ON CONGRUENCES AND HOMOMORPHISMS ON SOME NON-DETERMINISTIC ALGEBRAS

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Abstract: Starting with the underlying motivation of developing a general theory of L-fuzzy sets where L is a multilattice (a particular case of non-deterministic algebra), we study the relationship between the crisp notions of congruence, homomorphism and substructure on some non-deterministic algebras which have been used in the literature, i.e. hypergroups, and join spaces. Moreover, we provide suitable extensions of these notions to the fuzzy case.

1 INTRODUCTION

This paper follows the trend of developing fuzzy versions of crisp concepts in mathematics. Specifically, we focus on congruence relations, substructures and homomorphisms in the framework of several hyperstructures, which are non-deterministic algebraic structures in the sense that the operations are not single-valued but set-valued.

The study of congruence relations and homomorphisms between given hyperstructures plays an important role in the general theory of algebraic fuzzy systems. The underlying idea here is to apply methods of the algebraic theory of ordinary congruences and homomorphisms between classical structures in studying suitable extensions for specific hyperstructures, such as hypergroups, and join spaces.

One can find a number of extensions of classical algebraic structures to a fuzzy framework in the literature, all of them based more or less in similar ideas. However, the fuzzy extension of the notion of function has been studied from several standpoints, and this fact complicates the choice of the most suitable definition of fuzzy homomorphism: the most convenient definition seems to depend on particular details of the underlying algebraic structure under consideration.

The study of congruences is important both from a theoretical standpoint and for its applications in the field of logic-based approaches to uncertainty. Regarding applications, the notion of congruence is intimately related to the foundations of fuzzy reasoning and its relationships with other logics of uncertainty. More focused on the theoretical aspects of Computer Science, some authors (Bělohlávek, 2002),(Petković, 2006) have pointed out the relation between congruences, fuzzy automata and determinism.

More on the practical side, applications of the concept of congruence can be seen in the World Wide Web. Concerning web applications, some authors have argued on the convenience of using Answer Set Programming (ASP) in the Semantic Web. Congruence relations have been used in the study of modularization of ASP as a way to structure and ease the program development process. Specifically, composition of modules has been formalized in(Oikarinen and Janhunen, 2008; Janhunen et al., 2007) in terms of equivalence relations which are proper congruence relations.

The previous paragraphs have shown the usefulness of the theory of (crisp) congruences regarding practical applications. At this point, it is important to recall that the problem of providing suitable fuzzifications of crisp concepts is an important topic which has attracted the attention of a number of researchers. Since the inception of fuzzy sets and fuzzy logic, there have been approaches to consider underlying sets of truth-values more general than the unit interval; for instance, consider the L-fuzzy sets introduced by Goguen in (Goguen, 1967), where L is a complete lattice.

There are more general structures than a complete lattice which could host a suitable extension of the notion of fuzzy set, for instance, the multi-

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lattices and other general hyperstructures. The concepts of ordered and algebraic multilattice were introduced in (Benado, 1954): a multilattice is an algebraic structure in which the restrictions imposed on a (complete) lattice, namely, the "existence of least upper bounds and greatest lower bounds" are relaxed to the"existence of minimal upper bounds and maximal lower bounds". An alternative algebraic definition of multilattice was proposed in (Medina et al., 2005), which is more closely related to that of lattice, allowing for natural definitions of related structures such that multisemilattices and, in addition, is better suited for applications. For instance, a general approach to fuzzy logic programming based on a multilattice as underlying set of truth-values was presented in (Medina et al., 2007).

A number of papers have been published on the lattice of fuzzy congruences on different classical algebraic structures, and even in some hyperstructures, for instance (Cordero et al., 2008) studies congruences on a multilattice. In this paper, we will focus on congruences and homomorphisms in the more general setting of hyperstructures (Corsini and Leoreanu, 2003).

The structure of the paper is as follows: in Section 2, we introduce the definitions of some hyperstructures, together with some notational convention to be used in the rest of the paper; then, in Section 3, we concentrate on the definition of homomorphism between nd-groupoids and how it preserves the different subhyperstructures. Later, in Section 4, the relation between the introduced notion of nd-homomorphism and (crisp) congruence on a hyperstructure is investigated. It is in Section 5 where the notion of fuzzy homomorphism is defined and the canonical decomposition theorem is presented. Finally, we present some conclusions and prospects for future work.

2 PRELIMINARY DEFINITIONS

Firstly, let us introduce the preliminary concepts used in this paper:

Definition 1. *A **hypergroupoid** is a pair $(A,\cdot)$ consisting of a non-empty set A together with a hyperoperation $\cdot : A \times A \to 2^A \smallsetminus \varnothing$.*

Traditionally, authors working with hyperstructures have considered the natural restriction of the images of the operation to be non-empty sets, for instance, the structures of *hypergroup* or *join space* (Corsini and Leoreanu, 2003).

Definition 2. *A hypergroupoid $(A,\cdot)$ is said to be a **hypergroup** if the following properties hold:*

- ***Associativity:*** *$(ab)c = a(bc)$ for all $a,b,c \in A$.*
- ***Reproductivity:*** *$aA = Aa = A$ for all $a \in A$. That is, for all $a,b \in A$, the equations $ax = b$ y $xa = b$ have solutions.*

*$(A,\cdot)$ is said to be a **join space** if it satisfies associativity, reproductivity and, moreover, the two following properties hold:*

- ***Commutativity:*** *$ab = ba$ for all $a,b,c \in A$.*
- ***Transposition Property:*** *for all $a,b,c,d \in A$, $a/b \cap c/d \neq \varnothing$ implies $ad \cap bc \neq \varnothing$ where $a/b = \{x \mid a \in xb\}$.*

It is remarkable that in the context of multilattices it is admissible to consider the empty set in the range of the hyperoperation, hence interest arises in developing a suitable extension of the concept of hypergroupoid, the so-called *nd-groupoid*. Despite of the small change regarding the empty set, it is noticeable that the resulting theory differs substantially.

Definition 3. *An **nd-groupoid** is a pair $(A,\cdot)$ consisting of a non-empty set A together with an nd-operation $\cdot : A \times A \to 2^A$.*

Remark 4. *As usual, if $a \in A$ and $X \subseteq A$ then aX will denote $\{ax \mid x \in X\}$ and Xa will denote $\{xa \mid x \in X\}$. In particular, $a\varnothing = \varnothing a = \varnothing$.*

Note, as well, that in the rest of the paper we will frequently write singletons without braces.

3 ON THE DEFINITION OF ND-GROUPOID HOMOMORPHISM

This section introduces the extension of the existing results about homomorphisms to the more general framework of nd-grupoids. Firstly, we begin by discussing the different versions of the concept of homomorphism on hypergroupoids (also called multigroupoids) appearing in the literature. They are usually associated to particular classes of hypergroupoids such as those of hypergroups and join spaces.

Some authors that deal with these and other hyperstructures use the following definitions of homomorphism (Corsini, 2003).

Definition 5. *Let $(A,\cdot)$ and $(B,\cdot)$ be hypergroupoids. A map $h\colon A \to B$ is said to be:*

- ***Benado-homomorphism*** *if $h(ab) \subseteq h(a)h(b)$, for all $a,b \in A$.*
- ***Algebraic-homomorphism*** *if $h(ab) = h(a)h(b)$, for all $a,b \in A$.*

Recall that this definition extends without modification to the framework of nd-groupoids.

Regarding the terminology, we depart here a bit from the usual one. The first one was the original definition by Benado (Benado, 1954), which has been used in several recent papers (Davvaz, 2000; Gentile, 2006; Corsini, 2003). However, it is noticeable that, finally, the authors concentrate mostly on the equality-based definition. This choice is partly due to the excessive generality of Benado's definition, which limits the possibility of obtaining interesting theoretical results. The terminology used in those papers is to call homomorphism to Benado's ones and call *good* (or *strong*) homomorphism to algebraic ones.

We have adopted the term *algebraic* instead of good or strong because this type of homomorphism immediate allows the lifting of classical homomorphisms to the so-called powerset extension. Obviously, the advantage of using algebraic homomorphisms is that one can transfer properties from the powerset to the nd-groupoid very easily, so that the presentation of multivalued (namely, non-deterministic) concepts is greatly simplified.

Theorem 6. *Let $(A,\cdot)$ and $(B,\cdot)$ be nd-groupoids and $h\colon A \to B$ be a Benado-homomorphism. Let $(2^A,\cdot)$, $(2^B,\cdot)$ and $H : 2^A \to 2^B$ be the usual power extension of h, i.e.*

$$X \cdot Y = \bigcup_{\substack{x\in X\\ y\in Y}} xy \qquad \text{and} \qquad H(X) = \{h(x) \mid x \in X\}$$

Then, h is an algebraic-homomorphism if and only if H is a homomorphism.

Proof. Straightforward. □

It should be noticed that the definition of algebraic homomorphism, when applied to the lifted powerset version, collapses to the classical algebraic results on the powerset. Hence, the condition limits too much the non-deterministic interpretation of the concept of morphism in that does not provide any added value to the theory.

Moreover, the term homomorphism should induce the properties of the initial hypergroupoid on the image set. It can be easily checked that this is the case for algebraic-homomorphisms but, in general, it is not true for Benado-homomorphisms as the following examples show.

Example 7. *Let $(A,\cdot)$ and $(B,\cdot)$ be the hypergroupoids $A = \{a,b,c\}$ and $B = \{a,b,c,d\}$ with the hyperoperations defined as in the tables. The inclusion map i from A to B is a Benado-homomorphism. Let us consider the restriction of the operation in B to the codomain $i(A) = A$.*

$(A,\cdot)$

$\cdot$	a	b	c
a	a	b	c
b	b	b	c
c	c	c	c

$(B,\cdot)$

$\cdot$	a	b	c	d
a	a	b	b,c	A
b	b	b	a,c	A
c	c,d	c,d	a,c	A
d	A	A	A	A

$(i(A),\cdot)$

$\cdot$	a	b	c
a	a	b	b,c
b	b	b	a,c
c	c	c	a,c

The initial hypergrupoid $(A,\cdot)$ is commutative, idempotent and associative. However $(i(A),\cdot)$ is neither commutative nor idempotent nor associative (for instance, note that $(ab)c = \{a,c\} \neq a(bc) = \{a,b,c\}$). □

When searching for a suitable definition which allows to transfer properties to the codomain, it is noticeable the following interesting property of multisemilattices homomorphism in the case of the operations are actually hyperoperations (not nd-operations).

Theorem 8. *Let $(A,\cdot)$ and $(B,\cdot)$ be multisemilattices and $h\colon A \to B$ be a Benado-homomorphism. If $ab \neq \varnothing$, for all $a,b \in A$, then $h(ab) = h(a)h(b) \cap h(A)$, for all $a,b \in A$.*

Based on the previous result, we propose a new definition of homomorphism on nd-grupoids, which is stronger than Benado's one and weaker than the algebraic one. Furthermore, the underlying idea follows the categorical meaning of morphism.

Definition 9. *Let $(A,\cdot)$ and $(B,\cdot)$ be nd-groupoids. A map $h\colon A \to B$ is said to be* ***nd-homomorphism*** *if, for all $a,b \in A$, $h(ab) = h(a)h(b) \cap h(A)$.*

Note that, h is nd-homomorphism if, for all $a,b,c \in A$, the following conditions hold:

1. $c \in ab$ implies $h(c) \in h(a)h(b)$, that is, h is a Benado-homomorphism.
2. $h(c) \in h(a)h(b)$ implies that there exists $c' \in ab$ such that $h(c') = h(c)$.

The following examples show that the notion of nd-homomorphism is different from the two other definitions given previously.

Example 10. *Consider the hypergroupoid $(A,\cdot)$ being $A = [0,1]$ and the hyperoperation $\cdot : A \times A \to 2^A$ given by $a \cdot b = [0, \max\{a,b\}]$. Notice that this hypergroupoid is a* join space *(it is commutative, associative, reproductive and verifies the transposition axiom). Let us consider a homomorphism $h\colon A \to A$*

defined by

$$h(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

we have that

$$h(a \cdot b) = \begin{cases} \{0\} & \text{if } a = b = 0 \\ \{0,1\} & \text{otherwise} \end{cases}$$

and

$$h(a) \cdot h(b) = \begin{cases} \{0\} & \text{if } a = b = 0 \\ [0,1] & \text{otherwise} \end{cases}$$

and as a result, one obtains the equality $h(a \cdot b) = h(a) \cdot h(b) \cap h(A)$ *but, in general, the proper inclusion* $h(a \cdot b) \subsetneq h(a) \cdot h(b)$ *holds.*

Therefore, h is an nd-homomorphism but it is not an algebraic-homomorphism. □

Example 11. *Let us consider the set* $H = \{a,b,c\}$ *with the hyperoperation defined by the following table:*

$\cdot$	a	b	c
a	c	c	c
b	c	c	c
c	c	c	H

$(H,\cdot)$ *is a commutative hypergrupoid. Let* $h: H \to H$ *be the homomorphism*

$$h(x) = \begin{cases} c & \text{if } x = a \\ x & \text{otherwise} \end{cases}$$

$Im(h) = \{b,c\}$ *and we have that:*

$$\begin{aligned}
&h(a \cdot a) = h(c) = c \subsetneq h(a) \cdot h(a) \cap Im(h) = Im(h)\\
&h(a \cdot b) = h(c) = c = h(a) \cdot h(b) \cap Im(h)\\
&h(a \cdot c) = h(c) = c \subsetneq h(a) \cdot h(c) \cap Im(h) = Im(h)\\
&h(b \cdot b) = h(c) = c = h(b) \cdot h(b) \cap Im(h)\\
&h(b \cdot c) = h(c) = c = h(b) \cdot h(c) \cap Im(h)\\
&h(c \cdot c) = h(H) = \{b,c\} = h(c) \cdot h(c) \cap Im(h)
\end{aligned}$$

Therefore, h is a Benado-homomorphism but it is not an nd-homomorphism. □

We next discuss the different possible generalizations of the substructures of an nd-groupoid that connect with the corresponding definitions of homomorphism.

The more general definition of subalgebra in a non-deterministic setting was introduced by Benado(Benado, 1954) for multilattices, and later used by several authors (Davvaz, 2000; Gentile, 2006; Corsini, 2003). A second definition follows the line of the algebraic homomorphism and corresponds to the embedding of the classical (deterministic) notion of subgroupoid into the framework of nd-subgroupoids. We introduce a third alternative definition below.

Definition 12. *Let* $(A,\cdot)$ *be an nd-groupoid, X be a non-empty subset of A and* $*: X \times X \to 2^X$ *an nd-operation defined on X.*

- $(X,*)$ *is a* ***Benado-subgroupoid*** *of* $(A,\cdot)$ *if* $a * b \subseteq a \cdot b$*, for all* $a,b \in X$.
- $(X,*)$ *is a* ***Birkhoff-subgroupoid*** *of* $(A,\cdot)$ *if* $a * b = a \cdot b$*, for all* $a,b \in X$.
- $(X,*)$ *is an* ***nd-subgroupoid*** *of* $(A,\cdot)$ *if* $a * b = a \cdot b \cap X$*, for all* $a,b \in X$.

Therefore, $(X,*)$ is a (Benado, Birkhoff, nd)-subgroupoid if the inclusion map $i: X \to A$ is a (Benado, algebraic, nd)-homomorphism. Likewise, the image space of a (Benado, algebraic, nd)-homomorphism is a (Benado, Birkhoff, nd)-subgroupoid of the codomain.

Lemma 13. *Let* $(A,\cdot)$ *and* $(B,\cdot)$ *be nd-groupoids and* $h: A \to B$ *be an nd-homomorphism. Let us consider* $h(A)$ *and the operation* $*$ *defined as* $a' * b' = a'b' \cap h(A)$.

1. *If* $(A,\cdot)$ *is commutative then* $(h(A),*)$ *is commutative.*
2. *If* $(A,\cdot)$ *is associative then* $(h(A),*)$ *is associative.*
3. *If* $(A,\cdot)$ *is idempotent then* $(h(A),*)$ *is idempotent.*
4. *If* $(A,\cdot)$ *is reproductive then* $(h(A),*)$ *is reproductive.*
5. *If* $(A,\cdot)$ *satisfies the transposition property then* $(h(A),*)$ *satisfies the transposition property.*

Proof. It is a matter of systematic calculations. As an example we provide the proof of the case of associativity:

$$\begin{aligned}
\big(h(a) * h(b)\big) * h(c) = {} & \big(h(a)h(b) \cap h(A)\big) * h(c) = \\
& = h(ab) * h(c) = h(ab)h(c) \cap h(A) = \\
& = h\big((ab)c\big) = h\big(a(bc)\big) = \\
& = h(a)h(bc) \cap h(A) = h(a) * h(bc) = \\
& = h(a) * \big(h(b)h(c) \cap h(A)\big) \\
& = h(a) * \big(h(b) * h(c)\big)
\end{aligned}$$

□

Proposition 14. *Let* $(A,\cdot)$ *and* $(B,\cdot)$ *be nd-groupoids and* $h: A \to B$ *be an nd-homomorphism. Let us consider* $h(A)$ *and the operation* $*$ *defined as* $a' * b' = a'b' \cap h(A)$.

1. *If* $(A,\cdot)$ *is a hypergroup then* $(h(A),*)$ *is a hypergroup.*
2. *If* $(A,\cdot)$ *is a join space then* $(h(A),*)$ *is a join space.*

Proof. It is a consequence of the previous lemma. $\square$

In general, given a Benado-homomorphism between nd-groupoids, the properties of the initial nd-groupoid cannot be induced on the codomain, even with the map being bijective. Furthermore, the inverse map of a bijective Benado-homomorphism needs not be a Benado-homomorphism.

Example 15. *Let $A = \{a,b,c\}$ and the two operations defined as $x \cdot y = \{x,y\}$ and $x \star y = A$. Obviously, the identity mapping, I, is a Benado-homomorphism from $(A,\cdot)$ to $(A,\star)$ which is not an nd-homomorphism. However, I^{-1} is not a Benado-homomorphism: $I^{-1}(a \star b) = I^{-1}(A) = A \nsubseteq I^{-1}(a) \cdot I^{-1}(b) = a \cdot b = \{a,b\}$.* $\square$

Theorem 16. *Let h be a bijective mapping between nd-groupoids. The following conditions are equivalent:*

(i) h and h^{-1} are Benado-homomorphisms.
(ii) h is an algebraic-homomorphism.
(iii) h is an nd-homomorphism.

Proof. Obviously items (ii) and (iii) are equivalent because the map is bijective.

$(i) \Rightarrow (ii)$ Since h^{-1} is a Benado-homomorphism, one certainly has that $h^{-1}\big(h(a) \cdot h(b)\big) \subseteq h^{-1}\big(h(a)\big) \cdot h^{-1}\big(h(b)\big) = ab$. Thus, for an element $x \in h(a) \cdot h(b)$, we have $h^{-1}(x) \subseteq a \cdot b$. As a consequence, $x = h\big(h^{-1}(x)\big) \in h(a \cdot b)$.

$(ii) \Rightarrow (i)$ Let $x \in h^{-1}(a'b')$. Then $h(x) \in a'b' = h\big(h^{-1}(a')\big) \cdot h\big(h^{-1}(b')\big)$ and, since h is algebraic homomorphism, $h\big(h^{-1}(a')\big) \cdot h\big(h^{-1}(b')\big) = h\big(h^{-1}(a') \cdot h^{-1}(b')\big)$. Thus, $x \in h^{-1}\Big(h\big(h^{-1}(a') \cdot h^{-1}(b')\big)\Big) = h^{-1}(a') \cdot h^{-1}(b')$. $\square$

As a consequence of the previous result, we will call *nd-isomorphism* to any bijective algebraic homomorphism.

4 CONGRUENCES AND HOMOMORPHISMS IN ND-GROUPOIDS

In this section, we concentrate on the relation between homomorphisms and congruence relations. To begin with, we recall below the notion of congruence relation on an nd-groupoid that we are interested in.

Definition 17. *Let $(A,\cdot)$ be an nd-groupoid. A **congruence** on A is an equivalence relation $\equiv$ which for all $a,b,c \in A$ satisfies that if $a \equiv b$, then $ac \equiv bc$ and $ca \equiv cb$, where $X \equiv Y$ if and only if for all $x \in X$ there exists $y \in Y$ such that $x \equiv y$ and for all $y \in Y$ there exists $x \in X$ such that $x \equiv y$.*

The following result is a direct consequence of the definition.

Proposition 18. *Let $(A,\cdot)$ and $(B,\cdot)$ be nd-groupoids and $h\colon A \to B$ be an nd-homomorphism. The kernel relation $\equiv_h$, defined as*

$$a \equiv_h b \text{ if and only if } h(a) = h(b),$$

is a congruence relation.

It is obvious that an algebraic-homomorphism provides a congruence relation as well. However, being an nd-homomorphism is not a necessary condition to provide a congruence relation.

Example 19. *Let $(\mathbb{Z},\cdot)$ with $a \cdot b = \{a,b\}$ and $(\mathbb{N},\star)$ with $a \star b = \{0,a,b\}$. Let us define $h\colon \mathbb{Z} \to \mathbb{N}$ as $h(a) = |a|$. Then, h is a surjective Benado-homomorphism but is neither an algebraic or nd-homomorphism. Straightforwardly, the* kernel *relation $a \equiv_h b$ iff $|a| = |b|$ is a congruence relation.* $\square$

Theorem 20. *Let $h\colon (A,\cdot) \to (B,\cdot)$ be a Benado-homomorphism between nd-groupoids. The kernel relation is a congruence relation if and only if $h(ab) = h\big(h^{-1}(h(a)) \cdot h^{-1}(h(b))\big)$, for all $a,b \in A$*

Proof. Assume that the kernel relation is a congruence. Since for all $a \in A$ we have $a \in h^{-1}(h(a))$, and h is a Benado-homomorphism, then $h(ab) \subseteq h\big(h^{-1}(h(a)) \cdot h^{-1}(h(b))\big)$.

For the other inclusion, consider $h(x) \in h(h^{-1}(h(a)) \cdot h^{-1}(h(b)))$, there exist $y \in h^{-1}(h(a))$ and $z \in h^{-1}(h(b))$ such that $x \in yz$. Since $h(y) = h(a)$ and $h(z) = h(b)$, under the assumption that the kernel relation is a congruence, $yz \equiv ab$. So, for $x \in yz$, there exists $x' \in ab$ such that $h(x) = h(x') \in h(ab)$.

Conversely, assume the equality and let us prove that the kernel is a congruence. It is sufficient to note the following chain of equalities for all $a,b,c,d \in A$ such that $h(a) = h(b)$ and $h(c) = h(d)$:

$$\begin{aligned} h(ac) &= h(h^{-1}(h(a)) \cdot h^{-1}(h(c))) \\ &= h(h^{-1}(h(b)) \cdot h^{-1}(h(d))) = h(bd) \end{aligned}$$

$\square$

Finally, it is worth to note that every homomorphism h defining a congruence relation can be canonically decomposed as $i \circ \overline{h} \circ p$, where

- p is the projection homomorphism $p(x) = [x]$ and $[a] \cdot [b] = \{[x] \mid x \in ab\}$.
- $\overline{h}$ is the isomorphism defined as $\overline{h}([x]) = h(x)$ and $a \cdot_h b = h(h^{-1}(a) \cdot h^{-1}(b))$.

- i is the inclusion homomorphism corresponding to the type of the homomorphism of h. That is, if h is a Benado-homomorphism (respectively, algebraic or nd-homomorphism) then $(h(A), \cdot_h)$ is a Benado-subgroupoid (respectively, Birkhoff or nd-subgroupoid) and i is a Benado-monomorphism (respectively, algebraic or nd-monomorphism).

5 FUZZY HOMOMORPHISMS ON ND-GROUPOIDS

A fuzzy relation is a mapping φ from $A \times B$ into $[0,1]$, that is to say, any fuzzy subset of $A \times B$. The powerset extension of a fuzzy relation is defined as, $\widehat{\varphi}\colon 2^A \times 2^B \to [0,1]$ with

$$\widehat{\varphi}(X,Y) = \Big(\bigwedge_{x\in X}\bigvee_{y\in Y} \varphi(x,y)\Big) \wedge \Big(\bigwedge_{y\in Y}\bigvee_{x\in X} \varphi(x,y)\Big)$$

The composition of fuzzy relations φ and ψ is defined as follows:

$$(\psi \circ \varphi)(a,c) = \bigvee_{b\in B} \varphi(a,b) \wedge \psi(b,c)$$

A fuzzy relation ρ on $A \times A$ is said to be

1. **reflexive** if $\rho(x,x) = 1$, for every $x \in A$
2. **symmetric** if $\rho(x,y) = \rho(y,x)$, for all $x,y \in A$
3. **transitive** if for all $x,a,y \in A$ we have

$$\rho(x,a) \wedge \rho(a,y) \leq \rho(x,y)$$

A reflexive, symmetric and transitive fuzzy relation on A is called a **fuzzy equivalence**. A fuzzy equivalence ρ on A is called a **fuzzy equality** if for any $x,y \in A$, $\rho(x,y) = 1$ implies $x = y$.

Definition 21 ((Cabrera, 2009))**.** *A fuzzy equivalence relation ρ on an nd-groupoid $(A,\cdot)$ is said to be a* **fuzzy congruence relation** *if $\widehat{\rho}(a_1a_2, b_1b_2) \geq \rho(a_1,b_1) \wedge \rho(a_2,b_2)$ for all $a_1,a_2,b_1,b_2 \in A$.*

The fuzzification of the concept of function that we adopt has been introduced in (Klawonn, 2000), and also studied in (Demirci, 2000; Demirci, 2003; Demirci, 2001), and more recently in (Ćirić et al., 2009). We will introduce the extension of the notion of perfect fuzzy function.

Definition 22 ((Demirci, 2000))**.** *Let ρ and σ be fuzzy equalities defined on the sets A and B, respectively. A* ***partial fuzzy function*** *φ from A to B is a mapping $\varphi\colon A \times B \to [0,1]$ satisfying the following conditions for all $a,a' \in A$ and $b,b' \in B$:*

ext1 $\varphi(a,b) \wedge \rho(a,a') \leq \varphi(a',b)$

ext2 $\varphi(a,b) \wedge \sigma(b,b') \leq \varphi(a,b')$

part $\varphi(a,b) \wedge \varphi(a,b') \leq \sigma(b,b')$

If, in addition, the following condition holds:

f1 *For all $a \in A$ there is $b \in B$ such that $\varphi(a,b) = 1$*

then we say that φ is a ***perfect fuzzy function****.*

It is not difficult to show that the element b in condition (f1) above is unique. As a result, every perfect fuzzy function defines a crisp mapping from A to B called **crisp description of** φ.

Definition 23. *Let $(A,\cdot)$ and $(B,\cdot)$ be nd-groupoids endowed with fuzzy equalities ρ and σ, respectively.*

A perfect fuzzy function $\varphi \in [0,1]^{A\times B}$ is said to be a ***fuzzy homomorphism*** *if for all $a,a' \in A$ and $b,b' \in B$, the following inequality holds:*

compat $\varphi(a,b) \wedge \varphi(a',b') \leq \widehat{\varphi}(aa',bb')$

Moreover, φ is said to be ***complete*** *if the two following conditions hold:*

1. *if $\bigvee_{y\in Y} \varphi(a,y) = 1$, then there exists $y \in Y$ such that $\varphi(a,y) = 1$.*
2. *if $\bigvee_{x\in X} \varphi(x,b) = 1$, then there exists $x \in X$ such that $\varphi(x,b) = 1$.*

Remark: Hereafter, unless stated otherwise, we will always consider that we are working with a complete fuzzy homomorphism φ between nd-groupoids A and B and fuzzy equalities ρ and σ, respectively.

Proposition 24. *Given φ between A and B, the crisp description h of φ is an algebraic homomorphism.*

Proof. Let $a_1,a_2 \in A$. As $\varphi(a_1,h(a_1)) \wedge \varphi(a_2,h(a_2)) \leq \widehat{\varphi}(a_1a_2, h(a_1)h(a_2))$, we have that

$$\bigwedge_{a\in a_1a_2}\ \bigvee_{b\in h(a_1)h(a_2)} \varphi(a,b) \wedge \bigwedge_{b\in h(a_1)h(a_2)}\ \bigvee_{a\in a_1a_2} \varphi(a,b) = 1$$

and so, for all $a \in a_1a_2$ we have that $\bigvee_{b\in h(a_1)h(a_2)} \varphi(a,b) = 1$, and by completeness, there exist $b \in h(a_1)h(a_2)$ such that $\varphi(a,b) = 1$ and, therefore, $b = h(a)$. So $h(a_1a_2) \subseteq h(a_1)h(a_2)$.

Conversely for all $b \in h(a_1)h(a_2)$, we have that $\bigvee_{a\in a_1a_2} \varphi(a,b) = 1$ and by completeness, there exist $a \in a_1a_2$ such that $h(a) = b$, and so $h(a_1)h(a_2) \subseteq h(a_1a_2)$. $\square$

The notion of fuzzy homomorphism between nd-groupoids behaves properly with respect to the composition of fuzzy relations, in that the composition of fuzzy homomorphisms is a fuzzy homomorphism. Furthermore, the composition is associative and there exists an identity for this composition. As a result, the class of nd-groupoids together with the fuzzy homomorphisms between them forms a category.

Let us concentrate now on the relationship between fuzzy homomorphism and congruences.

Definition 25. *The **fuzzy kernel relation** induced by* φ *in A is defined as* $\rho_\varphi(a,a') = \varphi(a,h(a'))$.

We adopt here the term *kernel* as an extension of the crisp case because of the inequality

$$\varphi(a,b) \wedge \varphi(a',b) \leq \rho_\varphi(a,a')$$

which can be checked by direct computation.

Theorem 26. *Consider* φ *between A and B. The fuzzy kernel relation* ρ_φ *is a congruence relation which includes the fuzzy equality* ρ *in A.*

Proof. Let us see that ρ_φ is a congruence relation.

$$\begin{aligned}
&\widehat{\rho}_\varphi(a_1a_3,a_2a_4) = \\
&= \bigwedge_{a\in a_1a_3}\bigvee_{a'\in a_2a_4} \rho_\varphi(a,a') \wedge \bigwedge_{a'\in a_2a_4}\bigvee_{a\in a_1a_3} \rho_\varphi(a,a') \\
&= \bigwedge_{a\in a_1a_3}\bigvee_{a'\in a_2a_4} \varphi(a,h(a')) \wedge \bigwedge_{a'\in a_2a_4}\bigvee_{a\in a_1a_3} \varphi(a,h(a')) \\
&= \widehat{\varphi}(a_1a_3,h(a_2a_4)) \\
&= \widehat{\varphi}(a_1a_3,h(a_2)h(a_4)) \qquad \text{by Prop. 24} \\
&\geq \varphi(a_1,h(a_2)) \wedge \varphi(a_3,h(a_4)) \\
&= \rho_\varphi(a_1,a_2) \wedge \rho_\varphi(a_3,a_4)
\end{aligned}$$

Now, let us show that $\rho \leq \rho_\varphi$:

$$\begin{aligned}
\rho(a,a') &= \rho(a,a') \wedge \varphi(a',h(a')) \\
&\leq \varphi(a,h(a')) = \rho_\varphi(a,a') \qquad \text{by (ext1)}
\end{aligned}$$

□

In the rest of this section we will show the canonical decomposition theorem for a complete fuzzy homomorphism and a fuzzy congruence relation. For suitable extensions on the notions of injectivity and surjectivity we will rely on the definitions given in (Demirci, 2000).

Definition 27. *A perfect fuzzy function* $\varphi \in [0,1]^{A\times B}$ *is said to be:*

- ***surjective*** *if, for all* $b \in B$ *there exists* $a \in A$ *such that* $\varphi(a,b) = 1$.
- ***injective*** *if, for all* $a,a' \in A$ *and* $b \in B$ *we have* $\varphi(a,b) \wedge \varphi(a',b) \leq \rho(a,a')$.
- ***bijective*** *if it is injective and surjective.*

The image set is $\mathrm{Im}\ \varphi = \{b \in B \mid$ *there exists* $a \in A$ *with* $\varphi(a,b) = 1\}$.

In order to define the different homomorphisms involved in the decomposition theorem, we have to introduce the quotient set associated to a fuzzy equivalence relation.

Definition 28. *Let* $(A,\cdot)$ *be an nd-groupoid and* ρ *be a fuzzy equivalence relation in A. An **equivalence class** of an element* $a \in A$ *is defined as*

$$\rho(a) \in [0,1]^A \quad \textit{with} \quad \rho(a)(a') = \rho(a,a')$$

*The **quotient set** is defined as* $A/_\rho = \{\rho(a) \mid a \in A\}$ *and a fuzzy equality* $\overline{\rho}$ *can be defined in* $A/_\rho$ *as* $\overline{\rho}(\rho(a),\rho(a')) = \rho(a,a')$.

*The **fuzzy projection*** π *from A to* $A/_\rho$ *is defined as* $\pi(a,\rho(a')) = \rho(a,a')$.

Proposition 29. *Let* $(A,\cdot)$ *be an nd-groupoid,* ρ *a fuzzy equality in A and* ρ_A *be a fuzzy congruence relation in A that includes* ρ*. The **fuzzy projection*** π *from A to* $A/_{\rho_A}$ *is a surjective fuzzy homomorphism where the nd-operation in* $A/_{\rho_A}$ *is given by*

$$\rho_A(a_1)\cdot\rho_A(a_2) = \{\rho_A(d) \mid d \in a_1a_2\}$$

and the fuzzy equality is $\overline{\rho_A}$.

Proof. We will only prove properties (ext1) and (compat), as the rest of the required properties are more or less straightforward computations.

ext1 We will use that $\rho \leq \rho_A$ and ρ_A is symmetric and transitive: $\pi(a_1,\rho_A(a_2)) \wedge \rho(a_1,a_3) = \rho_A(a_1,a_2) \wedge \rho(a_1,a_3) \leq \rho_A(a_1,a_2) \wedge \rho_A(a_1,a_3) \leq \rho_A(a_3,a_2) = \pi(a_3,\rho_A(a_2))$.

compat $\pi(a_1,\rho_A(a_2)) \wedge \pi(a_3,\rho_A(a_4)) = \rho_A(a_1,a_2) \wedge \rho_A(a_3,a_4) \leq \widehat{\rho_A}(a_1a_3,a_2a_4) = \widehat{\pi}(a_1a_3,\rho_A(a_2)\rho_A(a_4))$ □

Remark: In order to prove that the canonical inclusion from the image of a homomorphism is an injective fuzzy homomorphism, we recall the following result from (Demirci, 2000): given φ between A and B, there exists a unique crisp function f such that $\varphi(a,b) = \sigma(f(a),b)$. This f actually coincides with the crisp description h of φ, which satisfies $\varphi(a,h(a)) = 1$.

Lemma 30. *Given* φ *between A and B, then the inclusion* ι *from* $\mathrm{Im}\ \varphi$ *to B defined as* $\iota(b,b') = \sigma(b,b')$ *is an injective fuzzy homomorphism.*

Proof. We only prove property (compat):

Let $b_1,b_2 \in \mathrm{Im}\ \varphi$ and $a_1,a_2 \in A$ such that $h(a_1) = b_1$ and $h(a_2) = b_2$. Then $\iota(b_1,b_3) \wedge \iota(b_2,b_4) = \sigma(b_1,b_3) \wedge \sigma(b_2,b_4) = \sigma(h(a_1),b_3) \wedge \sigma(h(a_2),b_4) = \varphi(a_1,b_3) \wedge \varphi(a_2,b_4) \leq \widehat{\varphi}(a_1a_2,b_3b_4)$. Now, by Proposition 24, $\widehat{\varphi}(a_1a_2,b_3b_4) = \widehat{\iota}(h(a_1a_2),b_3b_4) = \widehat{\iota}(h(a_1)h(a_2),b_3b_4) = \widehat{\iota}(b_1b_2,b_3b_4)$. □

Theorem 31. *Any complete fuzzy homomorphism* φ *from A to B can be canonically decomposed as* $\varphi = \iota\circ\overline{\varphi}\circ\pi$ *where* π *is the fuzzy projection from A to* $A/_{\rho_\varphi}$*,* ι *is the inclusion from* $\mathrm{Im}\ \varphi$ *to B, and* $\overline{\varphi}$ *is the isomorphism from* $A/_{\rho_\varphi}$ *to* $\mathrm{Im}\ \varphi$ *defined as* $\overline{\varphi}(\rho_\varphi(a),b) = \varphi(a,b)$*, and the nd-operation and the fuzzy equality in* $\mathrm{Im}\ \varphi$ *being the corresponding restrictions of those in B.*

Proof. Firstly, let us prove (ext1), (inj) and (surj), the rest of properties are straightforward:

ext1 $\overline{\varphi}(\rho_\varphi(a),b) \wedge \overline{\rho_\varphi}(\rho_\varphi(a),\rho_\varphi(a')) = \varphi(a,b) \wedge \rho_\varphi(a,a') = \varphi(a,b) \wedge \varphi(a,h(a')) = \varphi(a,b) \wedge \varphi(a,h(a')) \wedge \varphi(a',h(a')) \leq \sigma(b,h(a')) \wedge \varphi(a',h(a')) \leq \varphi(a',b) = \overline{\varphi}(\rho_\varphi(a'),b)$

inj $\overline{\varphi}(\rho_\varphi(a),b) \wedge \overline{\varphi}(\rho_\varphi(a'),b) = \varphi(a,b) \wedge \varphi(a',b) \leq \rho_\varphi(a,a') = \overline{\rho_\varphi}(\rho_\varphi(a),\rho_\varphi(a'))$.

surj For all $b \in \mathrm{Im}\ \varphi$ there exists $a \in A$ such that $\varphi(a,b) = 1$ and then $\overline{\varphi}(\rho_\varphi(a),b) = 1$

Finally, let us check that $\varphi = \iota \circ \overline{\varphi} \circ \pi$:

$$
\begin{aligned}
(\iota \circ \overline{\varphi} \circ \pi)(a,b) &= \\
&= \bigvee_{\substack{\rho_\varphi(a') \in A/\rho_\varphi \\ b' \in \mathrm{Im}\ \varphi}} \Big(\pi(a,\rho_\varphi(a')) \wedge \overline{\varphi}(\rho_\varphi(a'),b') \wedge \iota(b',b)\Big) \\
&= \bigvee_{\substack{a' \in A \\ b' \in \mathrm{Im}\varphi}} \Big(\rho_\varphi(a,a') \wedge \varphi(a',b') \wedge \sigma(b',b)\Big) \\
&\overset{(ext2)}{\leq} \bigvee_{a' \in A} \Big(\rho_\varphi(a,a') \wedge \varphi(a',b)\Big) \\
&\overset{(def\rho_\varphi)}{=} \bigvee_{a' \in A} \Big(\varphi(a,h(a')) \wedge \varphi(a',b)\Big) \\
&\overset{(f1)}{=} \bigvee_{a' \in A} \Big(\varphi(a,h(a')) \wedge \varphi(a',h(a')) \wedge \varphi(a',b)\Big) \\
&\overset{(part)}{\leq} \bigvee_{a' \in A} \Big(\varphi(a,h(a')) \wedge \sigma(h(a'),b)\Big) \\
&\overset{(ext2)}{\leq} \varphi(a,b)
\end{aligned}
$$

Conversely, $\varphi(a,b) = \sigma(h(a),b) = \pi(a,\rho_\varphi(a)) \wedge \overline{\varphi}(\rho_\varphi(a),h(a)) \wedge \sigma(h(a),b) \leq (\iota \circ \overline{\varphi} \circ \pi)(a,b)$. □

6 CONCLUSIONS AND FUTURE WORK

We have studied the relationship between the crisp notions of congruence, homomorphism and substructure on some non-deterministic algebras which have been used in the literature, i.e. hypergroups, and join spaces, as a step towards developing a general theory of L-fuzzy sets where L is a multilattice (a particular case of non-deterministic algebra). Moreover, we have provided suitable extensions of these notions to the fuzzy case.

As future work, we will study homomorphisms and congruences in some algebraic nd-structures, such as hyper-rings and hyper-near-rings, with the aim of investigating the adequate notion of ideal for them, together with the possible interactions with several applications of these structures already published in the literature. Then, we will analyze the behaviour of the extension of a fuzzy homomorphism between nd-groupoids to the powerset structures. At the same time, we will investigate weaker definitions which extend that of nd-homomorphism, in order to to obtain a more abstract and flexible approach. The fuzzy notions introduced in this paper will be studied in a more general framework by substituting the unit interval with a lattice-based structure.

Finally, it is worth to note that the notion of function compatible with certain structure is crucial in the study of fields such as Functional Dependencies in Databases, Attribute Implication in Formal Concept Analysis, Association Rules in Datamining, etc. Thus, we will analyze the possible applications derived from our work on these research lines.

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