GENERAL SPANNING TREES AND CORE LABELING

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Abstract: The checking of graph reachability is an important operation in many applications, by which for two given nodes u and v in a directed graph G we will check whether u is reachable from v through a path in G or *vice versa*. In this paper, we focus ourselves on this issue. A new approach is proposed to compress transitive closure to support reachability. The main idea is the concept of general spanning trees, as well as a new labeling technique, called core labeling. For a graph G with n nodes and e edges, the labeling time is bounded by $O(n + e + t \cdot b)$, where t is the number of non-tree edges (edges that do not appear in the general spanning tree T of G) and b is the number of the leaf nodes of T. It can be proven that b equals G's width, defined to be the size of a largest node subset U of G such that for every pair of nodes $u, v \in U$, there does not exist a path from u to v or from v to u. The space and time complexities are bounded by $O(n + t \cdot b)$ and $O(\log b)$, respectively.

1 INTRODUCTION

Given two nodes u and v in a directed graph $G = (V, E)$, we want to know if there is path from u to v. The problem is known as *graph reachability*. In many applications, such as transportation network, internet traffic analyzing, semantic web, and computer vision, as well as metabolic network and XML query processing, graph reachability is one of the most basic operations, and therefore needs to be efficiently supported. Among the above applications, some use sparse graphs, such as XML documents which are a labeled tree plus several *IDREF/ID* links, and metabolic networks which are an evolution tree plus some genes' interactions.

A naive method is to precompute the reachability between every pair of nodes – in other words, to compute and store the transitive closure (*TC* for short) of a graph. Then, a reachability query can be answered in constant time. However, this requires $O(n^2)$ space, which makes it impractical for massive graphs. Another method is to compute the shortest path from u to v over such a large graph on demand, which results in high query processing cost.

In this paper, we propose a new approach to compress transitive closure and to speed up reachability queries for massive graphs. The main idea behind them is to recognize a subset of nodes of G and assign them labels in such a way that the reachability through non-tree edges can be determined by checking such labels only. For this purpose, we introduce the concept of *general spanning trees* and a new labeling technique, the so called *core labeling*. Based on them, the space overhead for storing a transitive closure can be reduced to $O(n + t \cdot b)$, where n is the number of the nodes in G, t is the number of non-tree edges (edges that do not appear in the general spanning tree T of G) *and* b is the width of G, defined to be the size of a largest node subset U of G such that for every pair of nodes $u, v \in U$, there does not exist a path from u to v or from v to u. *The query time is* bounded by $O(\log b)$; and the labeling time is bounded by $O(n + e + t \cdot b)$, where e is the number of the edges in G.

The remainder of the paper is organized as follows. In Section 2, we review the related work. Section 3 is devoted to the description of our algorithms. Section 4 is a short conclusion.

2 RELATED WORK

In the past two decades, many interesting labeling-based methods have been proposed to speed up reachability query evaluation, which can be roughly classified into two groups: strategies for sparse graphs and strategies for non-sparse graphs.

The *Dual* labeling discussed in (Wang et al., 2006) is a method for sparse graphs. The main idea of this method is to assign to each node v a dual

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label: (a_v, b_v) and (x_v, y_v, z_v), which can be used to check reachability. The size of all labels is bounded by $O(n + t^2)$ and can be produced in $O(n + e + t^3)$ time, where t is the number of non-tree edges. The method proposed by Cohen *et al.* (2004) is another strategy for sparse graphs and labels a graph based on the so-called 2-*hop covers*. However, to find a well-constructed 2-hop cover is a *NP*-hard problem.

There are a bunch of strategies for non-sparse graphs, such as those discussed in (Agrawal et al., 1989; Jagadish, 1990). In (Jagadish, 1990), the reachablity problem is transformed to a *min flow* problem, which needs $O(n^3)$ time to label a graph. In (Agrawal et al., 1989), Agrawal *et al*. proposed a method based on interval labeling, which requires $O(n)$ query time in the worst case.

There are also some other graph labeling methods, such as the methods using signatures (Teuhola, 1996), *PE-Encoding* (Cohen, 1991) and *PQ-Encoding* (Zibin and Gil, 2001). The idea of the signature-based method is to assign to each node a signature (which is in fact a bit string) generated using a set of hash functions. The space complexity is $O(l \cdot n)$, where l is the length of a signature. But this encoding method suffers from the so-called signature conflicts (two nodes are assigned the same signature). Moreover, in the case of DAGs, a graph needs to be decomposed into a series of trees; and no formal decomposition was reported in that paper. The PE-Encoding and the PQ-Encoding are similar to the 2-hop labeling, but with higher computational complexities. The methods discussed in (Schenkel et al., 2004; Schenkel et. al., 2006) reduce 2-hop's labeling complexity from $O(n^4)$ to $O(n^3)$, but are still not applicable to massive graphs. The method proposed in (Cheng et al., 2006) is a geometry-based algorithm to find high-quality 2-hop covers. It also improves the 2-hop labeling by avoiding computation of transitive closures, which is required by Cohen's to find 2-hop covers. However, it has the same theoretical computational complexities as Cohen's method (Cohen, 2004). Finally, the method discussed in (Thorup, 2004) is suitable only for planar graphs with $O(n\log n)$ labeling time and $O(n\log n)$ space. The query time is $O(1)$.

3 NEW LABELING APPROACH

In this section, we present our labeling approach. The input is a directed graph G with n nodes and e edges. We assume that it is acyclic. If not, we will find all the *strongly connected components* (*SCCs*) of G and collapse each of them into a representative node. Obviously, each node in an *SCC* is equivalent to its representative node as far as reachability is concerned. This process takes $O(e)$ time using Tarjan's algorithm (Tarjan, 1972).

The main idea of our approach is to find a subset of nodes and assign them labels, which can be used to check reachability via non-tree edges. For this purpose, a new tree structure is generated for G, called the *core* of G, to figure out such nodes. In the following, we first show the tree labeling used in our approach in 3.1. Then, we define the core tree in 3.2. Next, we show our labeling scheme in 3.3.

3.1 General Spanning Trees and Tree Labeling

We begin our discussion by introducing the concept of general spanning trees, based on which a new labeling approach is developed.

Definition 1 (General Spanning Trees). Let G be a DAG (*acyclic directed graph*). A tree (forest) T is called a general spanning tree if the following two conditions are satisfied:

1. T covers G, i.e., for each node $v \in G$, we have $v \in T$.
2. For each edge $u \to v$ in T, there exists a path from u to v in G.

Since an edge $u \to v$ in G is also a path, a traditional spanning tree is a special case of general spanning trees.

As an example, consider the graph G shown in Fig. 1(a), for which a general spanning tree T can be found as shown in Fig. 2.

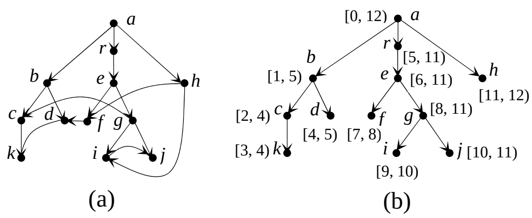


Figure 1: A graph and a spanning tree.

In T, special attention should be paid to the edge $h \to i$, which corresponds to a path from h to i in G. We also notice that the number of the leaf nodes in T is 5 while any (traditional) spanning tree of G has at least 6 leaf nodes (see Fig. 1(b) for illustration).

As demonstrated in (Chen, 2009), we can always find a general spanning tree with the least number of leaf nodes, which is bounded by b, the width of G.

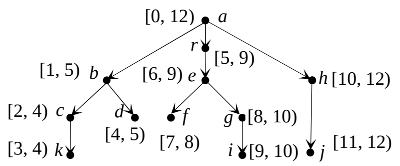


Figure 2: A general spanning tree.

Each node v is assigned an interval [*start*, *end*), where *start* is v's preorder number and *end* - 1 is the largest preorder number among all the nodes in $T[v]$. So another node u labeled [*start'*, *end'*) is a descendant of v (with respect to T) iff *start'* $\in$ [*start*, *end*) (Wang et al., 2006). Fig. 2 helps for illustration.

Let v and u be two nodes in T, labeled $[a, b)$ and $[a', b')$, respectively. If $a \in [a', b')$, we say, $[a, b)$ is subsumed by $[a', b')$. In this case, we must also have $b \leq b'$. Therefore, if v and u are not on the same path in T, we have either $a' \geq b$ or $a \geq b'$. In the former case, we say, $[a, b)$ is smaller than $[a', b')$, denoted $[a, b) \prec [a', b')$. In the latter case, $[a', b')$ is smaller than $[a, b)$.

3.2 Core of G

Now we define another important concept, the so-called *core* of G.

Let T be a general spanning tree of G. We denote by E' the set of all the non-tree edges, i.e., the edges not appearing in T. Denote by V' the set of all the end points of the non-tree edges. Then, $V' = V_{start} \cup V_{end}$, where V_{start} stands for a set containing all the start nodes of the non-tree edges and V_{end} for all the end nodes of the non-tree edges.

Definition 2 (Anti-subsuming Subset). A subset $S \subseteq V_{start}$ is called an *anti-subsuming set* iff $|S| > 1$ and no two nodes in S are related by ancestor-descendant relationship with respect to T.

As an example, consider the general spanning tree shown in Fig. 2 once again.

With respect to this spanning tree, $V_{start} = \{d, f, g, h, i\}$. We have altogether 11 anti-subsuming subsets as shown in Fig. 3.

$\{d, f\}$	$\{d, i\}$	$\{f, i\}$	$\{d, f, g\}$	$\{d, g, h\}$	$\{f, h, i\}$
$\{d, g\}$	$\{f, g\}$	$\{g, h\}$	$\{d, f, h\}$	$\{d, h, i\}$	$\{d, f, g, h\}$
$\{d, h\}$	$\{f, h\}$	$\{h, i\}$	$\{d, f, i\}$	$\{f, g, h\}$	$\{d, f, h, i\}$

Figure 3: Anti-subsumming subsets.

Definition 3 (Critical Node). A node v in a spanning tree T of G is *critical* if $v \in V_{start}$ or there exists an anti-subsuming subset $S = \{v_1, v_2, ..., v_k\}$ for $k \geq 2$ such that v is the lowest common ancestor of $v_1, v_2, ..., v_k$. We denote by $V_{critical}$ the set of all critical nodes.

In the general spanning tree shown in Fig. 2, node e is the lowest common ancestor of $\{f, g\}$, and node a is the lowest common ancestor of $\{d, f, g, h\}$. So e and a are two critical nodes. In addition, each $v \in V_{start}$ is a critical node. So all the critical nodes of G with respect to T are $\{d, f, g, h, i, e, a\}$. We call a critical node *trivial* if it belongs to V_{start}; otherwise, *non-trivial*.

Definition 4 (Core of G). Let $G = (V, E)$ be a directed graph. Let T be a spanning tree of G. The *core* of G with respect to T is a tree structure with the node set being $V_{critical}$, in which there is an edge from u to v ($u, v \in V_{critical}$) iff there is a path P from u to v in T and P contains no other critical nodes. The core of G with respect to T is denoted $G_{core} = (V_{core}, E_{core})$.

Example 1. Consider the graph G shown in Fig. 1(a) and the corresponding spanning tree T shown in Fig. 2. The core of G with respect to T is shown in Fig. 4.

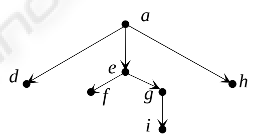


Figure 4: The core of G.

3.3 Graph Labeling

In this subsection, we show our graph labeling. The approach works in two steps. In the first step, we generate a data structure, called the *core label* (for G). It is in fact a set of interval sequences. In the second step, the core label is used to create non-tree labels for all the nodes in G.

3.3.1 Core Labeling

The core label for G is defined as below.

Definition 5. Let $V_{core} = \{v_1, ..., v_g\}$ be the node set of G_{core}. The core label for G is a set $\{L(v_1), ..., L(v_g)\}$, where each $L(v_l)$ ($l = 1, ..., g$) is an interval sequence associated with v_l, satisfying the following two properties:

(1) Let $L(v_l) = [a_{l_1}, b_{l_1}), ..., [a_{l_r}, b_{l_r})$ for some r. Then, for any $i, j \in \{1, ..., r\}$, $\leq$ if $i < j$. That is, $[a_{l_i}, b_{l_i}) \prec [a_{l_j}, b_{l_j})$ for $i < j$. (In this sense,

the intervals in $L(v_l)$ are considered to be sorted.)

(2) Let $[a, b)$ be the interval associated with a descendant of v_l with respect to G. There exists an interval $[a_{l_i}, b_{l_i})$ $(1 \le i \le r)$ in $L(v_l)$ such that $a \in [a_{l_i}, b_{l_i})$.

In order to generate the core label for G, we will first establish a graph, called a *link graph*, specified in the following definition.

Definition 6 (Link Graph). Let $G = (V, E)$ be a directed graph. Let T be a spanning tree of G. The *link graph* of G with respect to T is a graph, denoted G_{link}, with the node set being V' (the end points of all the non-tree edges) and the edge set $E' \cup E''$, where E' is the set of all the non-tree edges, and for any u, $v \in V'$, $(u, v) \in E''$ iff $u \in V_{end}$, $v \in V_{start}$, and there exists a path from u to v in T.

Example 2. In Fig. 5, we show the link graph of G (shown in Fig. 1(a)) with respect to the corresponding T shown in Fig. 2.

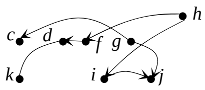
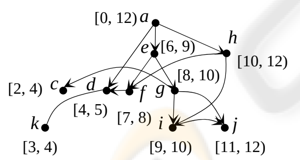


Figure 5: A G_{link}.

As the first step to generate the core label for G, we unite G_{core} and G_{link} to create a combined graph, denoted $G_{com} = (V_{com}, E_{com}) = G_{core} \cup G_{link}$ as shown in Fig. 6(a).



(a)

k	[3, 4)
j	[11, 12)
i	[9, 10)[11, 12)
c	[2, 4)
g	[2, 4)[8, 10)[11, 12)
d	[3, 4)[4, 5)
f	[3, 4)[4, 5)[7, 8)
e	[2, 4)[4, 5)[6, 9)[11, 12)
h	[3, 4)[4, 5)[7, 8)[9, 10)[10, 12)
a	[0, 12)

(b)

Figure 6: A combined graph and set of interval sequences.

Now we notice that by labeling T, each node in G_{com} will be initially associated with an interval as illustrated in Fig. 6(a). That is, if a node v is labeled with $[a, b)$ in T, it will be initially labeled with the same interval $[a, b)$ in G_{com}. Next, we will find a reverse topological sequence of the nodes in G_{com} such that $(v_i, v_j) \in E_{com}$ implies that v_j appears before v_i in the sequence. Then, scan the sequence from the beginning to the end and at each step merge the interval sequences of the children of a node into the interval sequence associated with that node. See Fig. 6(b) for illustration.

Using such interval sequences, the descendants of a node in G_{com} can be represented in an economical way. Let $L = [a_1, b_1), ..., [a_k, b_k)$ be an interval sequence and each $[a_i, b_i)$ is an interval labeling a node v_i $(i = 1, ..., k)$ in G_{com}. Then, L corresponds to the union of a set of subtrees $T[v_1]$, ..., and $T[v_k]$. For example, the interval sequence [2, 4)[4, 5)[6, 9)[11, 12) associated with e in Fig. 7(b) represents a union of 4 subtrees: $T[c]$, $T[d]$, $T[e]$ and $T[f]$, which contains all the descendants of h in G.

Now we consider all the nodes of G_{core}. Each node is associated with an interval sequence as shown in Fig. 7.

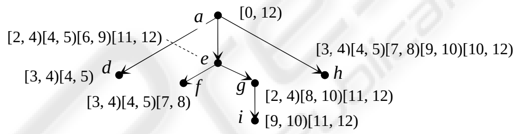


Figure 7: Interval sequences for critical nodes.

In this figure, we remark that the interval sequence for node a is [0, 12), which is the interval initially assigned to it. It is because when we merge all its children's interval sequences into it, they are all absorbed into [0, 12) (since all the intervals appearing in them are subsumed by [0, 12)).

Proposition 1. The time for generating the core label for G is bounded by $O(t \cdot b)$.

Proof. First, we note that the numbers of the edges in both G_{core} and G_{link} are bounded by $O(t)$. Second, the number of intervals in an interval sequence is smaller than or equal to b since among any $b + 1$ intervals we have at least two intervals that are appear on a same path in T. So, one of them is absorbed by the other. Thirdly, each interval sequence is sorted. Therefore, merging two sequences needs only $O(b)$ time. The total time is bounded by

$$O\Big(\sum_{v \in G_{com}} d_v \cdot b\Big) = O(t \cdot b)$$

where d_v represents the outdegree of v in G_{com}.

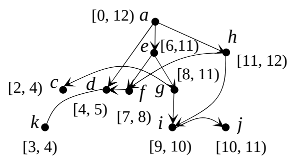
3.3.2 Comparison with Traditional Spanning Trees

In fact, the method described in 3.3.1 can be established based on traditional spanning trees. However, the size of a core label will be increased to

O($t \cdot \beta$), where β is the number of the leaf nodes of the corresponding spanning tree and is in general much larger than b.

To see this, consider the spanning tree shown in Fig. 1(b) once again, with which being used the combined graph will be slightly different from the one shown in Fig. 6(a). See Fig. 8(a) for illustration. Accordingly, the intervals associated with their nodes are also different.

In Fig. 8(b), we show the core label (i.e., the set of interval sequences generated for the core). We note that the length of the longest interval sequence is 6 while that shown in Fig. 6(b) is 5. As demonstrated in Proposition 1, the length of the longest interval sequence is bounded by the number of leaf nodes in a spanning tree.



k	[3, 4)
j	[11, 12)
i	[9, 10)[10, 11)
c	[2, 4)
g	[2, 4)[8, 11)
d	[3, 4)[4, 5)
f	[3, 4)[4, 5)[7, 8)
e	[2, 4)[4, 5)[6, 11)
h	[3, 4)[4, 5)[7, 8)[9, 10)[10, 11)[11, 12)
a	[0, 12)

(a) (b)

Figure 8: A set of interval sequences and a spanning tree.

3.3.3 Non-tree Labeling

Based on the core label of G, we assign non-tree labels to nodes, which would enable us to answer reachability queries quickly.

Find a general spanning tree T in G. Let v be a node in T. Consider the set of all the critical nodes in $T[v]$, denoted C_v. We then denote by v^- a critical node in C_v, which is closest to v. We further denote by v^* the lowest ancestor of v (in T), which has a non-tree incoming edge.

The following two lemmas are critical to our non-tree labeling method.

Lemma 2. Any critical node in C_v appears in $T[v^-]$.

Proof. Assume that there exists a critical node u in C_v, which does not appear in $T[v^-]$. Let $u_1, ..., u_k$ be all the critical nodes in $T[v^-]$. Consider the lowest common ancestor node of $u, u_1, ..., u_k$. It must be an ancestor node of v^-, which is closer to v than v^-, contradicting the fact that v^- is the closest critical node (in $T[v]$) to v.

Lemma 3. Let u be a node, which is not an ancestor of v in T; but v is reachable from u via some non-tree edges. Then, any way for u to reach v must be through v^*.

Proof. This can be seen from the fact that any node which reaches v via some non-tree edges is through v^* to reach v.

Let $V_{core} = \{v_1, ..., v_g\}$. We store the core label of G as a list: $s_1 = L(v_1), ..., s_g = L(v_g)$ (see Fig. 9(a) for illustration). Then, we define a function ϕ: $V_{core} \rightarrow \{1, ..., g\}$ such that for each $v \in V_{core}$ $\phi(v) = i$ iff $s_i = L(v)$.

Definition 7 (Non-tree Labels). Let v be a node in G. The non-tree label of v is a pair <x, y>, where

- $x = i$ if v^- exists and $\phi(v^-) = i$. If v^- does not exists, let x be the special symbol "-".
- $y = [a, b)$ if v^* exists and labeled $[a, b)$. If v^* does not exist, let y be "-".

Example 3. Consider G and T shown in Fig. 1. The core label of G with respect to T is shown in Fig. 9(a). The values of the corresponding ϕ-function are shown in Fig. 9(b).

$s_1 = L(i) =$ [9, 10)[11, 12)	$\phi(i) = 1$
$s_2 = L(g) =$ [2, 4)[8, 10)[11, 12)	$\phi(g) = 2$
$s_3 = L(d) =$ [3, 4)[4, 5)	$\phi(d) = 3$
$s_4 = L(f) =$ [3, 4)[4, 5)[7, 8)	$\phi(f) = 4$
$s_5 = L(e) =$ [2, 4)[4, 5)[6, 9)[11, 12)	$\phi(e) = 5$
$s_6 = L(h) =$ [3, 4)[4, 5)[7, 8)[9, 10)[10, 12)	$\phi(h) = 6$
$s_7 = L(a) =$ [0, 12)	$\phi(a) = 7$

(a) (b)

Figure 9: Core label of G.

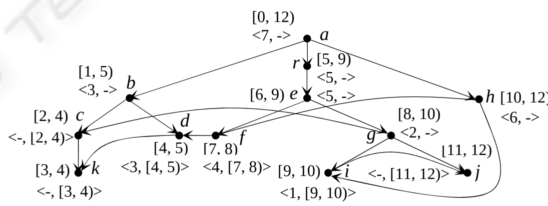


Figure 10: Graph with non-tree labelling.

Fig. 10 shows both the tree labels and the non-tree labels. For instance, the non-tree label of node r is <5, -> because (1) $r^- = e$; (2) $\phi(r^-) = \phi(e) = 5$ (see Fig. 9(b)); and (3) r^* does not exist. Similarly, the non-tree label of node b is <3, ->. Now we check the non-tree label of node d: <3, [4, 5)>. First, we note that d^- is d itself. So $\phi(d^-) = \phi(d) = 3$. Furthermore, e^* is also itself. Therefore, the tree label of e^* is the tree label of e itself.

Proposition 4. Assume that u and v are two nodes in G, labeled ($[a_1, b_1)$, <x_1, y_1>) and ($[a_2, b_2)$, <x_2, y_2>), respectively. Node v is reachable from u iff one of the following conditions holds:

(i) $[a_2, b_2)$ is subsumed by $[a_1, b_1)$, or

(ii) There exists an interval $[a, b)$ in such that $[a_2, b_2)$ is subsumed by $[a, b)$.

Proof. The proposition can be derived from the following two facts:

(1) v is reachable from u by tree edges iff $[a_2, b_2)$ is subsumed by $[a_1, b_1)$.

(2) In terms of Lemma 6, v is reachable from u via non-tree edges iff v^- exists and its interval sequence contains an interval $[a, b)$ which subsumes $[a_2, b_2)$. Furthermore, in terms of Lemma 7, $[a, b)$ subsumes $[a_2, b_2)$ iff v^* exists and its interval is subsumed by $[a, b)$.

Now we consider node c and e in the graph shown in Fig. 10. To check whether node c (labeled [2, 4), <-, [2, 4)>) is a descendant of node e (labeled ([6, 9), <5, ->), we will first check whether $2 \in [6, 9)$. Since $2 \notin [6, 9)$, we will check whether there is an interval in $L(e)$ = [2, 4)[4, 5)[6, 9)[11, 12) (note that $\phi(e) = 5$), which subsumes [2, 4). Since [2, 4) in $L(e)$ subsumes [2, 4), we know that node c is reachable from node e.

Finally, we notice that each interval sequence in the core table of G contains only the intervals not on the same path in T and they are also increasingly ordered. Therefore, to check a given interval is subsumed by an interval in $L(v)$ for some node v, we need only $O(\log|L(v)|)$ time. But $|L(v)|$ is bounded by b, so we require only $O(\log b)$ time for reachability checking.

Proposition 5. Let v and u be two nodes in G. It needs $O(\log b)$ time to check whether u is reachable from v via non-tree edges or *vice versa*.

Proof. See the above analysis.

4 CONCLUSIONS

In this paper, a new approach is proposed to compress transitive closure. Its main idea is to recognize a subset of nodes in G and assign them labels in such a way that the reachability via non-tree edges can be determined by checking such labels only. This is achieved by finding a general spanning tree with the least number of leaf nodes, based on which a core label can be established. Using this method, the labeling can be done in $O(n + e + t \cdot b)$ time, where t is the number of non-tree edges (edges that do not appear in the general spanning tree T of G) and b is the number of the leaf nodes of T. It can be proven that b equals G's width, defined to be the size of a largest node subset U of G such that for every pair of nodes $u, v \in U$, there does not exist a path from u to v or from v to u. The space and time complexities are bounded by $O(n + t \cdot b)$ and $O(\log b)$, respectively.

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