ANALYTICAL KINEMATICS FRAMEWORK FOR THE CONTROL OF A PARALLEL MANIPULATOR

A Generalized Kinematics Framework for Parallel Manipulators

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Abstract: Forward and inverse kinematics operations are important in the operational space control of mechanical manipulators. In case of a parallel manipulator, the forward kinematics function relates the joint variables of the active joints to the position of end-effector. This paper finds analytically forward kinematics function by exploiting the position-closure property. Using the proposed function along with the analytical Jacobian presented in the literature, the forward and the inverse kinematics blocks are formulated for a prospective operational space control scheme. Finally, an example is presented for a 3-R$\underline{P}$R robot.

1 INTRODUCTION

The end-effector of a parallel manipulator is connected to its *base* via a number of serial manipulators in parallel. In these manipulators, there are always more joints than the number of degrees of freedom (DOF) of the end-effector. This places constraints on the structure such that all the joints cannot be actuated at the same time. If the end-effector has l DOF, then there are l active joints where $l \leq 6$. All the other joints are passive and their motion is dependant on the motion of the active joints. The most famous family of such manipulators are called *Stewart-Gough* platforms (Bhattacharya et al., 1997). These platforms are widely used in simulators (Yamane et al., 2005), low impact docking systems for space vehicles (Timmons and Ringelberg, 2008), and in form of a hexapod for precise machining (Warnecke et al., 1998).

Figure 1 shows a 3-R$\underline{P}$R robot, which has three joints in each serial link. R stands for a rotatory joint and P stands for a prismatic joint whereby the underline signifies the joint which is actuated (Siciliano and Khatib, 2007).

The forward kinematics function of a parallel has been studied in detail in the literature, especially for a 3-R$\underline{P}$R robot. (Kong, 2008) derived algebraic expres-

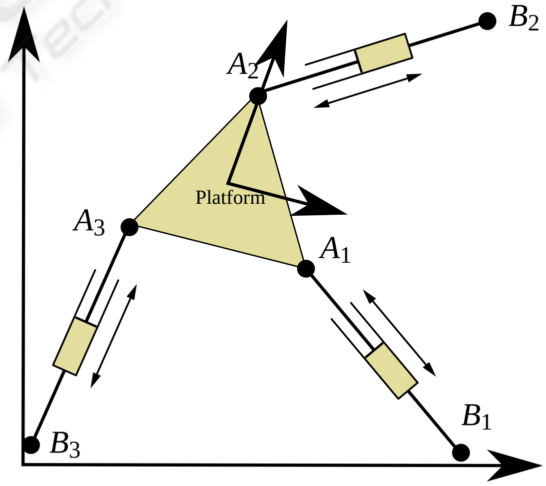

Figure 1: A 3-R$\underline{P}$R planar parallel manipulator. B_1, B_2, and B_3 are connected to a stationary base.

sions for the forward kinematics of a 3-R$\underline{P}$R robot and analyzed its singularities. (Collins, 2002) used planar quaternions to formulate kinematic constraints in equations for a 3-R$\underline{P}$R robot. (Murray et al., 1997) used coefficients of a *constraint manifold*, which are functions of the locations of the base and platform joints and the distance between them, for the kine-

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matics synthesis of a 3-R<u>P</u>R robot. (Wenger et al., 2007) studied the *degeneracy* in the forward kinematics of a 3-R<u>P</u>R robot. (Kim et al., 2000; Dutre et al., 1997) found the analytical Jacobian for a parallel manipulator. However, there is no attempt in literature to formulate analytically the forward kinematics function for non-redundant parallel manipulators. The forward kinematics function relates the joint variables of the active joints to the position of the end-effector.

In the following section , the structure of the forward and inverse kinematics blocks is layed out. Then forward kinematics function of a parallel manipulator is derived using the position-closure property. The analytical Jacobian of a parallel manipulator is also obtained as described in the literature. Finally, a framework to control a parallel manipulator is proposed, followed by an example for the forward kinematics function of a 3-R<u>P</u>R robot.

2 KINEMATICS FRAMEWORK

If the task is given in operational space then it becomes inevitable to cater for the non-linearities introduced by the forward and inverse kinematics functions. First, the joint variables are translated into operational space. The resultant is compared to the reference trajectory and the error is then converted back to joint space, as shown in Figure 2.

Suppose there are n serial manipulators in a parallel manipulator that has n_a active joints and n_p passive joints such that the total number of joints is $n_c = n_a + n_p$. If x is the end-effector position and F is the forward kinematics function then the following definitions can be introduced;

$$x = F \tag{1}$$

$$\dot{x} = \frac{F}{\partial q_a}\frac{\partial q_a}{\partial t} = J_c \dot{q}_a \tag{2}$$

$$\ddot{x} = J_c \ddot{q}_a + \dot{J}_c \dot{q}_a \tag{3}$$

where $\dot{\bullet}$ signifies differentiation with respect to time, J_c is the systems Jacobian, $\dot{J}_c$ is its time-derivate, and q_a is a vector of active joint variables. These variables are in radians if the joint is revolute or in meters if the joint is prismatic.

Equations (1), (2), and (3) can be combined as follows;

$$\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} F_c & 0 & 0 \\ 0 & J_c & 0 \\ 0 & \dot{J}_c & J_c \end{bmatrix} \begin{bmatrix} q_a \\ \dot{q}_a \\ \ddot{q}_a \end{bmatrix} \tag{4}$$

or

$$\begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} = N_1 y \tag{5}$$

where $F_c \triangleq \frac{F}{q_a}$ is the forward kinematics function that relates the active joints to the end-effector position and

$$N_1 \triangleq \begin{bmatrix} F_c & 0 & 0 \\ 0 & J_c & 0 \\ 0 & \dot{J}_c & J_c \end{bmatrix} \in \Re^{3n_c \times 3l} \tag{6}$$

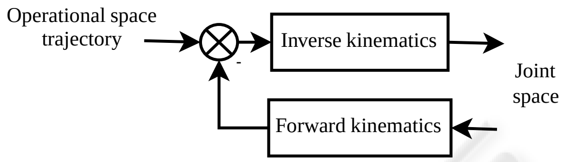

Figure 2: Operational space control of a parallel manipulator.

The above matrix produces large values for small values of q_a. To avoid this situation, a limit is imposed here on the value of each component of q_a so that there is always a valid solution available.

The difference between the reference operational space trajectory and the output of the forward kinematics block is referred to in this paper as system error. Let $\left[\Delta x, \Delta\dot{x}, \Delta\ddot{x}\right]^T$ be this error in operational space. If this error is small, then (2) can be approximated to

$$\Delta x \approx J_c \Delta q_a \tag{7}$$

However, it can be stated, without any approximation, that

$$\Delta\dot{x} = J_c \Delta\dot{q}_a \tag{8}$$

$$\Delta\ddot{x} = \dot{J}_c \Delta\dot{q}_a + J_c \Delta\ddot{q}_a \tag{9}$$

It is a common practice that when end-effector trajectory is formulated in operational space, Δx is chosen in (7) such that the approximate movement of the end-effector partially matches the target velocities in (8) (Whitney, 1969). Equation (7) is only valid for a small value of Δx. If the target position is too distant, it is important to bring the target closer. This way, the manipulator reaches its final target in smaller steps. For this reason, Δx needs to be clamped such that

$$clamp(\Delta x, D_{max}) = \begin{cases} \Delta x \text{ if } ||\Delta x|| < D_{max} \\ D_{max}\frac{\Delta x}{||\Delta x||} \text{ otherwise} \end{cases} \tag{10}$$

where $||\bullet||$ is the Euclidean norm. The value of the scalar D_{max} should be at least several times larger than what end-effector moves in a single step and less than half the length of a typical link. This heuristic approach has also been reported to reduce oscillations in the system, which allows the designer to use a smaller value for damping constant. This usually results in a quicker response (Buss and Kim, 2005). To calculate the error in joint space, (7), (8), and (9) can be written as

$$\begin{aligned}\Delta q_a &= J_c^{\dagger}\Delta x &(11)\\ \Delta\dot{q}_a &= J_c^{\dagger}\Delta\dot{x} &(12)\\ \Delta\ddot{q}_a &= J_c^{\dagger}(\Delta\ddot{x}-\dot{J}_c\Delta\dot{q}_a)\\ &= J_c^{\dagger}\Delta\ddot{x}-J_c^{\dagger}\dot{J}_cJ_c^{\dagger}\Delta\dot{x} &(13)\end{aligned}$$

In matrix form, these equations can be written as

$$\begin{bmatrix}\Delta q_a\\ \Delta\dot{q}_a\\ \Delta\ddot{q}_a\end{bmatrix} = \begin{bmatrix}J_c^{\dagger} & 0 & 0\\ 0 & J_c^{\dagger} & 0\\ 0 & -J_c^{\dagger}\dot{J}_cJ_c^{\dagger} & J_c^{\dagger}\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta\dot{x}\\ \Delta\ddot{x}\end{bmatrix} \tag{14}$$

or alternatively

$$\Delta y = N_2\begin{bmatrix}\Delta x\\ \Delta\dot{x}\\ \Delta\ddot{x}\end{bmatrix} \tag{15}$$

where $J_c^{\dagger}$ is the pseudoinverse of J_c. Pseudoinverse is defined for all matrices including the ones which are not square or are not full rank. It also gives the best solution in terms of least squares. Except near singularities, the pseudoinverse gives a stable solution even in those cases when the target end-effector position doesn't lie in the work volume of the mechanical manipulator. The resulting solution is the closest location to its target which minimizes $||J_c\Delta q-\Delta x||^2$. In the vicinity of singularity, the pseudoinverse creates large changes in joint variables, even for very small changes in the end-effector position, resulting in an unstable system. One important feature of pseudoinverse is that the term $(I-J_c^{\dagger}J_c)$ projects on the null space of J_c. This feature can be exploited for redundant manipulators. It is possible to generate internal motions in a redundant manipulator, i.e., $\dot{q}_0$, without changing its end-effector position (Sciavicco and Siciliano, 2000). For redundant manipulators, (2) can be written as

$$\dot{x} = J_c\dot{q}+(I-J_c^{\dagger}J_c)\dot{q}_0 \tag{16}$$

However, in this paper it is assumed that the parallel manipulator is not redundant, i.e., number of active joints is equal to the DOF of the end-effector.

The damped least-squares (DLS) method, which is also referred to the Levenberg-Marquardt method, solves many problems related to pseudoinverse. The method gives a numerically stable solution near singularities, and was first used in inverse kinematics by (Wampler, 1986) and (Nakamura and Hanafusa, 1986). It was also used for theodolite calibration by (Sultan and Wager, 2002).

Not only does DLS minimize the term $||J_c\dot{q}_a-\dot{x}||^2$ but it also minimizes the joint velocities with a damping factor, i.e., $\lambda^2||\dot{q}_a||^2$ where $\lambda\in\Re$ and $\lambda\neq 0$. The function to be minimized can be written as

$$\min_{\dot{q}_a}\left\{||J_c\dot{q}_a-\dot{x}||^2+\lambda^2||\dot{q}_a||^2\right\} \tag{17}$$

The DLS solution is equal to (Buss and Kim, 2005)

$$\dot{q}_a = (J_c^TJ_c+\lambda^2I)^{-1}J_c^T\dot{x} \tag{18}$$

or alternatively

$$\dot{q}_c = J_c^T(J_cJ_c^T+\lambda^2I)^{-1}\dot{x} \tag{19}$$

Equation (18) requires an inversion of an $n\times n$ matrix, while (19) requires an inversion of only an $l\times l$ matrix, which is computationally more efficient. In terms of SVD, the singular values change from $\frac{1}{\sigma^i}$ for $J_c^{\dagger}$ to $\frac{\sigma^2}{\sigma^2+\lambda^2}$ for $(J_cJ_c^T+\lambda^2I)^{-1}$ (Buss and Kim, 2005). If $\sigma_i\to 0$, $\frac{1}{\sigma_i}\to\infty$, while in the other case, $\frac{\sigma^2}{\sigma^2+\lambda^2}\to\frac{1}{\lambda^2}$ when $\sigma_i\to 0$. Therefore, a stable solution is observed even near singularities for $\forall\lambda:\lambda\neq 0$. Using (19), N_2 can be redefined as

$$N_2 \triangleq \begin{bmatrix}J_c^* & 0 & 0\\ 0 & J_c^* & 0\\ 0 & -J_c^*\dot{J}_cJ_c^* & J_c^*\end{bmatrix}\in\Re^{3l\times 3n_c} \tag{20}$$

where $J_c^* = J_c^T(J_cJ_c^T+\lambda^2I)^{-1}$. The value of λ is set by the designer. Large values can result in a slower convergence rate and very small values can reduce the effectiveness of the method. In literature, there are many methods proposed to select the value of λ dynamically (Mayorga et al., 1990; Nakamura and Hanafusa, 1986; Chiaverini et al., 1994).

3 FORWARD KINEMATICS FUNCTION

In order to formulate the forward and inverse kinematics matrices, N_1 and N_2, it is important to formulate *analytically* the forward kinematics function of a parallel manipulator. The derivation is somewhat similar to the derivation of the analytic Jacobian of a parallel manipulator by (Dutre et al., 1997), which was derived using the velocity-closure property. The derivation is given as follows;

As all the manipulators are connected to the same end-effector, it can be stated, using the position-closure property, that

$$F_cq_a = F_1q_1 = F_2q_2 = \cdots = F_nq_n \tag{21}$$

where q_j is the vector of joint variables of j^{th} manipulator and $F_j(q)\triangleq\frac{F_j(q)}{q_j}$ is the forward kinematics function of the j^{th} manipulator.

Each column of the function F_c corresponds to rotational angle or displacement of an active joint, depending on whether the joint is rotatory or prismatic. Hence

$$F_c^i = F_jq_j^i \tag{22}$$

where $F_c^i \in \Re^{n_a}$ is the i^{th} column of F_c and q_j^i is a vector of joint variables of the j^{th} manipulator when the i^{th} active joint is moved one unit while all the other active joints are locked. If q_c is the vector of all the joint variables, i.e.,

$$q_c = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{n_c} \end{bmatrix} \in \Re^{n_c} \tag{23}$$

then (22) can be written as

$$F_c^i = F_j S_j q_c^i \tag{24}$$

where q_c^i is a vector of all the joints when the i^{th} active joint is moved one unit while all the other active joints are locked and S_j is a selection matrix to select the variables of the j^{th} manipulator, i.e.,

$$S_j = \begin{bmatrix} 0 & \dots & 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 & \dots & 0 \end{bmatrix} \in \Re^{n_j \times n_c}$$

where n_j is the number of joints in the j^{th} manipulator. Let q_p be the vector of passive joint variables and q_a be the vector of active joint variables such that

$$\begin{aligned} q_p &= S_p q_c & (25) \\ q_a &= S_a q_c & (26) \end{aligned}$$

where $q_p \in \Re^{n_p}$ and $q_a \in \Re^{n_a}$ and S_p and S_a are selection matrices for passive and active joints, respectively. Typical values of S_p and S_a can be written as

$$S_p = \begin{bmatrix} \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots \end{bmatrix} \in \Re^{n_p \times n_c}$$

and

$$S_a = \begin{bmatrix} \dots & 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots \end{bmatrix} \in \Re^{n_a \times n_c}$$

Both of these matrices are sparse and orthogonal, i.e., $S_p S_p^T = I$ and $S_a S_a^T = I$, which implies

$$\begin{aligned} q_{c_p} &= S_p^T q_p & (27) \\ q_{c_a} &= S_a^T q_a & (28) \end{aligned}$$

where q_{c_p} is equivalent to q_c except that the active joints are set to zero and similarly, q_{c_a} is equivalent to q_c except that the passive joints are set to zero such that

$$q_c = q_{c_p} + q_{c_a} \tag{29}$$

Substituting (27) and (28) in (29) yields

$$q_c = S_p^T q_p + S_a^T q_a \tag{30}$$

In reference to the position-closure property (21), let

$$A q_c = 0 \tag{31}$$

where

$$A = \begin{bmatrix} \frac{F_1}{q_1} & -\frac{F_2}{q_2} & 0 & \dots & 0 \\ \frac{F_1}{q_1} & 0 & -\frac{F_3}{q_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{F_1}{q_1} & 0 & 0 & \dots & -\frac{F_n}{q_n} \end{bmatrix} \in \Re^{n_a(n-1) \times n_c} \tag{32}$$

Substituting the value of q_c from (30) gives

$$\begin{aligned} A q_c &= A(q_{c_p} + q_{c_a}) \\ &= A S_p^T q_p + A S_a^T q_a \\ &= A_p q_p + A_a q_a & (33) \end{aligned}$$

Applying (31)

$$q_p = -A_p^{\dagger} A_a q_a \tag{34}$$

Substituting this expression in (30) yields

$$q_c = S_a^T q_a - S_p^T A_p^{\dagger} A_a q_a \tag{35}$$

As q_c^i is defined for a unit displacement of the i^{th} active joint, hence, q_a can be replaced with a column of S_a which corresponds to the i^{th} active joint, denoted by $(S_a)^i$, to evaluate q_c^i, i.e.,

$$q_c^i = S_a^T (S_a)^i - S_p^T A_p^{\dagger} A_a (S_a)^i \tag{36}$$

Substituting the above value in (24) gives

$$F_c^i = F_j S_j q_c^i \tag{37}$$

or

$$F_c = F_j S_j \begin{bmatrix} q_c^1 & q_c^2 & \dots & q_c^{n_a} \end{bmatrix} \tag{38}$$

4 ANALYTICAL JACOBIAN AND ITS DERIVATIVE

(Dutre et al., 1997) evaluated the analytical Jacobian for a parallel manipulator using the velocity-closure property. The Jacobian can also be derived by replacing F_c in (38) by J_c and q_c^i by $\dot{q}_c^i$, i.e.,

$$J_c = J_j S_j \begin{bmatrix} \dot{q}_c^1 & \dot{q}_c^2 & \dots & \dot{q}_c^{n_a} \end{bmatrix} \tag{39}$$

where J_c is the analytical Jacobian that relates the velocities of the active joints to the end-effector velocity. J_j and $\dot{q}_j$ are the Jacobian and the vector of joint velocities of the j^{th} manipulator, respectively. $\dot{q}_c^i$ can be stated using (36) as follows;

$$\dot{q}_c^i = S_a^T (S_a)^i - S_p^T B_p^{\dagger} B_a (S_a)^i \tag{40}$$

where $B_p = BS_p^T$, $B_a = BS_a^T$, and

$$B = \begin{bmatrix} J_1 & -J_2 & 0 & \dots & 0 \\ J_1 & 0 & -J_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_1 & 0 & 0 & 0 & -J_n \end{bmatrix} \in \Re^{n_a(n-1)\times n_c} \tag{41}$$

Using B, the velocity-closure property of a parallel manipulator can be written as

$$B\dot{q}_c = 0$$

The derivative of the closed-loop Jacobian (J_c) given in (39) is

$$\dot{J}_c = \dot{J}_j S_j \begin{bmatrix} \ddot{q}_c^1 & \ddot{q}_c^2 & \dots & \ddot{q}_c^{n_a} \end{bmatrix} \tag{42}$$

where $\ddot{q}_c^i$ can be calculated by differentiating (40), i.e.,

$$\ddot{q}_c^i = -S_p^T \left(\frac{\partial B_p^\dagger}{\partial q_i} B_a + B_p^\dagger \frac{\partial B_a}{\partial q_i} \right) (S_a)^i \tag{43}$$

where q_i is the i^{th} driving joint.

The time derivative of a Jacobian column for a serial manipulator is the sum of the partial derivatives of this column with respect to joint variables, multiplied by the time-derivates of these variables (Bruyninckx and De Schutter, 1996). As such, time-derivative of the i^{th} column of the Jacobian is given by

$$\dot{J}^i = \sum_{j=1}^{n} \frac{\partial J^i(q)}{\partial q^j} \frac{\partial q^j}{\partial t} = \sum_{j=1}^{n} \frac{\partial J^i(q)}{\partial q^j} \dot{q}^j \tag{44}$$

Similarly, the derivative of the Jacobian of each manipulator of a parallel manipulator can be expressed using (44), i.e.,

$$\dot{J}_j = \sum_{i=1}^{n_a} \frac{\partial J_j}{\partial q_i} \dot{q}_i = \sum_{i=1}^{n_a} \left(\sum_{k=1}^{n_j} \frac{\partial J_j}{\partial q_{j,k}} \frac{\partial q_{j,k}}{\partial q_i} \right) \dot{q}_i \tag{45}$$

where k is a joint of the j^{th} manipulator and $\frac{\partial J_j}{\partial q_{j,k}}$ represents Jacobian derivative of the j^{th} serial manipulator. The factor, $\frac{\partial q_{j,k}}{\partial q_i}$ in (45), is the k^{th} component in $S_j \dot{q}_c^i$.

5 PROPOSED CONTROL FRAMEWORK

Figure 3 shows the structure of the proposed control framework for parallel manipulators. As only active joints are actuated, it is important to incorporate the contribution of the passive joints onto the active joints. If joint friction is ignored, the relationship between the torque of active joints and passive joints is given by the following equation (Cheng et al., 2003);

$$\tau_c = \tau_a + \left(\frac{\partial q_p}{\partial q_a} \right)^T \tau_p \tag{46}$$

where $\tau_p \in \Re^{n_p}$ is the torque measured from strain gauges on passive joints, $\tau_a \in \Re^{n_a}$ is the torque produced by the actuators on active joints, and $\tau_c \in \Re^{n_a}$ is the torque measured by strain gauges mounted on active joints. From (Dutre et al., 1997), it can be inferred that

$$\frac{\partial q_p}{\partial q_a} = B_p^\dagger B_a$$

Using the above value in (46) yields

$$\tau_c = \tau_a - (B_p^\dagger B_a)^T \tau_p$$

or

$$\tau_c = \tau_a - B_a^T (B_p^\dagger)^T \tau_p \tag{47}$$

The passive joints project torque onto the active joints with a factor of $-B_a^T (B_p^\dagger)^T$. This will be used as the exogenous force disturbance signal in the hybrid controller, as shown in Figure 3.

To ease the implementation of the *clamp* block, it can be taken out of the closed loop. This can be done by redefining r_k using

$$r_k = clamp(r_{original} - N_1 y_k) + N_1 y_k \tag{48}$$

6 EXAMPLE

As the proposed kinematics framework is evaluated analytically, it can be applied on any non-redundant parallel manipulator. However, in this section, for the sake of demonstration, a simple case of a 3-R<u>P</u>R robot is presented, shown in Figure 1.

The forward kinematics function for the first manipulator can be stated as

$$F_1 = \begin{bmatrix} (x_{1,1} + q_{1,2} + x_{1,2}) \cos(q_{1,1}) + x_{1,3} \cos(q_{1,1} + q_{1,3}) \\ (x_{1,1} + q_{1,2} + x_{1,2}) \sin(q_{1,1}) + x_{1,3} \sin(q_{1,1} + q_{1,3}) \\ q_{1,1} + q_{1,3} \end{bmatrix} \tag{49}$$

where $x_{1,2}$ and $q_{1,2}$ denote the length of the second link and the second joint variable, respectively. The expressions for other links can be written in the same way. Using this kinematic model for the values given in Table 1, the end-effector position was found to be at

$$x_{end_c} = \begin{bmatrix} 1.5 \\ 2.5981 \\ 1.0472 \end{bmatrix}$$

where the first two elements represent the position in $x-y$ plane and the third element represents the angular rotation of the end-effector.

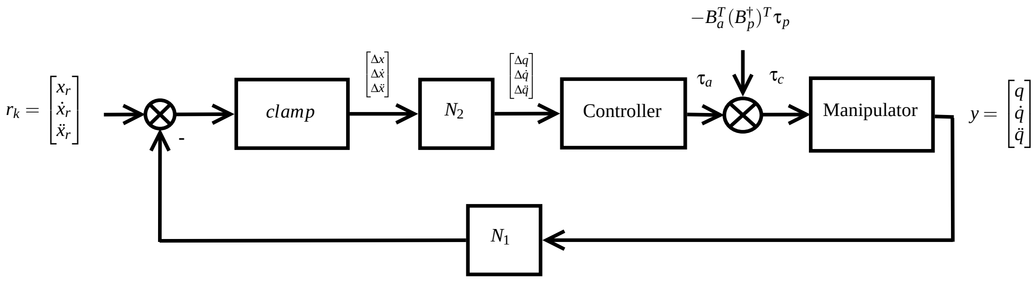

Figure 3: Operational space control of a parallel manipulator.

Table 1: Assumed values for a 3-RPR robot.

Manipulator 1	Manipulator 2	Manipulator 3
$q_{1,1} = \pi/3$	$q_{2,1} = 2\pi/3$	$q_{3,1} = 4\pi/3$
$q_{1,2} = 1$	$q_{2,2} = 1$	$q_{3,2} = 1$
$q_{1,3} = 0$	$q_{2,3} = -\pi/3$	$q_{3,3} = -\pi$
$x_{1,1} = 0.5$	$x_{2,1} = 0.5$	$x_{3,1} = 0.5$
$x_{1,2} = 0.5$	$x_{2,2} = 0.5$	$x_{3,2} = 0.5$
$x_{1,3} = 1$	$x_{2,3} = 1$	$x_{3,3} = 1$

The forward kinematics function, F_c, gives the following end-effector position for the active joints $[1,1,1]^T$;

$$x_{end} = \begin{bmatrix} 1.498 \\ 2.597 \\ 1.048 \end{bmatrix}$$

7 CONCLUSIONS

Similar to the analytical Jacobian for a parallel manipulator, which is a function of joint variables and relates the velocity of the active joints to the velocity of the end-effector, the analytical forward kinematics function is also a function of the joint variables that relates the position of the active joints to the position of the end-effector. The generality of the proposed technique allows the forward kinematics function to be used in a variety of applications. A control configuration is also described in this paper as a prospective application of the proposed technique.

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