Semiglobal Asymptotic Stabilization of a Class of Nonlinear Sampled-data Systems using Emulated Controllers

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Abstract. Considering nonlinear sampled-data systems, it has been shown in [14] - that emulating a continuous-time controller that ensures some global asymptotic stability properties in continuous-time. In this study, we provide a similar result, for a general class of systems, using a hybrid formulation that allows deriving explicit bounds on the maximum allowable sampling period.

1 Introduction

A number of researches focused on the stabilization problem of nonlinear sampled-data systems during the last decades (see the overview [13] and [14] and the references cited therein). A common approach consists in emulating a known continuous-time controller using a sample-and-hold device. Based on discrete-time model approximations and using results of [17], it has been shown in [14] that, by choosing a sufficiently small sampling period, asymptotic stability properties are recovered in an appropriate practical sense, under mild conditions. Practical state convergence might be an issue in practice, especially when the sampling period cannot be taken small enough. It is also important for engineers to know an explicit bound on the sampling period that can be taken so that designed controllers ensure the desired asymptotic state convergence. Thus, a number of papers propose solutions for the asymptotic stabilization of nonlinear sampled-data systems and the knowledge of an explicit bound on $\mathrm{T}_{\mathrm{MASP}}$. In most of these works, global asymptotic stability properties are studied. Two exceptions are however available in the literature. First in [9] where a hybrid stabilization method is proposed for some classes of systems: it consists in decomposing the state space in a number of regions for which a controller is designed in order to reach the next region that is closer to the origin. A semiglobal asymptotic stability property is shown to hold for system in output feedback form in [21] but no explicit bound on $\mathrm{T}_{\mathrm{MASP}}$ is given. Concerning results on global asymptotic stability properties for nonlinear systems, some papers are available in the literature. In [4], global Lipschitz conditions on system and static state-feedback nonlinearities are supposed to apply, thus the global exponential stability of the system origin is recovered under sampling. In [1], considering the Euler approximation of a dynamic feedback controller, Lyapunov stability results for impulsive systems are applied, under similar conditions than in [4]. A small gain theorem for a class of hybrid systems that does not satisfy the classical semi-group property is developed in [7] that allows to

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design discrete-time controllers for classes of nonlinear systems. The same authors in [9] derive an analytic bound on $\mathrm{T}_{\mathrm{MASP}}$ when using emulated controllers, by modeling sampled-data systems as time-delay systems. Recently, techniques firstly developed for networked systems have been applied to the stabilization problem of nonlinear sampled-data systems [16]. Writing nonlinear sampled-data systems with emulated controllers as hybrid systems in the modeling framework of [3, 2], sufficient Lyapunov-type conditions are proposed and an explicit bound on the T_{MASP} is given.

In this study, considering a known controller that is supposed to ensure the input-to-state stability w.r.t. measurement errors of the closed-loop system in continuous-time, it is shown that the emulated controller will ensure the asymptotic stability of system origin if the sampling period satisfy an explicit boundedness condition. Similarly to [16], the system is written as the interconnection of the continuous-time closed-loop system and the 'error' system due to the sampling. The stability analysis relies on trajectory based arguments and the Lyapunov-like analysis to ensure bounds on the state and sampling error.

2 Notations

The Euclidean norm of a vector is denoted by $|\cdot|$, for a function $f:\mathbb{R}\to\mathbb{R}^n$ and $t_1\le t_2\in\mathbb{R}$, $\|f\|_{[t_1,t_2)}$ stands for $\sup_{\tau\in[t_1,t_2)}|f(\tau)|$. Let $C(\mathbb{R}^p,\mathbb{R}^q)$, $p,\ q\in\mathbb{N}$, denote the space of all continuous mapping $\mathbb{R}^p\to\mathbb{R}^q$. $B_d\subset ofR^n$ denotes the open ball centered at 0 and of radius d. For initial conditions we use notations $t_\circ\ge 0$, $x_\circ=x(t_\circ)$, $e_\circ=e(t_\circ)$, finally, to simplify the notations we sometimes omit the arguments and when it is clear from the context, we write $V(x(t))$, or even $V(t)$ in place of of $V(x(t,t_\circ,x_\circ))$.

3 Problem Statement

Consider a system:

$$\dot{x}_p = f_p(x_p,u), \tag{1}$$

$$y = h_p(x_p), \tag{2}$$

where $x_p\in\mathbb{R}^{n_{x_p}}$ denotes the state vector of the plant, $u\in\mathbb{R}^{n_u}$ the input vector, $y\in\mathbb{R}^{n_y}$ the output vector, $n_{x_p},n_u,n_y\in\mathbb{N}$, $f_p\in\mathbb{R}^{n_{x_p}}\times\mathbb{R}^{n_u}\to\mathbb{R}^{n_{x_p}}$ is locally Lipschitz with $f_p(0,0)=0$, and $h_p\in\mathbb{R}^{n_{x_p}}\to\mathbb{R}^{n_y}$ is differentiable, its partial derivatives are locally Lipschitz and $h_p(0)=0$.

The following dynamic output-feedback controller is considered for the system (1)-(2):

$$\dot{x}_c = f_c(x_c,y), \tag{3}$$

$$u = h_c(x_c,y), \tag{4}$$

where $x_c\in\mathbb{R}^{n_{x_c}}$ denotes the state vector of the controller, $n_{x_c}\in\mathbb{N}$, $f_c\in\mathbb{R}^{n_{x_c}}\times\mathbb{R}^{n_y}\to\mathbb{R}^{n_{x_c}}$ is locally Lipschitz with $f_c(0,0)=0$ and $h_c\in\mathbb{R}^{n_{x_c}}\to\mathbb{R}^{n_u}$ is differentiable with locally Lipschitz partial derivatives and $h_c(0)=0$. For the sake of generality, all the results are stated for the system (1)-(4), but they apply also for the case of output or state static feedbacks. Denoting $x = [x_p^\top, x_c^\top]^\top \in \mathbb{R}^{n_x}$, $n_x = n_{x_p} + n_{x_c}$, the following assumption is supposed to apply throughout the paper.

Assumption A 1. *The origin $x = 0$ is globally asymptotically stable for the closed-loop system (1)-(4).*

Attention is focused on the case where the input u and the measure vector y are sampled at the same instants $\{t_k\}_{k\in\mathbb{N}}$ using a sample-and-hold device. In the sequel we will use the following assumption on the sampling instants.

Assumption A 2. *Sequence of sampling instants $\{t_k\}$, $k \in \mathbb{N}$ satisfies the following:*
(i) There exist positive constants $\upsilon, T_{max} \in \mathbb{R}_{>0}$ such that $\upsilon \leq t_{k+1} - t_k \leq T_{max}$ for all $k \geq 0$.
(ii) The sequence $\{t_k\}_{k\in\mathbb{N}_0}$ is unbounded.

Remark. Assumption A2 allows the sampling sequence to be non-uniform. The lower boundedness condition on the sampling periods is not restrictive since υ can be taken arbitrarily small.

Considered sampled-data system can be rewritten in the following way, for $k \in \mathbb{N}$ and $t \in (t_k, t_{k+1}]$,

$$\dot{x} = f(x, e), \tag{5}$$
$$\dot{e} = g(e, x), \tag{6}$$

for $t = t_k$,

$$x(t_k^+) = x(t_k), \tag{7}$$
$$e(t_k^+) = 0, \tag{8}$$

where $e = x - x(t_k)$, $f = [f_p(x_p, h_{ce})^\top, f_c(x_c, h_{pe})^\top]^\top$, $h_{pe}(x, e) = h_p(x(t_k)) = h_p(x_p - e_p)$, $h_{ce}(x, e) = h_c(x_c(t_k), y_k) = h_c(x_c - e_c, h_{pe})$ and $g(e, x) = f(x, e)$. Due to the properties of the functions f_p, f_c, h_p, h_c thus introduced functions f and g are locally Lipschitz. Since by assumption A2 the sampling sequence is not generated independently, the system (5)-(8) satisfies the classical semigroup property (see Example 2.12 in [7]).

The proposed presentation of the sampled data system is similar to this of [16] with the difference in the definition of the variable e.
Our objective is to establish certain stability properties of the system (5-8) in case where Assumptions A1-2 are satisfied. Namely we are interested in semi-global stability property defined next.

Definition 1. *System (5-8) is said to be* ***Semi-Globally Asymptotically Stable*** *(SGAS) with respect to T if for all $\Delta \in \mathbb{R}_{>0}$, there exist $T_{\max \in \mathbb{R}_{>0}}$, $\beta \in \mathcal{KL}$ such that for all $T \in [\upsilon, T)_{\max}$, $x(t_\circ) \in B_\Delta$ and for all $t \in [t_\circ, \infty)$ the following inequality holds:*

$$|[x(t)^\top, e(t)^\top]| \leq \beta(|[x(t_\circ)^\top, e(t_\circ)^\top]|, t - t_\circ). \tag{9}$$

If (9) holds for $\Delta = \infty$, then system (5-8) is said to be ***Globally Asymptotically Stable*** *(GAS).*

The approach we use is quite similar to the one proposed in [8] for design of hybrid observers for sampled-data systems. Indeed, similar to [8] an ISS-like property with respect to the measurement errors is exploited for the stability analysis. Actually, we base our analysys on the following theorem which is similar to the result given in Theorem 2 of [20] but in our case the bound on the possible input does not depend on the system initial condition but rather on the radius of the ball of initial conditions for the state $\varDelta$ and a chosen overshoot.

Theorem 1. *Consider the system*

$$\dot{x} = f(x, u), \tag{10}$$

where $x \in R^n$ *and function* $f : R^n \times R^m \to R^n$ *is continuous and locally Lipschitz. Let* $\varDelta \in \mathbb{R}_{>0}$ *be arbitrary and* $x_\circ \in B_\Delta$*. If the system (10) is GAS with the input* $u \equiv 0$*, then there exist function* $\beta \in \mathcal{KL}$*, a continuous positive definite function* $\delta : R_{\geq 0} \to R_{\geq 0}$ *and for each* $\varDelta > 0$ *there exists function* $\gamma_\Delta \in \mathcal{K}$ *such that for any* $t_\circ,\, t \geq 0,\, t \geq t_\circ$ *and each measurable, essentially bounded input* $u(\cdot)$ *for which*

$$\|u\|_{[t_\circ,\, t)} < \delta(\varDelta), \tag{11}$$

the solution of (10) exists at least for $\tau \in [t_\circ, t)$ *and satisfies on this interval the following bound*

$$|x(\tau) \leq \beta(|x|_\circ, \tau - t_\circ) + \gamma_\Delta(\|u\|_{[t_\circ,\, t)}). \tag{12}$$

Proof. Since the origin $x = 0$ is GAS for the system $\dot{x} = f(x, 0)$, then it follows from Proposition 13 in [18] (see also [22]) that there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{K}$ and a Lyapunov function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ such that for all $x \in \mathbb{R}^n$ we have

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \frac{\partial V}{\partial x} f(x, 0) \leq -V(x), \quad |\frac{\partial V}{\partial x}(x)| \leq \alpha_3(|x|).$$

Then, for the system (10) we have

$$\frac{\partial V}{\partial x} f(x, u) \leq -V(x) + \frac{\partial V}{\partial x}\left[f(x, u) - f(x, 0)\right] \leq -V(x) + \alpha_3(|x|)|f(x, u) - f(x, 0)|$$

Since the function f is continuous, it follows from the Lemma 2 that there exist a strictly increasing function $c \in C(\mathbb{R}, [1, \infty))$ and a function $d \in \mathcal{K}$ such that $|f(x, u) - f(x, 0)| \leq c(|x|)d(|u|)$ and therefore we have

$$\frac{\partial V}{\partial x} f(x, u) \leq -V(x) + c_1(|x|)d(|u|),$$

where $c_1(s) = \alpha_3(s)(s)$. Notice that the function $c_1 \in \mathcal{K}$
Let $\varDelta$, $\epsilon > 0$ and $x_\circ \in B_\Delta$ be fixed and arbitrary otherwise, define functions δ_1 and $\psi \in \mathcal{K}_\infty$ as follows

$$\psi(s) = (1 + \epsilon)\alpha_1^{-1} \circ \alpha_2(s), \quad \delta(s) = \frac{\alpha_1 \circ \psi(s) - \alpha_2(s)}{c\left(\psi(s)\right)}.$$

Functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_2(s) \geq \alpha_1(s)$ hence we have that function $\psi \in \mathcal{K}_\infty$ and $\psi(s) > \alpha_1^{-1} \circ \alpha_2(s) > s$ for all $s > 0$, therefore $\alpha_1 \circ \psi(s) - \alpha_2(s) > 0$ for all $s > 0$.

Since $c(s) \geq 1$ for all $s \geq 0$, function δ defined above is a continuous, positive definite function.
Claim 1. If the input satisfies the bound (11) for $\tau \in [t_\circ,\, t)$, then it holds that

$$\|x\|_{[t_\circ,\, t)} \leq \psi(\Delta). \tag{13}$$

Proof of the Claim 1. We proceed by contradiction. Assume that there exists $t^* \in [t_\circ,\, t)$ such that $|x(t^*)| = \psi(\Delta)$ and let $t_1 = \inf\{\tau \in [t_\circ,\, t) \,:\, |x(\tau)| = \psi(\Delta)\}$.

Then for all $\tau \in [t_\circ, t_1]$ we have that

$$\dot{V}_(10) = \frac{\partial V}{\partial x} f(x, e) \leq -V(x) + c(\psi(\Delta))d(|e|),$$

using the comparison principle [10] we obtain that for all $\tau \in [t_\circ, t_1]$ we have

$$\begin{aligned} V(x(\tau)) &\leq V(x_\circ)e^{-(\tau - t_\circ)} + \int_{t_\circ}^{\tau} c(\psi(\Delta))d(||e||_{[t_\circ,\tau)}) \exp(-\tau)d\tau \\ &\leq V(x_\circ)e^{-(\tau - t_\circ)} + c(\psi(\Delta))d(||e||_{[t_\circ,\tau)}). \end{aligned} \tag{14}$$

Combining the last inequality with (13) we obtain that for all $\tau \in [t_0, t_1]$

$$V(x(\tau)) \leq V(x_\circ) + \alpha_1(\psi(\Delta)) - \alpha_2(\Delta) < \alpha_1(\psi(\Delta)).$$

Thus, $V(x(t_1)) < \alpha_1(\psi(\Delta))$ which implies that $|x(t_1)| < \psi(\Delta)$ and we came to the contradiction with the initial assumption that $|x(t_1)| = \psi(\Delta)$ and hence Claim 1 is proved.

Next, since for any $\tau \in [t_\circ, t)$ we have that $|x(\tau)| \leq \psi(\Delta)$ then it follows from (14) and properties of the function $V(x)$ that on the same interval

$$\begin{aligned} \alpha_1(\|x(\tau)\|) \leq V(x(\tau)) \leq V(x_\circ)e^{-(t-t_\circ)} + c(\psi(\Delta))d(||e||_{[t_\circ,\tau)}) \\ \leq \alpha_2(\|x_\circ\|)e^{-(t-t_\circ)} + c(\psi(\Delta))d(||e||_{[t_\circ,\tau)}) \end{aligned}$$

and therefore

$$\begin{aligned} \|x(\tau)\| &\leq \alpha_1^{-1}\left(\alpha_2(\|x_\circ\|)e^{-(t-t_\circ)} + c(\psi(\Delta))d(||e||_{[t_\circ,\tau)})\right) \\ &\leq \alpha_1^{-1}\left(2\alpha_2(\|x_\circ\|)e^{-(t-t_\circ)}\right) + \alpha_1^{-1}\left(2c_\Delta d(||e||_{[t_\circ,\tau)})\right), \end{aligned}$$

where $c_\Delta = c(\psi(\Delta))$.

Since $\alpha_1,\, \alpha_2 \in \mathcal{K}_\infty$ and $d \in \mathcal{K}$, it is clear from the last inequality that there exists function $\beta \in \mathcal{KL}$ and for each $\Delta > 0$ there exists function $\gamma_\Delta \in \mathcal{K}$ such that for all $t \in [t_\circ,\, t)$ the bound (12) is satisfied.

□

4 Main Results

As mentioned in the Introduction, it is well known that the sampling of the system output and the control input is usually source of instability and that the only possibility to overcome this issue consists in restricting the upper bound on the sampling period.

The effect of the sampling is mostly due to the dynamics of the variable e. Thus, it is interesting to estimate an upper bound of this variable taking into account the fact that $e(t_k^+) = 0$, $k \geq 1$, i.e. we start every sampling period with zero initial condition for this variable

Lemma 1. *Consider the system (10) and assume that the function f is continuous, locally Lipschitz and $f(0,0) = 0$. Then, for any $\mu \in \mathbb{R}_{>0}$ there exist a C^1 function $W : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with bounded $\partial W(x)/\partial x$ and a C^1 function $\gamma \in \mathcal{K}$ such that for all $(x, e) \in R^n \times R^m$*

$$\frac{\partial W}{\partial x}(x), f(x,u) \leq \mu W(x) + \gamma(|u|). \tag{15}$$

The proof of the lemma 1 is presented in the appendix. It shows that the function ρ is not necessarily unbounded. Thus, according to Lemma 1, for any $\mu \in \mathbb{R}_{>0}$, there exist $\bar{\alpha} \in \mathbb{R}_{>0} \cup \{\infty\}$ such that $\rho : \mathbb{R}_{\geq 0} \to [0, \bar{\alpha})$ of class $\mathcal{K}$ ($\mathcal{K}_\infty$ if $\bar{\alpha} = \infty$).

Remark. Lemma 1 is similar to Lemma 11 in [18], but here, instead of finding an exponentially decreasing positive definite function of the state, an exponentially increasing one is obtained.

In the remaining part of the paper we assume that for the system (6) a function W is constructed according to Lemma 1 with a constant $\mu \in \mathbb{R}_{>0}$ given. Note that, since W is locally Lipschitz, using the arguments given in the footnote 8 in [15], this holds for almost all $(x, e) \in \mathbb{R}^{n_x + n_e}$, along solutions to (6):

$$\dot{W}(e) \leq \mu W(e) + \gamma(|x|). \tag{16}$$

The following proposition considers the case when subsystem (5) is ISS and gives the conditions under which there exists $T_{\max}$ such that the system (5)-(8) is GAS if the maximal sampling period is less than $T_{\max}$.

We start with introduction of the following assumption which will be used to ensure that the solutions of the sampled data system do not explose during the first sampling period.

Assumption A 3. *The system*

$$\dot{x} = f(x, x + c_e) \tag{17}$$

is forward complete for any parameter $c_e \in R^n$.

Remark 1. From the Theorem 2, [23] it follows that assumption A3 is equivalent to assuming existence of a proper and smooth function function $\Psi(x) : R^n \to R_{\geq 0}$ such that along solutions of (17) we have

$$\dot{\Psi} \leq a\Psi \tag{18}$$

for any $c_e \in R^n$.

Remark 2. Assumption A3 can actually be replaced by the equivalent assumption on forward completeness for the system $\dot{e} = g(e, e + c_x)$. Choice of the assumption depends rather on the simplicity to verify the assumptions for these two systems.

Theorem 2. *Consider the system (5)-(8) and let assumptions* **A1- A3** *hold. Suppose that for the system (5)-(6) there exist positive definite functions V, $W : R^n \to R_{\geq 0}$, functions α_{iv}, g_v, $g_w \in \mathcal{K}_\infty$, $\alpha_{iw} \in \mathcal{K}$, $i = 1, 2$ and positive constants μ and σ such that along solutions of the system (5)-(6) we have*

$$\alpha_{1v}(|x|) \leq V(x) \leq \alpha_{2v}(|x|) \tag{19}$$

$$\alpha_{1w}(|e|) \leq W(e) \leq \alpha_{2w}(|e|) \tag{20}$$

$$\dot{V} \leq -\sigma V + g_v(|e|) \tag{21}$$

$$\dot{W} \leq \mu W + g_w(|x|), \tag{22}$$

and functions g, α satisfy the following linear gain *conditions*

$$g_v \circ \alpha_{1w}^{-1}(s) \ \leq k_1 s \tag{23}$$

$$g_w \circ \alpha_{1v}^{-1}(s) \ \leq k_2 s, \tag{24}$$

where k_1, k_2 are positive constants. Then if $T_{\max}$ from the assumption **A2** *satisfies the inequality $T_{\max} < T_*$, where $T_* = \frac{1}{\mu+\sigma} \ln\left(1 + \frac{\sigma(\sigma+\mu)}{k_1 k_2}\right)$, then the system (5)-(8) is GAS.*

Proof. We start the proof with the remark that there is important difference between the first sampling interval and the rest of the sequence since it is only at the beginning of the 1st sampling interval we can have that $e(t_\circ) \neq 0$ while for all other intervals ($k \geq 1$) we have $e(t_k^+) = 0$, see (8). Therefore, we will teat here these two cases separately and later combine the results together. We start with the case of the first sampling interval.
<u>Case I.</u> $\mathbf{k} = \mathbf{0}$. On the interval $[t_\circ, t_1)$ we can rewrite the system (5)-(6) as follows:

$$\dot{x} = f(x, x + e_\circ)$$
$$\dot{e} = g(e, e + x_\circ).$$

Due to assumption A3 there exists a function $\Psi : R^n \to R_{\geq 0}$ such that (18) is satisfied, hence for any initial conditions $(x_\circ, e_\circ)$ we have that $\dot{\Psi} \leq \Psi$ and therefore, during the interval $[t_\circ, t_1) \subset [t_\circ, t_\circ + T_*)$ we have that $\Psi(x(t, x_\circ, e_\circ), e_\circ) \leq \Psi(x_\circ, , e_\circ)e_*^T$. Since function Ψ is proper and positive definite, there exist functions $\alpha_{i\psi} \in \mathcal{K}_\infty$, $i = 1, 2$ such that $\alpha_{1\psi}(|x|) \leq V(|x|) \leq \alpha_{2\psi}(|x|)$, thus for all $t \in [t_\circ, t_1]$

$$|x(t), e(t)| \leq \alpha_{1\psi}^{-1}\left(\alpha_{2\psi}(|x_\circ,\ e_\circ|)e^{T_*}\right). \tag{25}$$

<u>Case II.</u> $\mathbf{k} \geq \mathbf{1}$. This part of the proof is based on the following two observations:
– starting with $k = 1$ we have that at the beginning of each sampling period $e(t_k^+) = 0$ and therefore we can use (22) to estimate the error $e(t)$ during the sampling period.
– to ensure asymptotic stability it is enough to show that there exists a Lyapunov function $V(x)$ such that for any $k \geq 1$ and any $t \in (t_k, t_{k+1}]$ we have

$$V(x(t)) \leq V(x(t_k)) \tag{26}$$

and moreover, there exists $\varepsilon > 0$ such that

$$V(x_{k+1}) \leq \varepsilon V(x(t_k)). \tag{27}$$

Notice that condition (26) insures Lyapunov stability of solutions, while (27) ensures decreas of the Lyapunov function during each sampling period and thus it's convergence to zero. From convergence to zero of the sequence $V(x_k)$ follows convergence to zero of the $x(t_k)$, hence of $x(t)$ and therefore of the differences $e(t) = x(t) - x(t_k)$.

Thus we only need to ensure that conditions of the theorem guarantee that during any sampling period of the lenght less than T_* inequalities (26), (27) are satisfied. In order to prove (26) we proceed by contradiction. We assume that there exists $k \geq 1$ such that (26) is not true and $t_* \in (t_k, t_{k+1})$ is the first moment such that $V(x(t_*)) = V(x(t_k))$.

Let $t \in (t_k, t_*]$. Since $e(t_k^+) = 0$, then from (22) it follows that

$$\begin{aligned} W(e(t)) &\leq e^{\mu(t-t_k^+)} \int_{t_k}^{t} e^{-\mu\tau} g_w(|x(\tau)|) d\tau \\ &\leq e^{\mu(t-t_k^+)} \int_{t_k}^{t} e^{-\mu\tau} g_w \circ \alpha_{1v}^{-1}(V(x(\tau))) d\tau \leq k_2 e^{\mu(t-t_k^+)} \int_{t_k}^{t} e^{-\mu\tau} g_w V(x(\tau)) d\tau \end{aligned}$$

By assumption, for $\tau \in (t_k, t_*]$ we have that $V(x(\tau)) \leq V(x(t_k))$ and therefore we conclude that

$$W(t) \leq \frac{k_2}{\mu} V(x(t_k)) \left(e^{\mu(t-t_k)} - 1 \right). \tag{28}$$

In a similar way, from (21) we obtain that

$$\begin{aligned} V(t) &\leq V(t_k) e^{-\sigma(t-t_k)} + e^{-\sigma(t-t_k)} \int_{t_k^+}^{t} e^{\sigma\tau} g_v(|e(\tau)|) d\tau \\ &\leq V(t_k) e^{-\sigma(t-t_k)} + k_1 e^{-\sigma(t-t_k)} \int_{t_k^+}^{t} e^{\sigma\tau} W(e(\tau)) d\tau \\ &\leq V(t_k) e^{-\sigma(t-t_k)} + \frac{k_1 k_2}{\mu} V(t_k) e^{-\sigma(t-t_k)} \int_{t_k^+}^{t} e^{\sigma\tau} \left(e^{\mu(\tau-t_k)} - 1 \right) d\tau, \end{aligned}$$

where we used (28) in the last inequality.
After simple but tedious calculations we obtain that

$$V(t) \leq V(t_k) f(t), \tag{29}$$

where

$$f(t) = \frac{k_1 k_2}{\mu(\mu+\sigma)} e^{\mu(t-t_k)} + \left(1 + \frac{k_1 k_2}{\mu(\mu+\sigma)} \right) e^{-\sigma(t-t_k)} - \frac{k_1 k_2}{\mu\sigma}. \tag{30}$$

Notice that $f(t_k^+) = 1$, while during the interval $[t_k^+, t_k^+ + T_*)$ derivative of $f(t)$ satisfies the following bound

$$\begin{aligned} f'(t) &\leq e^{-\sigma(t-t_k^+)} \left(-(\sigma + \frac{k_1 k_2}{\mu+\sigma}) + \frac{k_1 k_2}{\mu+\sigma} e^{(\mu+\sigma)T_{\max}} \right) \\ &< e^{-\sigma(t-t_k^+)} \left(-(\sigma + \frac{k_1 k_2}{\mu+\sigma}) + \frac{k_1 k_2 + \sigma(\mu+\sigma)}{\mu+\sigma} \right) = 0 \end{aligned}$$

and therefore for all $t \in (t_k, t_k + T_*)$ we have that $f(t) < 1$[1]. Now, since $t^* \in (t_k, t_{k+1})$, we have that $t^* \leq t_k + T_{\max} < t_k + T_*$ and therefore from (29) it follows that

$$V(t^*) \leq V(t_k^+) f(t^*) < V(t_k^+) = V(t_k)$$

and we came to the contradiction. Hence the estimate (26) is satisfied during any sampling interval $(t_k, t_{k+1}]$. Next, let $\varepsilon = f(T_{\max})$. Since $T_{\max} < T_*$, we have that $\varepsilon < 1$ and then from (29) we obtain that on any sampling interval

$$V(t_{k+1}) \leq V(t_k) f(t_{k+1}) \leq V(t_k) f(T_{\max}) \leq \varepsilon V(t_k)$$

and so the bound (27) is satisfied for any sampling period $(t_k, t_{k+1}]$. □

5 Conclusions

In this paper, for a general class of nonlinear systems we presented a result on asymptotic stability of a continuous time system in a closed loop with an emulated controller. We use a hybrid formulation that allows to give explicit bounds on the maximum allowable sampling period.

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[1] Notice that actually we need a T_* which corresponds to the positive solution of the equation $f(t) = 1$. Expression for T_* used in the theorem corresponds to the interval where $f'(t)$ is negative. This is done to give a simple expression for T_*. However we can use (30) to get numerically a better estimate for T_*

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Appendix

Proof of Lemma 1. Let $\mu > 0$ be arbitrary and define an auxiliary function $\Phi(x) = \|x\|$. Similar to Lemma 11 in [18] this function will serve the basis to construct function W which satisfies inequality (15). Taking derivative of Φ along the solutions of (10) we obtain

$$\frac{\partial \Phi}{\partial x}(x)\, f(x,u) \leq \|f(x,u)\|\,. \tag{31}$$

Notice that function f satisfies the assumptions of Lemma 2 and therefore there exist C^1 functions λ_i, C^1 functions $\varkappa \in \mathcal{K}$ and *positive* constants $c_i > 0$, $i = 1, 2$ such that

$$\lambda_i(s) = (\varkappa_i(s) + c_i)\, s, \tag{32}$$

and

$$\|f(x,u)\| \leq \lambda_1(\|x\|) + \lambda_2(\|u\|). \tag{33}$$

It follows then that

$$\frac{\partial \Phi}{\partial x}(x)\, f(x,u) \leq \lambda_1(|x|) + \lambda_2(|u|) = \lambda_1(\Phi(x)) + \lambda_2(\|u\|).$$

Next we define the function ρ as

$$\begin{cases} \rho(\tau) = \exp\left(\int_1^\tau \frac{a}{\lambda_1(s)} ds\right) \quad \text{for all} \quad \tau \in \mathbb{R}_{>0} \\ \rho(0) = 0 \end{cases} \tag{34}$$

where $a = \max\{\mu, 2(c_1 + \varkappa_1(1)\}$.

Claim 1. Thus defined function ρ is a continuous, locally Lipschitz function and there exists a constant $c > 0$ such that $\rho'(s) \leq c$ for all $s > 0$.

We will prove this Claim a little bit later while for now we assume that it is true and define function W as $W = \rho \circ \Phi$. Function W is locally Lipschitz (as a composition of 2 locally Lipschitz functions) and we have

$$\begin{aligned} \frac{\partial W}{\partial x}(x) f(x,u) &= \frac{a}{\lambda_1(\Phi(x))} W(x) \frac{\partial \Phi}{\partial x}(x) f(x,u) \\ &\leq \frac{a}{\lambda_1(\Phi(x))} W(x) \left(\lambda_1(\Phi(x)) + \lambda_2(|u|)\right) \\ &\leq \mu W(x) + \frac{\mu W(x)}{\lambda_1(\Phi(x))} \lambda_2(|u|) \\ &= \mu W(x) + \mu \rho'(\Phi(x)) \lambda_2(|u|) \leq \mu W(x) + c\mu \lambda_2(|u|). \end{aligned} \tag{35}$$

This would end the proof of the lemma.

Proof of the Claim. Function ρ defined in (34) is continuous on $\mathbb{R}_{>0}$ and strictly increasing. From (32) we have that for all $s \in [0,1]$

$$c_i s \leq \lambda_1(s) \leq (c_1 + \varkappa_1(1))\, s \tag{36}$$

and therefore $\int_1^\tau \frac{a}{\lambda_1(s)} ds \leq \int_1^\tau \frac{a}{c_1 + \varkappa_1(1)} \frac{ds}{s}$. Since the last integral diverges to $-\infty$ as τ goes to zero we have that function ρ is continuous on $R_{\geq 0}$ and therefore $\rho \in \mathcal{K}$.

In contrast with [18] we can not guarantee that thus constructed function ρ belongs to $\mathcal{K}_\infty$. Actually, this function will belong to $\mathcal{K}_\infty$ only under certain conditions.

Next we will prove that the function ρ is locally Lipschitz. Since it is a C^2 function on $\mathbb{R}_{>0}$, it is enough for us to show that $\lim_{\tau \to 0^+} \rho'(\tau)$ exists and is bounded.[2]

For $\tau \neq 0$ we have

$$\rho'(\tau) = \frac{a}{\lambda_1(\tau)} \rho(\tau), \qquad \rho''(\tau) = \left(\frac{a^2}{\lambda_1^2(\tau)} - \frac{a\lambda_1'(\tau)}{\lambda_1^2(\tau)}\right) \rho(\tau). \tag{37}$$

From (32) it follows that $\lambda_1'(0) = c_1$ and $\lambda'(\tau) > 0$ for all $\tau \geq 0$. Thus there exists a constant $\delta > 0$ such that for $0 < \tau < \delta$ we have $\lambda_1'(\tau) \leq 2c_1$ and we have that on the

[2] In doing this we mostly retrace the steps of proof of Lemma 11, [18]

interval $(0, \delta)$ the function ρ' is positive and strictly increasing and hence $\lim_{\tau\to 0^+} \rho'(\tau)$ exists.

Next we show that this limit is bounded. From the first inequality in (37) we have that on the interval $(0, 1)$

$$\begin{aligned}\rho'(\tau) &= \frac{a}{\lambda_1(\tau)} \exp\left(-\int_\tau^1 \frac{a}{\lambda_1(s)} ds\right) \\ &\leq \frac{a}{c_1\tau} \exp\left(-\int_\tau^1 \frac{a}{c_1 + \varkappa_1(1)} \frac{ds}{s}\right) = \frac{a}{c_1\tau} \exp\left(\frac{a}{c_1 + \varkappa_1(1)} \ln \tau\right) \\ &= \frac{a\tau^{\frac{a}{c_1+\varkappa_1(1)}}}{c_1\tau} \leq \frac{a}{c_1}\tau, \qquad (38)\end{aligned}$$

where we used definition of the constant a in the last inequality. From (38) it follows trivially that $\lim_{\tau\to 0^+} \rho'(\tau) = 0$ and therefore we proved that the function ρ is locally Lipschitz. [3]

To prove boundedness of ρ' on $\mathbb{R}_{\geq 0}$ we are left only with the case $\tau \geq 1$. From (32) it follows that there exists $\tau_\star > 0$ such that $\varkappa(\tau_\star) + c_1 = a$. Without loss of generality we can assume that $\tau_\star > 1$. Using lower estimate $\lambda_1(\tau) \geq c_1\tau$ and (37) we obtain that for all $\tau \in [1, \tau_\star]$ the following holds

$$\rho'(\tau) \leq \frac{a}{c_1\tau} \exp\left(\int_1^\tau \frac{a}{c_1 s}\, \mathrm{d}s\right) = \frac{a}{c_1\tau} \exp\left(\ln \tau^{\frac{a}{c_1}}\right) = \frac{a}{c_1}\tau^{\left(\frac{a}{c_1}-1\right)} \leq \Delta,$$

where $\Delta = \frac{a}{c_1}\tau_\star^{\left(\frac{a}{c_1}-1\right)}$ Finally for all $\tau \geq \tau_\star$ the following holds

$$\begin{aligned}\rho'(\tau) &\leq \frac{a}{c_1\tau} \exp\left(\int_1^{\tau_\star} \frac{a}{c_1 s}\, \mathrm{d}s + \int_{\tau_\star}^{\tau} \frac{a}{as}\, \mathrm{d}s\right) = \frac{a\Delta}{c_1\tau} \exp\left(\int_{\tau_\star}^{\tau} \frac{ds}{s}\right) \\ &\leq \frac{a\Delta}{c_1\tau} \exp\left(\ln \frac{\tau}{\tau_\star}\right) \leq \frac{a}{c_1}\tau_\star^{\left(\frac{a}{c_1}-2\right)},\end{aligned}$$

and therefore the function ρ' is bounded on $R_{>0}$. □

Lemma 2. *Let $n,\, m,\, l \in \mathbb{N} -\ 0$ and $F : \mathbb{R}^{n+m} \to \mathbb{R}^l)$ be continuous function then the following statements are correct for all $(x, y) \in \mathbb{R}^{n+m}$*

A1. *There exists a function $\alpha \in \mathcal{K}$ and a continuouslu differentiable, strictly increasing function $c : R_{\geq 0} \to [1, +\infty)$ such that the following inequality holds*

$$|F(x, y) - F(x, 0)| \leq c(|x|)d(|y|). \qquad (39)$$

A2. *If in addition, function F is locally Lipschitz and $F(0, 0) = 0$, then there exist continuously differentiable functions $\gamma_i \in \mathcal{K}$ and nonnegative constants $c_i \geq 0$ $(i = 1, 2)$ such that*

$$|F(x, y)| \leq \lambda_1(|x|) + \lambda_2(|y|), \qquad (40)$$

where $\lambda_i(s) = [c_i + \gamma_i(s)]\, s$, $i = 1,\, 2$.

[3] Actually, following reasoning of Lemma 11 of [18] and slightly increasing the constant a we can ensure that ρ is a C^1 function. However, since function Φ is only locally Lipschitz, in general we can not expect to find a C^1 function W.

Proof.

A1. From Lemma A.1, [12] we have that there exist functions $\gamma_0,\ \gamma_1 \in \mathcal{K}_\infty$, $\gamma_1 \in C^1$ such that, for all $(x, y) \in \mathbb{R}^{n+m}$,

$$|F(x,y) - F(x,0)| \le \gamma_0(2|y|)\left(1 + \gamma_1(|x|^2 + |y|^2)\right).$$

Using properties of class $\mathcal{K}_\infty$ functions and denoting $c(s) = \left(1 + \gamma_1(2s^2)\right)$, $d(s) = \gamma_0(s)\left(1 + \gamma_1(2s^2)\right)$ we obtain

$$\begin{aligned} |F(x,y) - F(x,0)| &\le \gamma_0(|y|)\left(1 + \gamma_1(2|x|^2) + \gamma_1(2|y|^2)\right) \le \\ &\le \gamma_0(|y|)\left(1 + \gamma_1(2|x|^2)\right) + \gamma_0(|y|)\gamma_1(2|y|^2) \\ &\le \left(\gamma_0(|y|) + \gamma_0(|y|)\gamma_1(2|y|^2)\right)\left(1 + \gamma_1(2|x|^2)\right) = c(\|x\|)d(\|y\|). \end{aligned}$$

Continuous differentiability of the function c and other properties follow straightforward from the definitions of the functions c and d and the fact that $\gamma_1 \in C^1$.

A2. Define $z \in \mathbb{R}^{n+m}$ as $z = (x^\top, y^\top)^\top$ and let $\tilde{F}(z) = F(x, y)$. Since function $\tilde{F}$ is locally Lipschitz in z, hence there exists a continuous function $L : R^{n+m} \to R_{\ge 0}$ such that $\left\|\tilde{F}(z)\right\| \le L(z)\,\|z\|$. Based on $L(x)$ we define function $l_0 : R_+ \to R_+$ as follows $l(s) = \sup_{\{z:\|z\| \le s\}} L(z)$ and $l_0(0) = L(0)$. Since $L(z)$ is continuous, the function $l_0(s)$ is well defined, continuous at $s = 0$ and nondecreasing. It is easy to show that we can always upperbound function l_0 by a strictly increasing continuously differentiable function, i.e there always exists a C^1 function $l_1 \in \mathcal{K}$ and a constant $c_1 \ge 0$ such that $l_0(s) \le l_1(s) + c_1$ for all $s \ge 0$.
Notice that $\|z\| \le \|x\| + \|y\|$ and $l_l(s_1)s_2 \le l_1(s_1)s_1 + l_1(s_2)s_2$ for any $s_1, s_2 \ge 0$; the last one is due to the fact that $l_1 \in \mathcal{K}$. Using this inequalities we obtain that for all $s \ge 0$

$$\begin{aligned} \|F(x,y)\| = \left\|\tilde{F}(z)\right\| &\le (l_1(\|z\|) + c_1)\,\|z\| \le (l_1(\|x\| + \|y\|) + c_1)\,(\|x\| + \|y\|) \le \\ &\le (l_1(2\,\|x\|) + l_1(2\,\|y\|) + c_1)\,(\|x\| + \|y\|) \\ &\le (3l_1(2\,\|x\|) + c_1)\,\|x\| + (3l_1(2\,\|y\|) + c_1)\,\|y\| \end{aligned}$$

□