ON THE LINEAR SCALE FRACTIONAL SYSTEMS
An Application of the Fractional Quantum Derivative

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Abstract: The Linear Scale Invariant Systems are introduced for both integer and fractional orders. They are defined by the generalized Euler-Cauchy differential equation. It is shown how to compute the impulse responses corresponding to the two regions of convergence of the transfer function. This is obtained by using the Mellin transform. The quantum fractional derivatives are used because they are suitable for dealing with this kind of systems.

1 INTRODUCTION

Braccini and Gambardella (1986) introduced the concept of "form-invariant" filters. These are systems such that a scaling of the input gives rise to a scaling of the output. This is important in detection and estimation of signals with unknown size requiring some type of pre-processing: for example edge sharpening in image processing or in radar signals. However in their attempt to define such systems, they did not give any formulation in terms of a differential equation. The Linear Scale Invariant Systems (LSIS) were really introduced by Yazici and Kashyap (1997) for analysis and modelling 1/f phenomena and in general the self-similar processes, namely the scale stationary processes. Their approach was based on an integer order Euler-Cauchy differential equation. However, they solved only a particular case corresponding to the all pole case. To insert a fractional behaviour, they proposed the concept of pseudo-impulse response. Here we avoid this procedure by presenting a fractional derivative based general formulation of the LSIS. We assume that the fractional LSIS is described by the general Euler-Cauchy differential equation

$$\sum_{i=0}^{N} a_i \, t^{\alpha_i} . y^{(\alpha_i)}(t) = \sum_{i=0}^{M} b_i \, . \, t^{\beta_i} . y^{(\beta_i)}(t) \qquad (1)$$

This equation is difficult to solve for any values for N or M and any derivative orders. However, when the derivative orders have the format

$$\alpha_i = \alpha + i \quad i=0, 1, 2, \dots, N$$

and

$$\beta_i = \beta + i \quad i=0, 1, 2, \dots, N$$

we obtain a simpler equation

$$\sum_{i=0}^{N} a_i \, t^{\alpha+i} . y^{(\alpha+i)}(t) = \sum_{i=0}^{M} b_i \, . \, t^{\beta+i} x^{(\beta+i)}(t) \qquad (2)$$

that we can solve with the help of the Mellin transform and using the fractional quantum derivative (Ortigueira, 2007, 2008). As we will show, the above equation allows us to obtain two transfer functions. Each of them has two terms that lead to two inverse functions. The impulse response is obtained by using the multiplicative convolution defined by (Bertran et al, 2000):

$$f(t)Vg(t) = \int_{0}^{\infty} f(t/u)g(u)\frac{du}{u} \qquad (3)$$

Before going into the solution of equation (2), we are going to obtain the solution of the integer order equation corresponding to put $\alpha=\beta=0$ in (2). Then we will solve equation (2) for any α and β. This will be done in section 1. Other interesting results will be introduced in section 3. Finally we will present some conclusions.

2 THE EULER-CAUCHY EQUATION

2.1 The Integer Order Case

Consider a linear system represented by the differential equation

$$\sum_{i=0}^{N} a_i\, t^i . y^{(i)}(t) = \sum_{i=0}^{M} b_i . t^i\, x^{(i)}(t) \qquad (4)$$

where $x(t)$ is the input, $y(t)$ the output, and N and M are positive integers ($M \leq N$). Usually a_N is chosen to be 1. We will assume that this equation is valid for every $t \in R^+$. Applying the Mellin transform to both sides of (3) we obtain (Gerardi, 1959;Bertran et al, 2000)

$$\sum_{i=0}^{N} a_i\, (-1)^i (s)_i\, Y(s) = \sum_{i=0}^{M} b_i . (-1)^i (s)_i\, X(s), \qquad (5)$$

from where we obtain a transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{i=0}^{M} b_i\, (-1)^i (s)_i}{\sum_{i=0}^{N} a_i\, (-1)^i (s)_i} \qquad (6)$$

In this expression we need to transform both numerator and denominator into polynomials in the variable s. To do it we use the well known relation (Abramowitz and Stegun, 1972)

$$(x)_k = \sum_{i=0}^{k} (-1)^{k-i} s(k,i)\, x^i \qquad (7)$$

where s(,) represent the Stirling numbers of first kind that verify the recursion

$$s(n+1,m) = s(n,m-1) - ns(n,m) \qquad (8)$$

for $1 \leq m \leq n$ and with

$$s(n,0) = \delta_n \text{ and } s(n,1) = (-1)^{n-1}(n-1)!$$

With some manipulation, we obtain:

$$\sum_{i=0}^{N} a_i\, (-1)^i (x)_i = \sum_{i=0}^{N} \sum_{k=i}^{N} a_k\, (-1)^k s(k,i)\, x^i \qquad (9)$$

$$= \sum_{i=0}^{N} A_i\, x^i$$

with the A_i coefficients given by

$$A_i = \sum_{i=k}^{N} a_k\, (-1)^k s(k,i) \qquad (10)$$

or in a matricial format

$$\mathbf{A} = \mathbf{S.a} \qquad (11)$$

where

$$\mathbf{A} = [A_0\ A_1\ \ldots\ \ldots\ A_N]^T \qquad (12)$$

$$\mathbf{S} = [\ s(i,j),\ i,j = 0,1,\ \ldots,N] \qquad (13)$$

and

$$\mathbf{a} = [a_0\ a_1\ \ldots\ \ldots\ a_N]^T \qquad (14)$$

With this formulation, the transfer function is given by:

$$H(s) = \frac{\sum_{i=0}^{M} B_i\, s^i}{\sum_{i=0}^{N} A_i\, s^i} \qquad M \leq N \qquad (15)$$

that is the quotient of two polynomials in s. In general H(s) has the following partial fraction decomposition

$$H(s) = \frac{B_M}{A_N} + \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{a_{ij}}{(s-p_i)^j} \qquad (16)$$

The constant term only exists when M=N and its inversion gives a delta at t=1:

$$\mathcal{M}^{-1}[\frac{B_M}{A_N}] = \frac{B_M}{A_N}\delta(t-1) \qquad (17)$$

For inversion of a given partial fraction, we must fix the region of convergence $\text{Re}(s) > \text{Re}(p_i)$ or $\text{Re}(s) < \text{Re}(p_i)$ similar to identical situation found in the usual shift invariant systems with the Laplace transform. Let us assume that the poles are simple. Accordingly to each region of convergence we have (Bertran *et al*, 2000) respectively

$$\mathcal{M}^{-1}[\frac{1}{(s-p)}] = u(1-t).t^p \qquad (18)$$

and

$$\mathcal{M}^{-1}[\frac{1}{(s-p)}] = u(t-1).t^p \qquad (19)$$

By successive derivation in order to p we obtain the solution for higher order poles

$$\mathcal{M}^{-1}[\frac{1}{(s-p)^k}] = u(1-t).\frac{(-1)^{k-1}[\log(t)]^{k-1}}{(k-1)!}t^{-p} \qquad (20)$$

and

$$\mathcal{M}^{-1}[\frac{1}{(s-p)^k}] = u(t-1).\frac{(-1)^{k-1}[\log(t)]^{k-1}}{(k-1)!}t^{-p} \qquad (21)$$

We conclude that the response corresponding to an input $\delta(t-1)$ is given by:

$$h(t)=\frac{B_M}{A_N}\delta(t-1)+\sum_{i=1}^{N}\sum_{k=1}^{m_i}a_{ik}.\frac{(-1)^{k-1}[\log(t)]^{k-1}}{(k-1)!}t^{-p_i}w(t) \qquad (22)$$

where $w(t)$ is equal to $u(1-t)$ or to $u(t-1)$, in agreement with the region of convergence adopted to invert (15). To compute the output to any function $x(t)$ we only have to use the multiplicative convolution.

We must call the attention to the fact the point of application of the impulse is $t=1$ and not $t=0$, as it is the case of the shift-invariant systems.

2.2 The Fractional Quantum Derivative

To consider a more general case we must introduce the notion of fractional quantum derivative. This was not needed in the previous section because in the integer order case we only have one Mellin transform for $t^K f^{(K)}(t)$. This is not the situation in the fractional case. In fact we have two fractional derivatives given by {see appendix}:

$$D_q^{\alpha}f(t) = \lim_{q\to 1}\frac{\sum_{j=0}^{\infty}\begin{bmatrix}\alpha\\j\end{bmatrix}_q(-1)^j q^{j(j+1)/2}q^{-j\alpha}f(q^j t)}{(1-q)^{\alpha}t^{\alpha}} \qquad (23)$$

and

$$D_{q^{-1}}^{\alpha}f(t) = \lim_{q\to 1}\frac{\sum_{j=0}^{\infty}\begin{bmatrix}\alpha\\j\end{bmatrix}_q(-1)^j q^{j(j-1)/2}f(q^{-j}t)}{(1-q^{-1})^{\alpha}t^{\alpha}} \qquad (24)$$

These derivatives have the same Mellin transform in the integer order case, but in the general their Mellin transforms are given by:

$$\mathcal{M}[D_q^{\alpha}f(t)] = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}F(s-\alpha) \qquad (25)$$

valid for $Re(s) < \min(0,\alpha)+1$, in the first case and by

$$\mathcal{M}[D_{q^{-1}}^{\alpha}f(t)] = (-1)^{\alpha}.\frac{\Gamma(s)}{\Gamma(s-\alpha)}F(s-\alpha) \qquad (26)$$

valid for $Re(s) > \max(0,\alpha)$, in the second case. It is interesting that the first corresponds to the anti-causal case when working in the Laplace transform context, while the second corresponds to the causal one.

2.3 The Fractional Order Equation

Consider now a linear system represented by the fractional differential equation

$$\sum_{i=0}^{N}a_i\,t^{\alpha+i}.y^{(\alpha+i)}(t) = \sum_{i=0}^{M}b_i\,.\,t^{\beta+i}x^{(\beta+i)}(t) \qquad (27)$$

where α and β are real numbers. With the Mellin transform we obtain two different transfer functions depending on the derivative we use, (23) or (24). From (23) we have:

$$H(s) = \frac{\sum_{i=0}^{M}b_i(-1)^i(s+\beta)_i}{\sum_{i=0}^{N}a_i(-1)^i(s+\alpha)_i}.\frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)}\frac{\Gamma(1-s)}{\Gamma(1-s-\beta)} \qquad (28)$$

Proceeding as in 2.1 we have

$$H(s) = \frac{\sum_{i=0}^{M}B_i(s+\beta)^i}{\sum_{i=0}^{N}A_i(s+\alpha)^i}.\frac{\Gamma(1-s-\alpha)}{\Gamma(1-s-\beta)} \qquad (29)$$

So, the transfer function in (29) has two parts; the first is similar to (25) aside a translation on the pole and zero positions. Its inverse has the format:

$$h(t)=\frac{B_M}{A_N}\delta(t-1)+t^{\alpha}\sum_{i=1}^{N}\sum_{k=1}^{m_i}C_{ik}.\frac{(-1)^{k-1}[\log(t)]^{k-1}}{(k-1)!}t^{-p_i}u(t-1) \qquad (30)$$

where the p_i, $i=1,2,\ldots,N$ are the poles. We must remark that it does not depend explicitly on β. The second factor in (29) leads to a new convolutional factor needed to compute the complete solution of (27). So, we have to compute the inverse Mellin transform of

$$H_a(s) = \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s-\beta)} \qquad (31)$$

To do it we can always choose an integration path on the left of all the poles. Computing this integral, we obtain:

$$h_a(t) = \frac{1}{\Gamma(\alpha-\beta)} t^{\beta}(t-1)^{\alpha-\beta-1} u(t-1) \qquad (32)$$

So, the impulse response corresponding to (29) is the convolution of (30) and (32). By simplicity, assume that all the poles are simple. In this case, the impulse response is given by:

$$h(t) = \frac{B_M}{A_N} \frac{1}{\Gamma(\alpha-\beta)} t^{\beta}(t-1)^{\alpha-\beta-1} u(t-1) +$$
$$+ t^{\alpha} \sum_{i=1}^{N} C_i \cdot \frac{\Gamma(1-p_i)}{\Gamma(\alpha-\beta-p_i+1)} t^{-p_i} u(t-1) \qquad (33)$$

Choosing the other region of convergence we have

$$H(s) = \frac{\sum_{i=0}^{M} B_i (s+\beta)^i}{\sum_{i=0}^{N} A_i (s+\alpha)^i} \cdot (-1)^{\beta-\alpha} \frac{\Gamma(s+\beta)}{\Gamma(s+\alpha)} \qquad (34)$$

The first factor has as inverse the expression:

$$h(t) = \frac{B_M}{A_N} \delta(t-1) +$$
$$+ t^{\alpha} \sum_{i=1}^{N} \sum_{k=1}^{m_i} C_{ik} \cdot \frac{(-1)^k [\log(t)]^{k-1}}{(k-1)!} t^{-p_i} u(1-t) \qquad (35)$$

For the second we proceed as before. Now the integration path is in the right half complex plane. We obtain

$$h_a(t) = -\frac{1}{\Gamma(\alpha-\beta)} t^{\beta}(t-1)^{\alpha-\beta-1} u(1-t) \qquad (36)$$

To compute the final impulse response we only have to convolve the two expressions as we did in the other case. We obtain, for the simple pole case

$$h(t) = -\frac{B_M}{A_N} \frac{1}{\Gamma(\alpha-\beta)} t^{\beta}(t-1)^{\alpha-\beta-1} u(1-t) -$$
$$t^{\alpha} \sum_{i=1}^{N} C_i \cdot \frac{\Gamma(\beta-\alpha+p_i)}{\Gamma(p_i)} t^{-p_i} u(1-t) \qquad (37)$$

It is interesting to verify that (33) and (37) behavior like the usual anti-causal and causal systems. When $Re(p_i) < 0$, (30) increases without bound while (35) decreases. If $Re(p_i) > 0$, we verify the reverse

situation. This means that we can use the well known Routh-Hurwitz test to study the stability of LSIS.

2.4 Particular Cases

2.4.1 $\alpha = \beta$

If $\alpha=\beta$, the second terms in (29) and (34) is equal to 1, implying that the complete impulse response is given by (30) and (35).

2.4.2 $\alpha = 0$ and $\beta \neq 0$

This case is very interesting since it is similar to the situation treated by Yazici and Kashyap. With $\alpha=0$, (30) and (35) do not depend explicitly on β and they are similar to the integer order case. The dependence on β appears only in the second therm.

2.4.3 $\alpha \neq 0$ and $\beta = 0$

This situation is more involved, since both terms of the impulse response depend on α. We can obtain the general impulse response by putting $\beta=0$ into (30), (32), (35), and (36).

3 THE EIGENFUNCTIONS AND FREQUENCY RESPONSE

Consider relation (3) and assume that one of the functions is the impulse response of the system (1) and the other is a power function $t^{-\sigma}$, $\sigma \in C$. It is not hard to show that

$$h(t) V t^{-\sigma} = H(\sigma) \cdot t^{-\sigma} \qquad (38)$$

Leading us to conclude that the power function is the eigenfunction of the LSIS. In particular we can write:

$$h(t) V t^{-jv} = H(v) \cdot t^{-jv} \qquad (39)$$

and $H(v)$ will be the frequency response of the system, considering that our "cisoids" have the format

$$c(t) = e^{-jv\log(t)} \qquad (40)$$

that verify:

$$c(t) = c(at) \qquad (41)$$

provided that

$$a = e^{2\pi/v} \qquad (42)$$

defining the scale periodicity. These results show that the output of a LSIS to a cisoid is a cisoid. For a

cosine signal, as input, the output y(t) is given by

$$y(t) = |H(\nu)|.\cos[2\pi\nu\log(t)+\varphi(\nu)] \qquad (43)$$

where $\varphi(\nu)$ is the phase spectrum of the system.

4 CONCLUSIONS

In this paper, we introduced the general formulation of the linear scale invariant systems through the fractional Euler-Cauchy equation. To solve this equation we used the fractional quantum derivative concept and the help of the Mellin transform. As in the linear time invariant systems we obtained two solutions corresponding to the use of two different regions of convergence. We presented other interesting features of the LSIS, namely the frequency response. We made also a brief study of the stability.

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APPENDIX - QUANTUM DERIVATIVE FORMULATIONS

- **Incremental Ratio Formulation**

The normal way of introducing the notion of derivative is by means of the limit of an incremental ratio that in the forward case reads

$$D_h f(t) = \lim_{h \to 0} \frac{f(t) - f(t\text{-}h)}{h} \qquad (a.1)$$

By repeated application, this definition leads to the derivative of any integer order that can be generalized to any real or complex order by the well known forward Grünwald-Letnikov fractional derivative (Ortigueira, 2006):

$$D_h^{\alpha} f(z) = \lim_{h \to 0+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z - kh)}{h^{\alpha}} \qquad (a.2)$$

An alternative derivative valid only for t>0 or t<0 is the so-called quantum derivative (Kac and Cheug, 2002). Let Δ_q be the following incremental ratio:

$$\Delta_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t} \qquad (a.3)$$

where q is a positive real number less than 1 and f(t) is assumed to be a causal type signal. The corresponding derivative is obtained by computing the limit as q goes to 1

$$D_q f(t) = \lim_{q \to 1} \frac{f(t) - f(qt)}{(1 - q)t} \qquad (a.4)$$

This derivative uses values of the variable below t. We can introduce another one that uses values above t. It is defined by

$$D_{q^{-1}} f(t) = \lim_{q \to 1} \frac{f(q^{-1}t) - f(t)}{(q^{-1} - 1)t} \qquad (a.5)$$

The repeated application of (a.3) followed by the limit computation leads to the Nth order derivative (Ash et al, 2002; Koornwinder, 1999):

$$D_q^N f(t) =$$
$$\lim_{q \to 1} \frac{\sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_q (-1)^j q^{j(j+1)/2} q^{-jN} f(q^j t)}{(1-q)^N t^N} \qquad (a.6)$$

where we introduced the q-binomial coefficients

$$\begin{bmatrix} \alpha \\ i \end{bmatrix}_q = \frac{[\alpha]_q!}{[i]_q![\alpha-i]_q!} \qquad (a.7)$$

with $[\alpha]_q$ given by

$$[\alpha]_q = \frac{1-q^\alpha}{1-q} \qquad (a.8)$$

Using the q-binomial theorem (Kac and Cheug, 2002), the Mellin transform, and the Pochhamer symbol we conclude that:

$$\mathcal{M} \left[\lim_{q \to 1} \frac{\sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_q (-1)^j q^{j(j+1)/2} q^{-jN} f(q^j t)}{(1-q)^N t^N} \right]$$
$$\qquad (a.9)$$
$$= (1-s)_N F(s - N)$$

$$= \frac{\Gamma(1-s+N)}{\Gamma(1-s)} F(s-N)$$

The previous results are readily generalised for the case of a real order, α, (Ortigueira, 2007; Ortigueira, 2008) leading to a Grunwald-Letnikov like fractional quantum derivative:

$$D_q^\alpha f(t) =$$
$$\lim_{q \to 1} \frac{\sum_{j=0}^{\infty} \begin{bmatrix} \alpha \\ j \end{bmatrix}_q (-1)^j q^{j(j+1)/2} q^{-j\alpha} f(q^j t)}{(1-q)^\alpha t^\alpha} \qquad (a.10)$$

that is similar to the one proposed by Salam (1966).

In (a.10) the fractional q-binomial coefficients are given by

$$\begin{bmatrix} \alpha \\ j \end{bmatrix}_q = \frac{[1-q^\alpha]_q^j}{[j]_q} \qquad (a.11)$$

The Mellin transform of (a.10) reads

$$\mathcal{M} [D_q^\alpha f(t)] = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha) \qquad (a.12)$$

valid for $Re(s) < \min(0,\alpha)+1$. This relation allows us to obtain an integral representation of the fractional quantum derivative, as we will see later. As referred before, in (a.10) we are using values of the variable less than t. In the following we will consider the other case. The repeated application of (a.5) leads to the Nth order derivative:

$$D_{q^{-1}}^N f(t) =$$
$$\lim_{q \to 1} \frac{\sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix}_q (-1)^j q^{j(j-1)/2} f(q^{-j} t)}{(1-q^{-1})^N t^N} \qquad (a.13)$$

The Mellin transform gives:

$$\mathcal{M} [D_{q^{-1}}^N f(t)] = (1-s)_N F(s-N) \qquad (a.14)$$

that coincides with (a.9) as expected. To generalize the above results for any order, we substitute α for N in the above expressions. We have from (a.10):

$$D_{q^{-1}}^\alpha f(t) =$$
$$\lim_{q \to 1} \frac{\sum_{j=0}^{\infty} \begin{bmatrix} \alpha \\ j \end{bmatrix}_q (-1)^j q^{j(j-1)/2} f(q^{-j} t)}{(1-q^{-1})^\alpha t^\alpha} \qquad (a.15)$$

and finally

$$\mathcal{M} [D_{q^{-1}}^\alpha f(t)] = (-1)^\alpha \cdot \frac{\Gamma(s)}{\Gamma(s-\alpha)} F(s-\alpha) \qquad (a.16)$$

valid for $Re(s) > \max(0,\alpha)$. Remark the difference relatively to (a.12) mainly in the region of convergence.

- **Integral Formulations**

The two Mellin transforms in (a.12) and (a.16) lead to different integral representation of fractional derivatives by computing the corresponding inverse

functions.

The inverse $h_b(t)$ of $\dfrac{\Gamma(s)}{\Gamma(s-\alpha)}$ is obtained from (Andrews *et al*,1999):

$$\frac{\Gamma(s)\Gamma(-\alpha)}{\Gamma(s-\alpha)} = \int_0^1 \tau^{s-1}(1-\tau)^{-\alpha-1}d\tau \qquad (a.17)$$

Provided that Re(s)>0 and Re(α)<0. This leads immediately to

$$h_b(t) = \frac{(-1)^\alpha}{\Gamma(-\alpha)}(1-t)^{-\alpha-1}u(1-t) \qquad (a.18)$$

u(t) is the Heaviside unit step. A similar procedure to obtain the inverse $h_a(t)$ of $\dfrac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}$ gives

$$\frac{\Gamma(1-s+\alpha)\Gamma(-\alpha)}{\Gamma(1-s)} =$$

$$\int_0^1 \tau^{1-s+\alpha}(1-\tau)^{-\alpha-1}d\tau \qquad (a.19)$$

With a variable change inside the integral, we obtain:

$$h_a(t) = \frac{1}{\Gamma(-\alpha)}(t-1)^{-\alpha-1}u(t-1) \qquad (a.20)$$

To compute in integral formulations of the derivatives corresponding to (a.12) and (a.16) we remark that the inverse Mellin transform of F(s-α) is given by:

$$\mathcal{M}^{-1}[F(s-\alpha)] = t^{-\alpha}f(t) \qquad (a.21)$$

and use the convolution (3). With (a.12) and (a.16) we obtain the following integral formulations, valid for Re(α) < 0.

$$D_b^\alpha f(t) = -\frac{t^{-\alpha}}{\Gamma(-\alpha)}\cdot\int_0^1 f(t/\tau)(1-\tau^{-1})^{-\alpha-1}d\tau \qquad (a.22)$$

and

$$D_a^\alpha f(t) = \frac{t^{-\alpha}}{\Gamma(-\alpha)}\int_1^\infty f(t/\tau)(\tau^{-1}-1)^{-\alpha-1}d\tau \qquad (a.23)$$

signals. Although we obtained these results for α<0, they remain valid for other values of α, since $\dfrac{\Gamma(s)}{\Gamma(s-\alpha)}$ and $\dfrac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}$ are analytic in the regions

of convergence and we can fix an integration path independent of α. This can be confirmed by expanding (a.22) and (a.23) and transforming each term of the series.