(w))_{i,j} = \langle s|g_i.w.d_j\rangle \tag{6}$$

*satisfies $\chi(w) = \chi(1)\mu(w)$ with $\chi(1)$ invertible.*

1. The method consists in extracting from the Hankel matrix $H(s)$ (whose rank is $N$) a system $B$ of $N$ row vectors $(L_{w_i})_{i\in I}$ (resp. $N$ column vectors $(C_{w_j})_{j\in J}$), indexed by some words of minimum length, such that their determinant is nonzero and such that every row (resp. every column) of $H(s)$ can be expressed as a linear combination of elements of $B$. These relations allow us to define $\forall X_k \in X$ the matrices $\mu(X_k)$ describing the action of the letter $X_k$ on the row vector $L_{w_i}$ (resp. the column vector $C_{w_j}$). The first row (resp. the first column) of $B$ defines $\lambda$. $\gamma$ is the initial vector $(1\ 0\cdots 0)^T$. The series $s$ can thus be written

$$s = \sum_{w\in X^*}\langle s|w\rangle = \sum_{w\in X^*}\lambda\mu(w)\gamma \tag{7}$$

2. We define, based on the basis $B$ and matrices $\mu(X_i)$, $\gamma$ and $\lambda$, a finite weighted (left or right) automaton $A = \{X, Q, I, A, \tau\}$ such that
   - X is the alphabet,
   - the state set is $Q = \{L_{w_i}\}_{i\in I}$ representing $\{s \triangleright w_i\}_{i\in I}$ (resp. $Q = \{C_{w_j}\}_{j\in J}$ representing $\{w_j \triangleleft s\}_{j\in J}$),
   - the first row (resp. the first column) $I$ of $B$ is the initial state,
   - every transition between states belonging to $\tau$ is labeled by a letter $X_i \in X$ and labeled by the coefficient appearing in the linear dependence relation,
   - $A$ is the final state set; it is the set of rows $L_w$ (resp. the columns $C_w$) of $B$ such that $\langle s|w\rangle \neq 0$.

# 3 DECOMPOSITION OF RATIONAL SERIES : PRINCIPLE

## 3.1 Theoretical Results

In his thesis (Fliess, 1977), M.Fliess gives the idea of a unique decomposition of the reduced matrix representation $\mu$ associated to a rational series $s$ into the direct sum of a finite number of simple representations. His idea is based on the Krull–Schmidt theorem.

Let us recall some definitions and notations (Berstel and Reutenauer, 1988; Fliess, 1977).

Let $s \in K\langle\langle X\rangle\rangle$ be a rational series. Let us denote by $\{N, \lambda, \mu(X^*), \gamma\}$, or rather by $\mu$, its reduced matrix representation. The coefficients of $s$ satisfy

$$\langle s|w\rangle = \lambda\mu(w)\gamma, \quad \forall w \in X^* \tag{8}$$

For a decomposition of $\mu$

$$\mu = \oplus_{i=1}^{k}\mu_i \tag{9}$$

the associated decompositions of the vectors $\lambda$ and $\gamma$ are

$$\lambda = \oplus_{i=1}^{k}\lambda_i, \qquad \gamma = \oplus_{i=1}^{k}\gamma_i \tag{10}$$

The series $s$ is then split up into $s = \sum_{i=1}^{k} s_i$, where every rational series satisfies

$$s_i = \sum_{w\in X^*}(\lambda_i\mu_i(w)\gamma_i)w \tag{11}$$

Among $\{s_i\}_{1\le i\le k}$ there can exist a subfamily with indices $J \subseteq \{1, \cdots, k\}$ such that $\forall j \in J$, the representation $\mu_j$ is nilpotent.

- *A representation $\mu_i$ is nilpotent* if and only if $\forall w \in X^+$, $\mu_j(w)$ is nilpotent.

Using Levitzki theorem (Kaplanski, 1969), the semi–group of nilpotent matrices $\{\oplus_{j\in J}\ \mu_j(w),\ w \in X^+\}$ is simultaneously triangulable. Particularly, for every word $w$ of sufficient length, $\oplus_{j\in J}\ \mu_j(w)$ is the zero matrix. Then the sum $\sum_{j\in J}\ s_j$ of the series associated to this decomposition into nilpotent matrices is a polynomial representing the polynomial part of $s$.

Let us consider now the representations which cannot be decomposed and which are not nilpotent.

- Such a representation $\mu_i$ is associated with a simple series $s_i$.
- Two series $s_1$ and $s_2$ are called relatively prime if and only if

$$\begin{gathered}\forall\alpha,\ \beta \in C\backslash\{0\},\\ rank(\alpha s_1 + \beta s_2) = rank(s_1) + rank(s_2)\end{gathered} \tag{12}$$

We can express the following theorem (Fliess, 1977)

**Theorem 3.1.** *K being a field, there is a unique way for decomposing every rational series $s \in K\langle\langle X\rangle\rangle$ into the sum of its polynomial part and of some simple rational relatively prime series.*

### 3.2 Approaches of the Simultaneous Decomposition of Matrices $\{A_i\}_{i\in I}$

We restrict the number of matrices to two in order to simplify the explanations. The problem is the following : to provide a simultaneous decomposition of $A_1$ and $A_2$ into a nilpotent part $A_{1_n}, A_{2_n}$ and a block–diagonalizable part $A_{1_d}, A_{2_d}$, in some basis.

This problem is difficult. We present some approaches from Gantmacher, Jordan, Dunford and Jacobi.

1. **First Approach : Gantmacher**

   Gantmacher considers the linear pencil $A_1 + \lambda A_2$ of the matrices $A_1, A_2$. By using elementary transformations, ((Gantmacher, 1966), tome 1, Chapter 2), the original regular/singular pencil can be reduced to a quasi–diagonal canonical form ((Gantmacher, 1966), tome 2, Chapter 12). The original pencil $A_1 + \lambda A_2$ and the canonical pencil $A'_1 + \lambda A'_2$ are then equivalent but generally not similar : there exist some regular matrices $P, Q$ such that $A'_1 + \lambda A'_2 = P(A_1 + \lambda A_2)Q$ but generally $Q \neq P^{-1}$.

2. **Second Approach : Jordan, Dunford**

   These methods are suitable for a single matrix. The Jordan's method consists in computing 2 regular matrices $P, Q$ and irreducible block diagonal matrices $A'_1, A'_2$ such that

   $$A_1 = P^{-1}A'_1P, \;\; A_2 = Q^{-1}A'_2Q. \qquad (13)$$

   So one can use the Jordan decomposition $A'_1$ and $A'_2$ of each matrix in order to initialize a simultaneous decomposition in block diagonal matrices of suitable size. The knowledge of the eigenspaces $(E_{1_i})$ and $(E_{2_i})$ of $A_1$ and $A_2$ allows to set some bounds on the size of the blocks.

   The Dunford decomposition into a diagonalizable part and a nilpotent part can be provided from the Jordan decomposition.

3. **Approach by Jacobi Algorithms**

   When the sizes of the decomposition blocks are known, the method consists in providing a joint block–diagonalizer. This matrix is iteratively computed as a product of Givens rotations. The convergence of this algorithm is proven but not necessary to obtain an optimal solution.

## 4 DECOMPOSITION OF RATIONAL SERIES IN PRACTICE

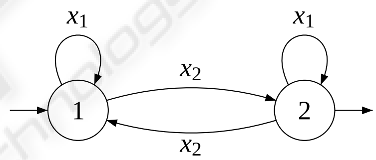
**Theorem 4.1.** *A rational series can be decomposed into a sum of simpler series using matrix joint block–decomposition.*

*Proof.* Let $s$ be a rational series $s = \sum_{w\in X^*} \langle s|w\rangle = \sum_{w\in X^*} \lambda\mu(w)\gamma$. For a simultaneous change of basis matrix $P$ for $\mu(x_{i_j})_{i_j}$, we have

$$\begin{aligned}\langle s|x_{i_1}\cdots x_{i_l}\rangle &= \lambda\mu(x_{i_1})\cdots\mu(x_{i_l})\gamma = \\ &= \lambda P\mu'(x_{i_1})P^{-1}\cdots P\mu'(x_{i_l})P^{-1}\gamma \\ &= (\lambda P)\mu'(x_{i_1})\cdots\mu'(x_{i_l})(P^{-1}\gamma) = \\ &= \lambda_P\mu_P(x_{i_1})\cdots\mu_P(x_{i_l})\gamma_P\end{aligned} \qquad (14)$$

Thus, when $\mu'(x_{i_1}), \cdots, \mu'(x_{i_l})$ are decomposed into block–diagonal matrices, we obtain the decomposition of $s$ into corresponding simpler series. ☐

**Example 1.** A representation of the series is given by the finite weighted automaton




The actions of the letters $x_1$ and $x_2$ are given by the matrices

$$\mu(x_1) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \text{and} \quad \mu(x_2) = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad (15)$$

The initial vector is

$$\gamma = \begin{pmatrix}1\\ 0\end{pmatrix} \qquad (16)$$

and the covector is

$$\lambda = \begin{pmatrix}0 & 1\end{pmatrix}. \qquad (17)$$

The eigenvalues of $\mu(x_2)$ are $\lambda_1 = 1$ and $\lambda_2 = -1$. In the basis $B$ of the eigenvectors, the matrices $\mu(x_1)$ and $\mu(x_2)$ are

$$\mu(x_1)_P = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \;\text{and}\; \mu(x_2)_P = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad (18)$$

The initial vector is now

$$\gamma_P = \begin{pmatrix}1/2\\ 1/2\end{pmatrix} \qquad (19)$$

and the covector is

$$\lambda_P = \begin{pmatrix}1 & -1\end{pmatrix}. \qquad (20)$$

Thus this series can be decomposed into series $s_1$ and $s_2$ : $s = s_1 + s_2$. The representation of $s_1$ is

$$\mu_1(x_1) = (1),\ \mu_1(x_2) = (1),\ \gamma_1 = (1/2),\ \lambda_1 = (1). \tag{21}$$

For $s_2$ we have

$$\mu_2(x_1) = (1),\ \mu_2(x_2) = (-1),\ \gamma_1 = (1/2),\ \lambda_1 = (-1). \tag{22}$$

**Example 2.** Now let us consider the series with the following representation. The actions of the letters $x_1$ and $x_2$ are given by the matrices

$$\mu(x_1) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mu(x_2) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{23}$$

The initial vector is

$$\gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{24}$$

and the covector is

$$\lambda = \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{25}$$

There is no decomposition of $s$.

**Example 3.** Finally, let us consider the series whose Hankel matrix is shown in Table 1.

The rank of this Hankel matrix is 6. We select the independent rows $\{L_1, L_{x_1}, L_{x_2}, L_{x_1x_2}, L_{x_2x_1x_2}, L_{x_1x_2x_1x_2}\}$ and the columns associated with the same words. This determinant has a maximal rank $= 6$.

The matrices $\mu(x_1)$ et $\mu(x_2)$ describe the action of the letters $x_1$ and $x_2$.

$$\mu(x_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{26}$$

and

$$\mu(x_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{27}$$

The initial vector is

$$\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \tag{28}$$

and the covector is

$$\lambda = \begin{pmatrix} 3 & 1 & 1 & 3 & 1 & 2 \end{pmatrix}. \tag{29}$$

By using the Jordan reduction on $\mu(x_1)$ (with Maple) we obtain

$$A = \mu(x_1)_P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{30}$$

where the change of basis matrix is

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{31}$$

By this change of basis, $\mu(x_2)$ becomes

$$B = \mu(x_2)_P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{32}$$

In this new basis

$$\lambda_P = \begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix} \tag{33}$$

and

$$\gamma_P = \begin{pmatrix} 0 & 1 & 1 & 0 & -1 & 0 \end{pmatrix}^T \tag{34}$$

In this case, we are lucky and the matrices $A$ and $B$ corresponding to $\mu(x_1)_P$ and $\mu(x_2)_P$ in the same basis directly present 3 diagonal blocks :

- the upper left block of size 2 corresponding to the series $s_1 = \dfrac{1}{1 - x_1x_2}$,
- the middle block of size 1 corresponding to the series $s_2 = \dfrac{1}{1 - (x_1 + x_2)}$,
- the lower right block of size 3 corresponding to the polynomial $s_3 = 1 + x_1x_2$. This last block is associated to a nilpotent representation.

## 5 AN APPLICATION TO DYNAMICAL SYSTEMS

**Definition 5.1.** *A bilinear dynamical system is a system of ordinary differential equations of the form*

$$\begin{cases} \dot{\mathbf{q}}(t) = \left(M_0 + \sum_{i=1}^{m} u_i(t) M_i\right) \mathbf{q}(t) \\ s(t) = \lambda \cdot \mathbf{q}(t), \end{cases} \tag{35}$$

*where*

Table 1: Hankel matrix of example 3.

| | 1 | $x_1$ | $x_2$ | $x_1^2$ | $x_1x_2$ | $x_2x_1$ | $x_2^2$ | $x_1^3$ | $x_1^2x_2$ | $x_1x_2x_1$ | $x_1x_2^2$ | $x_2x_1^2$ | $x_2x_1x_2$ | $x_2^2x_1$ | $x_2^3\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $1\cdots$ |
| $x_1$ | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | $1\cdots$ |
| $x_2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $1\cdots$ |
| $x_1^2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $1\cdots$ |
| $x_1x_2$ | 3 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $1\cdots$ |
| $x_2x_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $1\cdots$ |
| $x_2^2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $1\cdots$ |
| $x_1^3$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $1\cdots$ |
| $x_1^2x_2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $1\cdots$ |
| $x_1x_2x_1$ | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | $1\cdots$ |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |

1. $\mathbf{u}(t) = (u_1(t),\ldots,u_n(t)) \in \mathbb{R}^n$ *is the (partwise continuous) input vector,*
2. $\mathbf{q}(t) \in \mathcal{M}$ *is the current state, where* $\mathcal{M}$ *is a real differential manifold, usually* $\mathbb{R}^m$,
3. $s(t) \in \mathbb{R}$ *is the output function.*

**Definition 5.2.** *The generating series G of a bilinear dynamical system (Fliess, 1981) is a formal power series with the alphabet* $X = \{z_o, z_1, \ldots z_m\}$, *where* $z_i$ *for* $j > 0$ *correspond to the input* $u_i(t)$ *whereas* $z_0$ *corresponds to the drift. It is defined by*

$$\langle G | z_{j_0} \cdots z_{j_k} \rangle = \lambda \cdot M_{j_0} \cdots M_{j_k} \cdot \mathbf{q}(0). \tag{36}$$

**Theorem 5.1.** *The generating series of bilinear dynamical system are rational. Inversely, every rational series is a generating series of a bilinear dynamical system.*

*Proof.* We take $\mu$ such that $\mu(z_i) = M_i$ for $i \geq 0$ and we denote $\gamma = \mathbf{q}(0)$. It follows directly that $\langle \lambda, \mu, \gamma \rangle$ is a rational series. □

**Definition 5.3.** *The Chen series measures the input contribution (Chen, 1971), and is independent of the system. The coefficients of the Chen series are calculated recursively by integration using the following two relations :*

- $\langle \mathcal{C}_u(t) | 1 \rangle = 1$,
- $\langle \mathcal{C}_u(t) | w \rangle = \int_0^t \langle \mathcal{C}_u(\tau) | v \rangle u_j(\tau) d\tau$ *for a word* $w = z_j v$.

The causal functional $y(t)$ is then obtained locally as the product of the generating series and the Chen series :

$$y(t) = \langle G || \mathcal{C}_u(t) \rangle = \sum_{w \in X^*} \langle G | w \rangle \langle \mathcal{C}_u(t) | w \rangle \tag{37}$$

This formula is known as the *Peano–Baker formula*, as well as the *Fliess' fundamental formula*.

Now we apply the decomposition in the 3 above examples to the corresponding dynamical systems (identifying $z_0$ with $x_1$ and $z_1$ with $x_2$).

**Example 1.** The corresponding dynamical system is

$$\begin{cases} y_1'(t) = y_1(t) + u(t) y_2(t), & y_1(0) = 1, \\ y_2'(t) = y_2(t) + u(t) y_1(t), & y_2(0) = 0, \\ s(t) = y_2(t). \end{cases} \tag{38}$$

Maple gives its solution is some complicated form. However using our decomposition into two dynamical systems

$$\bar{y}_1'(t) = \bar{y}_1(t)(1 + u(t)), \quad \bar{y}_1(0) = \frac{1}{2}, \quad s_1(t) = \bar{y}_1(t) \tag{39}$$

and

$$\bar{y}_2'(t) = \bar{y}_2(t)(1 - u(t)), \quad \bar{y}_2(0) = \frac{1}{2}, \quad s_2(t) = -\bar{y}_2(t) \tag{40}$$

we can easily obtain that

$$\begin{aligned} s(t) &= s_1(t) + s_2(t) = \\ &= \frac{1}{2}\left( exp \int_0^t (1 + u(\tau)) d\tau) - exp \int_0^t (1 - u(\tau)) d\tau \right) \end{aligned} \tag{41}$$

**Example 2.** The corresponding dynamical system is

$$\begin{cases} y_1'(t) = u(t)(y_1(t) + y_2(t)), & y_1(0) = 1, \\ y_2'(t) = y_1(t) + y_2(t), & y_2(0) = 0, \\ s(t) = y_2(t). \end{cases} \tag{42}$$

We can compute its solution directly

$$s(t) = \int_0^t exp\left( \int_0^{\tau_1} (1 + u(\tau_2) d\tau_2 \right) d\tau_1. \tag{43}$$

$s(t)$ cannot be decomposed as a sum of two simpler expressions.

**Example 3.** The corresponding dynamical system cannot be solved directly. However, using the above decomposition we obtain $s(t) = s_1(t) + s_2(t) + s_3(t)$, where

$$\begin{aligned} s_1(t) =& 1 + \int_0^t \int_0^{\tau_1} u(\tau_2) d\tau_2 d\tau_1 + \\ & \int_0^t \int_0^{\tau_1} u(\tau_2) \int_0^{\tau_2} \int_0^{\tau_3} u(\tau_4) d\tau_4 d\tau_3 d\tau_2 d\tau_1 + \cdots \end{aligned} \tag{44}$$

corresponds to the first dynamical system and is the solution of the system

$$\begin{cases} y_1'(t) = s_2(t), & y_1(0) = 0, \\ y_2'(t) = u(t)s_1(t), & y_2(0) = 1, \\ s_1(t) = y_2(t). \end{cases} \tag{45}$$

whereas

$$s_2(t) = exp\left(\int_0^t (1 + u(\tau)) d\tau\right) \tag{46}$$

corresponds to the second dynamical system and

$$s_3(t) = 1 + \int_0^t \int_0^{\tau_1} u(\tau_2) d\tau_2 d\tau_1 \tag{47}$$

is the solution of the third system.

# 6 CONCLUSIONS

In this paper, we presented an approach to the problem of decomposition of rational series in noncommutative variables into some simple series. The study of the simultaneous block–diagonalization has yet to be improved. We present an application of this decomposition to dynamical systems.

There are numerous further applications of this decomposition to dynamical systems and automata :

- The study of the stability of bilinear systems can be approached by using its generating series (Benmakrouha and Hespel, 2007) : in some cases, the output can be explicitly computed or bounded. The decomposition of this series into simple series would simplify this study in the other cases.
- In a bilinear system, the dependence or the independence of subsystems can be studied via the decomposition of the generating series of the system.
- A finite weighted automaton being another representation of a rational series, the property of decomposition of a rational series into simpler series is transferred to the corresponding finite weighted automaton. So we can define a simpler finite weighted automaton.

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