SIMPLE DESIGN OF THE STATE OBSERVER FOR LINEAR TIME-VARYING SYSTEMS

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Abstract: A simple design method of the Luenberger observer for linear time-varying systems is proposed in this paper. The paper first propose the simple calculation method to derive the pole placement feedback gain vector for linear time-varying systems. For this purpose, it is shown that the pole placement controller can be derived simply by finding some particular ”output signal” such that the relative degree from the input to this output is equal to the order of the system. Using this fact, the feedback gain vector can be calculated directly from plant parameters without transforming the system into any standard form. Then, this method is applied to the design of the observer, i.e., because of the duality of linear time-varying system, the state observer can be derived by un-stabilization of the state error equation.

1 INTRODUCTION

The design of the pole placement and the state observer for linear time-varying systems is well established problem. As for the linear time-invariant case, if the system is controllable, the pole placement controller can be designed, and, if observable, the state observer can be designed. However, many of those design method need a complicated calculation procedure. In this paper, a simple design method of the Luenberger observer for linear time varying systems is proposed.

Since, the observer design problem is the dual problem of the pole placement, simplified calculation method to derive the pole placement feedback gain vector for linear time-varying systems should be considered first. Usually, the pole placement procedure needs the change of variable to the Flobenius standard form, and hence, is very complicated (e.g., Michael Valášek and Nejat Olgaç). To simplify this procedure, it will be shown that the pole placement controller can be derived simply by finding some particular ”output signals” such that the relative degree from the input to this output is equal to the order of the system. This is motivated from the fact that the input-output linearization of a certain type of nonlinear systems is equivalent to the entire state linearization, if the relative degree of the system is equal to the system order. Using this fact, the feedback gain vector can be calculated directly from plant parameters without transforming the system into any standard form.

Because of the duality of the linear time-varying system, the state observer can be derived by unstabilizing the state error eqation. This implies that the simplified pole placement technique can be applied to the design of the state observer for linear time-varying systems to obtain simpler design method than existing methods from the point of view of the calculational compexity.

In the sequel, the simple pole placement technique is proposed in Section 2, and then, this method is used to the observer design problem in Section 3.

2 POLE PLACEMENT OF LINEAR TIME-VARYING SYSTEMS

Consider the following linear time-varying system with a single input.

$$\dot{x} = A(t)x + b(t)u \tag{1}$$

Here, $x \in R^n$ and $u \in R^1$ are the state variable and the input signal respectively. $A(t) \in R^{n\times n}$ and $b(t) \in R^n$ are time-varying parameter matrices. The problem is to find the state feedback

$$u = k^T(t)x \tag{2}$$

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which makes the closed loop system equivalent to the time invariant linear system with arbitrarily stable poles.

Now, consider the problem of finding a new output signal $y(t)$ such that the relative degree from u to y is n. Here, $y(t)$ has the following form.

$$y(t) = c^T(t)x(t) \tag{3}$$

Then, the problem is to find a vector $c(t) \in R^n$ that satisfies this condition.

Lemma 1. Let $c_k^T(t)$ be defined by the following equation.

$$c_k^T(t) = \dot{c}_{k-1}^T(t) + c_{k-1}^T(t)A(t), \quad c_0^T(t) = c^T(t) \tag{4}$$

The relative degree from u to y defined by (3) is n, if and only if

$$\begin{aligned} c_0^T(t)b(t) &= c_1^T(t)b(t) = \cdots = c_{n-2}^T(t)b(t) = 0 \\ c_{n-1}^T(t)b(t) &= 1 \end{aligned} \tag{5}$$

(Here, $c_{n-1}^T(t)b(t) = 1$ without loss of generality.)
Proof : By differentiating y successively using (5), the following equations are obtained from (1) and (3).

$$\begin{aligned} y &= c^T(t)x \\ &= c_0^T(t)x \\ \dot{y} &= \left(\dot{c}^T(t) + c^T(t)A(t)\right)x + c^T(t)b(t)u \\ &= c_1^T(t)x + c_0^T(t)b(t)u \\ &= c_1^T(t)x \\ \ddot{y} &= \left(\dot{c}_1^T(t) + c_1^T(t)A(t)\right)x + c_1^T(t)b(t)u \\ &= c_2^T(t)x + c_1^T(t)b(t)u \\ &= c_2^T(t)x \qquad\qquad (6) \\ &\vdots \\ y^{(n-1)} &= c_{n-1}^T(t)x + c_{n-2}^T(t)b(t)u \\ &= c_{n-1}^T(t)x \\ y^{(n)} &= c_n^T(t)x + c_{n-1}^T(t)b(t)u \\ &= c_n^T(t)x + u \end{aligned}$$

This implies that the relative degree from u to y is n. ∇∇

Lemma 2. If $c^T(t)$ satisfies the condition that the relative degree from u to y is n, then we have the following equation.

$$\begin{aligned} &[c_0^T(t)b(t),\ c_1^T(t)b(t),\ \cdots,\ c_{n-1}^T(t)b(t)] \\ &\quad = [c^T(t)b_0(t),\ c^T(t)b_1(t),\ \cdots,\ c^T(t)b_{n-1}(t)] \end{aligned} \tag{7}$$

where $b_i(t)$ is defined by

$$b_i(t) = A(t)b_{i-1}(t) - \dot{b}_{i-1}(t), \quad b_0(t) = b(t) \tag{8}$$

Proof : First, the following is trivial.

$$c_0^T(t)b(t) = c^T(t)b(t) = c^T(t)b_0(t) \tag{9}$$

From (5), we have

$$\dot{c}_0^T(t)b(t) = -c_0^T(t)\dot{b}(t) \tag{10}$$

which implies

$$\begin{aligned} c_1^T(t)b(t) &= \dot{c}_0^T(t)b(t) + c_0^T(t)A(t)b(t) \\ &= -c_0^T(t)\dot{b}(t) + c_0^T(t)A(t)b(t) \\ &= c_0^T(t)b_1(t) \\ &= c^T(t)b_1(t) \end{aligned} \tag{11}$$

In a similar fashion, from (5) and (11), we have

$$\begin{aligned} \dot{c}_0^T(t)b_1(t) &= -c_0^T(t)\dot{b}_1(t) \\ \dot{c}_1^T(t)b(t) &= -c_1^T(t)\dot{b}(t) \end{aligned} \tag{12}$$

which implies

$$\begin{aligned} c_2^T(t)b(t) &= \dot{c}_1^T(t)b(t) + c_1^T(t)A(t)b(t) \\ &= -c_1^T(t)\dot{b}(t) + c_1^T(t)A(t)b(t) \\ &= c_1^T(t)b_1(t) \\ &= \dot{c}_0^T(t)b_1(t) + c_0^T(t)A(t)b_1(t) \\ &= -c_0^T(t)\dot{b}_1(t) + c_0^T(t)A(t)b_1(t) \\ &= c_0^T(t)b_2(t) \\ &= c^T(t)b_2(t) \end{aligned} \tag{13}$$

By continuing the same process, (7) is derived. ∇∇

From Lemma 2, (5) implies

$$\begin{aligned} &[c_0^T(t)b(t), c_1^T(t)b(t), \cdots,\ c_{n-1}^T(t)b(t)] \\ &\quad = [c^T(t)b_0(t),\ c^T(t)b_1(t),\ \cdots,\ c^T(t)b_{n-1}(t)] \\ &\quad = c^T(t)[b_0(t),\ b_1(t),\ \cdots,\ b_{n-1}(t)] \\ &\quad = c^T(t)U_c(t) \\ &\quad = [0,\ 0,\ \cdots,\ 1] \end{aligned} \tag{14}$$

Here,

$$U_c(t) = [b_0(t),\ b_1(t),\ \cdots,\ b_{n-1}(t)] \tag{15}$$

where, $U_c(t)$ the controllability matrix for linear time-varying system (1). If $U_c(t)$ is nonsingular for all $t \in [0, \infty)$, the system is said to be controllable. Hence, we have the following Theorem.

Theorem 1. If the system (1) is controllable, there exists a vector $c(t)$ such that the relative degree from u to $y = c^T(t)x$ is n. And, such a vector, $c(t)$ is given by

$$c^T(t) = [0,\ 0,\ \cdots,\ 1]U_c^{-1}(t) \tag{16}$$

∇∇

The next step is to derive the state feedback for the arbitrary pole placement. Let $q(p)$ be a desired stable polynomial of the differential operator, p, i.e.,

$$q(p) = p^n + \alpha_{n-1}p^{n-1} + \cdots + \alpha_0 \tag{17}$$

By multiplying $y^{(i)}$ by α_i $(i = 0, \cdots, n-1)$ and then summing them up, the following equation is obtained, using (5) and (6).

$$q(p)y = d^T(t)x + u \tag{18}$$

where $d(t) \in R^n$ is defined by the following.

$$d^T(t) = [\alpha_0, \alpha_1, \cdots, \alpha_{n-1}, 1] \begin{bmatrix} c_0^T(t) \\ c_1^T(t) \\ \vdots \\ c_{n-1}^T(t) \\ c_n^T(t) \end{bmatrix} \tag{19}$$

Hence, the state feedback,

$$u = -d^T(t)x + r \tag{20}$$

makes the closed loop system as follows.

$$q(p)y = r \tag{21}$$

where r is an external input signal. This method is regarded as an extension of Ackermann's pole placement method to the time-varying case.

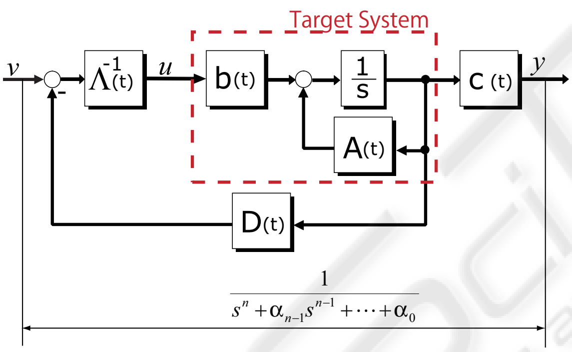

Figure 1: Blockdiagram of Pole Placement for a Linear Time-Varying System.

This control system can be summarized as follows. The given system is

$$\dot{x} = A(t)x + b(t)u \tag{22}$$

and, using (16) and (19), the state feedback for the pole placement is given by

$$u = -d^T(t)x. \tag{23}$$

Then, the closed loop system becomes

$$\dot{x} = (A(t) - b(t)d^T(t))x. \tag{24}$$

At the same time, we have (21) as another representation of the closed loop system. This can be explained as follows.

Let $T(t)$ be the time varying matrix defined by

$$T(t) = \begin{bmatrix} c_0(t)^T \\ c_1(t)^T \\ \vdots \\ c_{n-1}^T(t) \end{bmatrix} \tag{25}$$

and define the new state variable w by

$$x = T(t)w, \qquad w = \begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} \tag{26}$$

Theorem 2. If the system (1) is controllable, then, the matrix for the change of variable, $T(t)$, given by (25) is nonsingular for all t. $\nabla\nabla$

This theorem can be proved by simple calculation as for the time invariant case.

Then, (24) is transformed into

$$\begin{aligned} \dot{w} &= \{T(t)(A(t) - b(t)d^T(t))T^{-1}(t) - T(t)\dot{T}^{-1}(t)\}w \\ &= \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & 1 \\ -\alpha_0 & \cdots & \cdots & -\alpha_{n-1} \end{bmatrix} w = A^* w \end{aligned} \tag{27}$$

This implies that the closed loop system is equivalent to the time invariant linear system which has the desired closed loop poles. ($\det(pI - A^*) = q(p)$)

3 STATE OBSERVER

In this section, we consider the design of the observer for the following linear time-varying system.

$$\begin{aligned} \dot{x} &= A(t)x + b(t)u \\ y &= c^T(t)x \end{aligned} \tag{28}$$

Here, $y \in R$ is the output signal of this system. The problem is to design the full order state observer of (28). Consider the following system as a candidate of the observer.

$$\begin{aligned} \dot{z} &= F(t)z + b(t)u + h(t)y \\ &= F(t)z + b(t)u + h(t)c^T(t)x \end{aligned} \tag{29}$$

where $F(t) \in R^{n \times n}$, and $h(t) \in R^n$. Define the state error $e \in R^n$ by

$$e = x - z \tag{30}$$

Then, e satisfies the following error equation.

$$\dot{e} = F(t)e + (A(t) - F(t) - h(t)c^T(t))x \tag{31}$$

Hence, (29) is a state observer of (28) if $F(t)$ and $h(t)$ satisfy the following condition.

$$\begin{array}{rcl} F(t) & = & A(t) - h(t)c^T(t) \\ F(t) & : & \text{arbitrarily stable matrix} \end{array} \qquad (32)$$

Consider the pole placement control problem of the following system.

$$\dot{x} = -A^T(t)x + c(t)u \qquad (33)$$

From the property of the duality of the time varying system, if the pair $(A(t), c^T(t))$ is observable, the pair $(-A^T(t), c(t))$ is controllable. This implies that if the system (28) is observable, there is a state feedback for the arbitrary pole placement for the system (33). Let $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ be the set of desired stable closed loop poles.

Suppose that

$$u = k^T(t)x \qquad (34)$$

is the state feedback for (33) with the desired **unstable** closed loop poles, $(-\lambda_1, -\lambda_2, \cdots, -\lambda_n)$. The closed loop system is

$$\dot{x} = (-A^T(t) + c(t)k^T(t))x \qquad (35)$$

This implies that, using the appropriate change of variable, $x = P(t)w$, (35) can be transformed into the following time invariant system.

$$\begin{array}{rcl} \dot{w} & = & \{P^{-1}(t)(-A^T(t) + c(t)k^T(t))P(t) \\ & & \qquad - P^{-1}(t)\dot{P}(t)\}w \\ & = & -F^{*T}w \end{array} \qquad (36)$$

Here, the eigenvalues of $-F^*$ are $(-\lambda_1, -\lambda_2, \cdots, -\lambda_n)$.

It is also well known that if the fundamental matrices of (35) and its dual system,

$$\dot{x} = (A(t) - k(t)c^T(t))x \qquad (37)$$

are $\Phi(t, t_0)$ and $\Psi(t, t_0)$, respectively, then,

$$\Phi(t, t_0) = \Psi^T(t_0, t). \qquad (38)$$

Furthermore, by the change of variable,

$$x = (P^T(t))^{-1}\xi \qquad (39)$$

(35) is transformed into

$$\begin{array}{rcl} \dot{\xi} & = & (P^T(A(t) - k(t)c^T(t))(P^T)^{-1} - P^T(\dot{P}^T)^{-1})\xi \\ & = & (P^T(A(t) - k(t)c^T(t))(P^T)^{-1} + \dot{P}^T(P^T)^{-1})\xi \\ & = & \left(P^{-1}(A^T(t) - c(t)k^T(t))P + P^{-1}\dot{P}\right)^T\xi \\ & = & F^*\xi \end{array} \qquad (40)$$

Hence, by choosing

$$h(t) = k(t) \qquad (41)$$

(29) becomes the observer for (28), and the state error equation becomes

$$\dot{e} = F(t)e \qquad (42)$$

which is equivalent to (40). That is, if the system (28) is observable, it is possible to design $h(t)$ so that the state estimation error equation is equivalent to the time invariant homogeneous system which has the arbitrary stable poles.

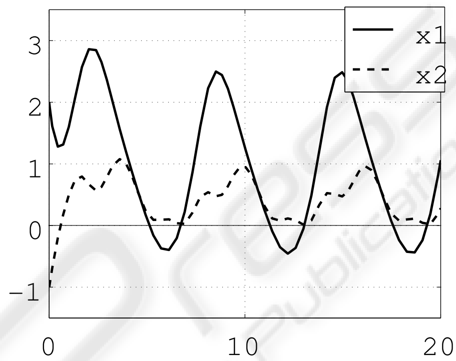

Figure 2: Responce of the state variable (x) of the system.

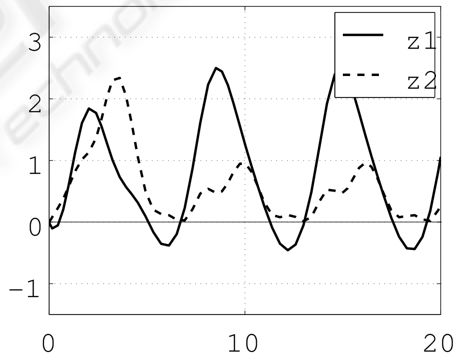

Figure 3: Responce of the state variable of the observer (z).

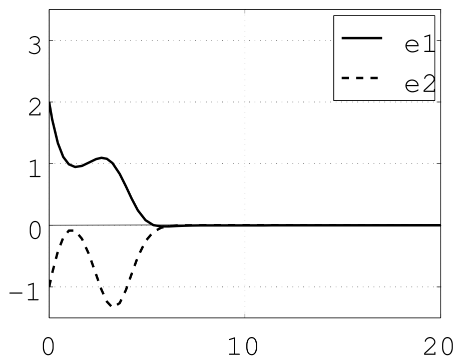

Figure 4: Responce of the state error ($e = x - z$).

Example 1. Consider the following system.

$$\begin{aligned} \dot{x} &= A(t)x + b(t)u \\ y &= c^T(t)x \end{aligned} \qquad (43)$$

where

$$\begin{aligned} A(t) &= \begin{bmatrix} -1, & 1 \\ -1+\sin 2t - \cos t, & -3+\cos t \end{bmatrix} \\ b(t) &= \begin{bmatrix} 2+\sin t \\ 0 \end{bmatrix} \\ c^T(t) &= \begin{bmatrix} 2+\sin 0.5t & 0 \end{bmatrix} \end{aligned} \qquad (44)$$

This is a stable time-varying observable system. Fig.2 shows the response of the state variable of this system with

$$u = \sin t \qquad (45)$$

The state observer is the following.

$$\dot{z} = (A(t) - h(t)c^T(t))z + b(t)u + h(t)y \qquad (46)$$

where we choose the desired observer poles as -1 and -2. (The numerical details are omitted in this draft paper.)

Fig.3 and 4 show the state variable of the observer and the state error.

4 CONCLUSIONS

In this paper, one design method for the state observer for linear time-varying systems is proposed. We first proposed the simple calculation method for the pole placement state feedback gain for liner time-varying system. Feedback gain can be derived directly from the plant parameter without the transformation into any standard form.

In this method, since the transformation of the given system into the Flobenius standard form is not required, the design procedure is very simple. It was shown that if the system is observable, then the state observer can be obtained with arbitrarily stable observer poles.

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