COONS TRIANGULAR BÉZIER SURFACES

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Abstract: In this paper we give some different surface generation methods starting out from prescribed boundary curves. If the boundary control points are known it is natural to think of Coons patches, a popular solution of the problem of finding a surface given its boundary curves. We have developed three methods to generate triangular patches given the boundary curves. First we give a discrete version of the triangular Coons patch. A second method lets us to find the extremals of a functional as a solution of a linear system of the control points. That functional is the one that minimizes the Coons patch. The third method makes it possible to build a Bézier triangle by means of a mask deduced from the characterization of cubical extremals of the functional.

1 INTRODUCTION

One of the oldest surface problems in CAGD is the following: given the boundary curves, find the parametric surface $\overrightarrow{x}$ with these as boundary curves with no other restriction. A popular solution of this problem is the Coons patch.

One of the aims of this paper is to find the extremals of a functional as a solution of a linear system of the control points. This functional is the one that minimizes the Coons patch.

Some other work about finding extremals of a functional was already done. For rectangular patches in (Monterde, 2003), (Monterde, 2004) and (Monterde and Ugail, 2004), the functional they work with is the Dirichlet functional. The Dirichlet functional is related with the theory of minimal surfaces due to the fact that is a linear functional having the same extremals as the area functional. For the triangular Bézier case (Arnal et al., 2003) worked with the Dirichlet functional too. When the geometric problem is get Bézier approximations to constant mean curvature surfaces the study of the appropriate functional appears in (Arnal et al., 2008). Finally, in a more general way, for rectangular patches in (Monterde and Ugail, 2006) a general quadratic functional was studied.

Before we present our study about triangular Coons patches, let us describe the more conventional rectangular Coons patch and its properties.

2 BACKGROUND ON COONS RECTANGULAR PATCHES

Coons first described this type of interpolant in (Coons, 1967). It is assumed that four boundary curves are given, which it is convenient to think of as coming from a surface denoted $\overrightarrow{x_0}$, and so the notation $\overrightarrow{x_0}(u,0)$, $\overrightarrow{x_0}(u,1)$, $\overrightarrow{x_0}(0,v)$ and $\overrightarrow{x_0}(1,v)$ is used to represent these boundary curves. The bilinearly blended Coons patch that interpolates to the given boundary curves is defined by:

$$\begin{aligned}&\overrightarrow{x}(u,v) = (1-u)\overrightarrow{x_0}(0,v) + u\overrightarrow{x_0}(1,v)\\&+(1-v)\overrightarrow{x_0}(u,0)+v\overrightarrow{x_0}(u,1)\\&-\begin{pmatrix}1-u & u\end{pmatrix}\begin{pmatrix}\overrightarrow{x_0}(0,0) & \overrightarrow{x_0}(0,1)\\ \overrightarrow{x_0}(1,0) & \overrightarrow{x_0}(1,1)\end{pmatrix}\begin{pmatrix}1-v\\ v\end{pmatrix}.\end{aligned}$$

The Coons rectangular patch interpolates four boundary curves and in addition is an extremal of the functional

$$\mathcal{F}(\overrightarrow{x}) = \int_U \|\overrightarrow{x_{uv}}\|^2 du\,dv, \qquad (1)$$

where $U = [0,1]\times[0,1]$, over all patches, $\overrightarrow{x} \in \mathcal{C}^\infty[u,v]$, with a prescribed boundary. The Coons patch was described in (Nielson et al., 1978) as the unique interpolant that minimizes the functional $\mathcal{F}(\overrightarrow{x})$.

In general if a surface, $\overrightarrow{x}$, is an extremal of a functional, then it satisfies the associated Euler-Lagrange equation, which, for this functional, is the PDE

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$$\overrightarrow{x}_{uuvv} = 0. \qquad (2)$$

Therefore the Coons patch can be considered as a PDE surface, since it is a solution of the equation above.

Instead of working with the general problem of finding extremals of the functional $\mathcal{F}$, we will consider a restricted problem, namely that of finding the polynomial patch that minimizes the functional among all polynomial patches with the same boundary.

Some work related with rectangular Coons patches was carried out in (Farin and Hansford, 1999). While the boundary curves $\overrightarrow{x_0}(u,0)$, $\overrightarrow{x_0}(u,1)$, $\overrightarrow{x_0}(0,v)$ and $\overrightarrow{x_0}(1,v)$ may be totally arbitrary, in the early days the boundary curves were considered as discretized curves with many points on them. In fact, in (Farin and Hansford, 1999) these boundary polygons are treated as Bézier border control points and a discrete version of the Coons patch is given. The interior control points $P_{i,j}$ are defined in terms of boundary points by the discrete Coons patch:

$$P_{i,j} = \left(1-\frac{i}{m}\right)P_{0,j} + \frac{i}{m}P_{m,j} + \left(1-\frac{j}{m}\right)P_{i,0} + \frac{j}{n}P_{i,n} - \begin{pmatrix} 1-\frac{i}{m} & \frac{i}{m} \end{pmatrix} \begin{pmatrix} P_{0,0} & P_{0,n} \\ P_{m,0} & P_{m,n} \end{pmatrix} \begin{pmatrix} 1-\frac{j}{n} \\ \frac{j}{n} \end{pmatrix}.$$

for $0 < i < m$ and $0 < j < n$. These control points define the discrete Coons patch which is the same patch as if Coons interpolation was applied to the Bézier curves associated to the boundary polygons.

The discrete Coons patch also minimizes the discrete version of the functional $\mathcal{F}$. In fact, the discrete Coons patch is a PDE Bézier surface satisfying the discrete version of $\overrightarrow{x}_{uuvv} = 0$.

3 TRIANGULAR COONS PATCHES

Now after introducing all these topics for rectangular surfaces, let us come back to triangular patches. The triangular Coons patch we will define first appeared in (Nielson et al., 1978). Similar to the rectangular Coons patch we consider the border curves $\overrightarrow{x_0}(u,0)$, $\overrightarrow{x_0}(0,v)$ and $\overrightarrow{x_0}(u,1-u)$, (or $\overrightarrow{x_0}(1-v,v)$), to denote the boundary curves and define the patch as

$$\begin{aligned}\overrightarrow{x}(u,v) = {} & (1-u-v)\left(\overrightarrow{x_0}(u,0)+\overrightarrow{x_0}(0,v)-\overrightarrow{x_0}(0,0)\right)\\ & + v\left(\overrightarrow{x_0}(0,u+v)+\overrightarrow{x_0}(u,1-u)-\overrightarrow{x_0}(0,1)\right)\\ & + u\left(\overrightarrow{x_0}(u+v,0)+\overrightarrow{x_0}(1-v,v)-\overrightarrow{x_0}(1,0)\right).\end{aligned}$$

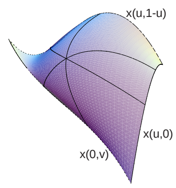

Figure 1: A representation of a triangular Coons patch.

Some differences with respect to the rectangular Coons patch must be pointed out. First let us remark that if we consider the border curves to be polynomial curves of degree n, then the associated triangular Coons patch is a degree $n+1$ polynomial surface. This increase in degree does not happen in the rectangular case.

In contrast to the rectangular case we find two more differences. First since the triangular patch is not linear in both variables, then $\overrightarrow{x}_{uuvv} \neq 0$. On the other hand the Triangular Coons patch is not an extremal of the functional $\mathcal{F}$. It can be proved that for the triangular case, being an extremal of such a functional is not equivalent to satisfying the associated Euler-Lagrange equation, as was true for the rectangular Coons patch: An extremal of the functional, described in Equation (4), would coincide with the solution of its associated Euler-Lagrange equation, $\overrightarrow{x}_{uuvv} = 0$, only under certain conditions on the control points.

Now, analogously to what was done in (Farin and Hansford, 1999) for rectangular patches, we have obtained the discrete version of the triangular Coons patch.

Definition 1. *The interior points $P_{i,j,k}$ with $i+j+k = n+1$, of the Triangular Discrete Coons patch are defined by*

$$\begin{aligned}P_{i,j,k} = {} & \frac{k}{n+1}\left(P_{i,0,n-i}+P_{0,j,n-j}-P_{0,0,n}\right)\\ & +\frac{j}{n+1}\left(P_{0,n-k,k}+P_{i,n-i,0}-P_{0,n,0}\right) \qquad (3)\\ & +\frac{i}{n+1}\left(P_{n-k,0,k}+P_{n-j,j,0}-P_{n,0,0}\right).\end{aligned}$$

The triangular Bézier surface with the previous interior control points coincides with the triangular Coons patch that would be obtained from the Bézier curves associated to the boundary control points.

In the following proposition we give a formula to express the functional of a Bézier triangular patch,

$$\mathcal{F}(\overrightarrow{x}) = \int_{\mathcal{T}} \|\overrightarrow{x_{uv}}\|^2 du\, dv, \qquad (4)$$

defined now in the region $\mathcal{T} = \{(u,v) \in \mathbb{R}^2 : 0 \leq$

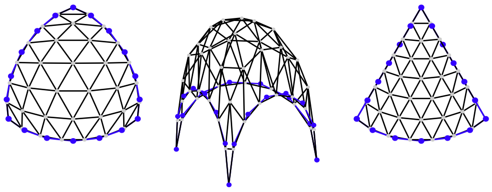

Figure 2: Three discrete triangular Coons patches.

$u, 0 \leq v, u+v \leq 1\}$, in terms of the control points $P_I = \left(x_I^1, x_I^2, x_I^3\right)$, where $I = (i, j, k)$.

Proposition 2. *The functional, $\mathcal{F}(\overrightarrow{x})$, of a triangular Bézier surface can be expressed by the formula*

$$\mathcal{F}(\overrightarrow{x}) = \sum_{a=1}^{3} \sum_{|I_0|=n} \sum_{|I_1|=n} C_{I_0 I_1} x_{I_0}^a x_{I_1}^a \tag{5}$$

where $|I| = i+j+k$, and with

$$\begin{aligned} C_{I_0 I_1} =& 2n(2n-1) \frac{\binom{n}{I_0}\binom{n}{I_1}}{\binom{2n}{I_0+I_1}} \Big(\frac{1}{2}\left(b_{12}^{12}+b_{13}^{13}+b_{23}^{23}+b_{33}^{33}\right) \\ & - b_{12}^{13} - b_{12}^{23} + b_{12}^{33} + b_{13}^{23} - b_{13}^{33} - b_{23}^{33}\Big), \end{aligned} \tag{6}$$

where the coefficients b_{rs}^{tl} satisfy the symmetry relation $b_{rs}^{tl} = b_{sr}^{tl} = b_{rs}^{lt} = b_{sr}^{lt}$, and are defined by

$$b_{rs}^{tl} = \begin{cases} \dfrac{I_0^r I_0^s I_1^t I_1^r + I_1^r I_1^s I_0^t I_0^r}{(I_0^r+I_1^r)(I_0^s+I_1^s)(I_0^t+I_1^t)(I_0^r+I_1^r-1)} & r = l \\ \dfrac{I_0^r I_0^s I_1^t (I_1^t-1) + I_1^r I_1^s I_0^t (I_0^t-1)}{(I_0^r+I_1^r)(I_0^s+I_1^s)(I_0^t+I_1^t)(I_0^t+I_1^t-1)} & t = l \\ \dfrac{2 I_0^r I_0^s I_1^r I_1^r}{(I_0^r+I_1^r)(I_0^r+I_1^r-1)(I_0^s+I_1^s)(I_0^s+I_1^s-1)} & r = t, s = l \\ \dfrac{2 I_0^r (I_0^r-1) I_1^t (I_1^t-1)}{(I_0^r+I_1^r)(I_0^r+I_1^r-1)(I_0^t+I_1^t)(I_0^t+I_1^t-1)} & r = s, t = l \\ \dfrac{I_0^r I_0^s I_1^r (I_1^r-1) + I_1^r I_1^s I_0^r (I_0^r-1)}{(I_0^r+I_1^r)(I_0^s+I_1^s)(I_0^r+I_1^r-1)(I_0^t+I_1^t-2)} & r = t = l \\ \dfrac{2 I_0^r (I_0^r-1) I_1^r (I_1^r-1)}{(I_0^r+I_1^r)(I_0^r+I_1^r-1)(I_0^r+I_1^r-2)(I_0^r+I_1^r-3)} & r = s = t = l. \end{cases} \tag{7}$$

Proof: The functional $\mathcal{F}$ is a second-order functional and, therefore, in order to obtain the coefficients $C_{I_0 I_1}$ we compute its second derivative, first from Equation (5):

$$\begin{aligned} \frac{\partial^2 \mathcal{F}(\overrightarrow{x})}{\partial x_{I_0}^a \partial x_{I_1}^a} &= \frac{\partial^2}{\partial x_{I_0}^a \partial x_{I_1}^a} \sum_{a=1}^{3} \sum_{|I|=n} \sum_{|J|=n} C_{IJ} x_I^{\bar{a}} x_J^{\bar{a}} \\ &= \frac{\partial}{\partial x_{I_1}^a} \sum_{|J|=n} 2 C_{I_0 J} x_J^a = 2 C_{I_0 I_1}. \end{aligned}$$

And now, we compute the first derivative from Equation (4):

$$\begin{aligned} \frac{\partial \mathcal{F}(\overrightarrow{x})}{\partial x_{I_0}^a} &= \int_{\mathcal{T}} \frac{\partial}{\partial x_{I_0}^a} \|\overrightarrow{x_{uv}}\|^2 du\, dv \\ &= \int_{\mathcal{T}} 2 < \frac{\partial \overrightarrow{x}_{uv}}{\partial x_{I_0}^a}, \overrightarrow{x}_{uv} > du\, dv \\ &= \int_{\mathcal{T}} 2 < (B_{I_0}^n)_{uv}, \overrightarrow{x}_{uv} > du\, dv, \end{aligned}$$

Let us denote by $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Then, the second derivative is given by:

$$\begin{aligned} \frac{\partial^2 \mathcal{F}(\overrightarrow{x})}{\partial x_{I_0}^a \partial x_{I_1}^a} &= 2 \int_{\mathcal{T}} < (B_{I_0}^n)_{uv}, \frac{\partial \overrightarrow{x}_{uv}}{\partial x_{I_1}^a} > du\, dv \\ =& 2 \int_{\mathcal{T}} < (B_{I_0}^n)_{uv}, (B_{I_1}^n)_{uv} > du\, dv \\ =& 2 \int_{\mathcal{T}} n^2 (n-1^2) \left(B_{I_0-e_1-e_2}^n - B_{I_0-e_1-e_3}^n \right. \\ & \left. - B_{I_0-e_2-e_3}^n + B_{I_0-2e_3}^n\right) \left(B_{I_1-e_1-e_2}^n \right. \\ & \left. - B_{I_1-e_1-e_3}^n - B_{I_1-e_2-e_3}^n + B_{I_1-2e_3}^n\right) du dv \\ =& 2n(2n-1) \frac{\binom{n}{I_0}\binom{n}{I_1}}{\binom{2n}{I_0+I_1}} \Big(\frac{1}{2}\left(b_{12}^{12}+b_{13}^{13}+b_{23}^{23}+b_{33}^{33}\right) \\ & - b_{12}^{13} - b_{12}^{23} + b_{12}^{33} + b_{13}^{23} - b_{13}^{33} - b_{23}^{33}\Big), \end{aligned}$$

where we have computed the integral of the Bernstein polynomials with the formula:

$$\int_{\mathcal{T}} B_{I_0+I_1}^{2n-2}(u,v)\, du dv = \frac{1}{(3n-2)(3n-3)},$$

and we have performed some simplifications like the following:

$$\begin{aligned} & \int_{\mathcal{T}} B_{I_0-e_1-e_2}^{n-1} B_{I_1-e_1-e_3}^{n-1} + B_{I_1-e_1-e_2}^{n-1} B_{I_0-e_1-e_3}^{n-1} du dv \\ &= \int_{\mathcal{T}} \frac{\binom{n-2}{I_0-e_1-e_2}\binom{n-2}{I_1-e_1-e_3} + \binom{n-2}{I_1-e_1-e_2}\binom{n-2}{I_0-e_1-e_3}}{\binom{2n-4}{I_0+I_1-2e_1-e_2-e_3}} \cdot \\ & \qquad B_{I_0+I_1-2e_1-e_2-e_3}^{2n-4} du dv \\ &= \frac{2n(2n-1)}{n^2(n-1^2)} \frac{\binom{n}{I_0}\binom{n}{I_1}}{\binom{2n}{I_0+I_1}} \frac{I_0^1 I_0^2 I_1^1 I_1^3 + I_0^1 I_0^3 I_1^1 I_1^2}{(I_0^1+I_1^1)(I_0^2+I_1^2)(I_0^1+I_1^1-1)(I_0^3+I_1^3)} \\ &= \frac{2n(2n-1)}{n^2(n-1^2)} \frac{\binom{n}{I_0}\binom{n}{I_1}}{\binom{2n}{I_0+I_1}} b_{12}^{13}. \end{aligned}$$

Therefore

$$\begin{aligned} C_{I_0 I_1} =& 2n(2n-1) \frac{\binom{n}{I_0}\binom{n}{I_1}}{\binom{2n}{I_0+I_1}} \Big(\frac{1}{2}\left(b_{12}^{12}+b_{13}^{13}+b_{23}^{23}+b_{33}^{33}\right) \\ & - b_{12}^{13} - b_{12}^{23} + b_{12}^{33} + b_{13}^{23} - b_{13}^{33} - b_{23}^{33}\Big) \end{aligned}$$

where b_{rs}^{tl} are defined in Equation (7) .

■

Let us remark that the formula we give in Equation (5) translates the functional, $\mathcal{F}$, into a function of the control points. This fact, will allow us to compute compute the gradient of the functional $\mathcal{F}$ with respect to the coordinates of a general control point $P_{I_0} = \left(x_{I_0}^1, x_{I_0}^2, x_{I_0}^3\right)$ to obtain an extremal of the functional among all Bézier surfaces with the same border as a solution of a linear system.

Proposition 3. *A triangular control net, $\mathcal{P} = \{P_I\}_{|I|=n}$, is an extremal of the functional, $\mathcal{F}$, among all triangular Bézier surfaces with a prescribed boundary if and only if:*

$$\sum_{|J|=n} C_{I_0 J} P_J = 0 \quad \text{for all} \quad |I_0 = (I_0^1, I_0^2, I_0^3)| = n \quad (8)$$

with $I_0^1, I_0^2, I_0^3 > 0$, where C_{IJ} are the coefficients defined in Equation (6).

Proof: The gradient of the functional with respect to the coordinates of an interior control point $P_{I_0} = \left(x_{I_0}^1, x_{I_0}^2, x_{I_0}^3\right)$ is given by

$$\begin{aligned}\frac{\partial \mathcal{F}(\vec{x})}{\partial P_{I_0}} &= \left(\frac{\partial \mathcal{F}(\vec{x})}{\partial x_{I_0}^1}, \frac{\partial \mathcal{F}(\vec{x})}{\partial x_{I_0}^2}, \frac{\partial \mathcal{F}(\vec{x})}{\partial x_{I_0}^3}\right)\\ &= 2\left(\sum_{|J|=n} C_{I_0 J} x_J^1, \sum_{|J|=n} C_{I_0 J} x_J^2, \sum_{|J|=n} C_{I_0 J} x_J^3\right)\\ &= 2\sum_{|J|=n} C_{I_0 J} P_J .\end{aligned}$$

■

Equivalently, a triangular control net, $\mathcal{P} = \{P_I\}_{|I|=n}$, is an extremal among all control nets with prescribed border control points if and only if

$$\begin{aligned}0 = \sum_{|I|=n} \frac{\binom{n}{I}}{\binom{2n}{I_0+I}} \Big(\frac{1}{2}\left(b_{12}^{12} + b_{13}^{13} + b_{23}^{23} + b_{33}^{33}\right)\\ - b_{12}^{13} - b_{12}^{23} + b_{12}^{33} + b_{13}^{23} - b_{13}^{33} - b_{23}^{33}\Big) P_I\end{aligned}$$

for all $|I_0 = (I_0^1, I_0^2, I_0^3)| = n$ with $I_0^1, I_0^2, I_0^3 > 0$, with the coefficients b_{rs}^{tl} given in Equation (7).

4 COONS MASKS AND TRIANGULAR PERMANENCE PATCHES

In general a condition that relates some control points can be written by means of a mask only if (considering a 3×3 triangular grid),

$$\begin{array}{c} P_{i-1,j-1,k+2}\ P_{i-1,j,k+1}\ P_{i-1,j+1,k}\ P_{i-1,j+2,k-1} \\ P_{i,j-1,k+1}\ P_{i,j,k}\ P_{i,j+1,k-1} \\ P_{i+1,j-1,k}\ P_{i+1,j,k-1} \\ P_{i+2,j-1,k-1} \end{array}$$

this condition relates the points on the grid in such a way, that the interior point can be expressed in terms of the boundary control points. The mask is then considered to be a stencil for the central point.

Some previous work related to masks can be found in (Farin and Hansford, 1999). The rectangular Coons patch, as well as the associated discrete Coons patch, satisfies a **Permanence Principle**: *Let two points (u_0, v_0) and (u_1, v_1) define a rectangle R in the domain U of the Coons patch. The four boundaries of this subpatch will map onto four curves on the Coons patch. The Coons patch for those four boundary curves is the original Coons patch restricted to the rectangle R.*

Moreover, as we said before, the rectangular Coons patch is a PDE surface satisfying $\vec{x}_{uuvv} = 0$ and the discrete version of this partial differential equation is verified exactly by the discrete Coons patch. Farin and Hansford, in the previously cited paper, (Farin and Hansford, 1999), deduced the following rectangular mask from this discrete PDE.

$$P_{i,j} = \frac{1}{4} \times \begin{array}{ccc} -1 & 2 & -1 \\ 2 & \star & 2 \\ -1 & 2 & -1 \end{array}$$

In that work, the authors generalized it by defining what they called permanence patches: A permanence patch is obtained from a control net

$$P_{i,j} = \begin{array}{ccc} \alpha & \beta & \alpha \\ \beta & \star & \beta \\ \alpha & \beta & \alpha \end{array}$$

with $4\alpha + 4\beta = 1$.

This kind of mask suggests the possibility of different choices for α and β, so in this sense Farin and Hansford, show how some choices of these values give different masks which are also the discrete form of a PDE, as the discrete version of the Euler-Lagrange PDE $\vec{x}_{uuvv} = 0$, gave the first rectangular mask $\alpha = \frac{-1}{4}$.

Moreover Farin and Hansford extended the permanence patches concept to the triangular case just by considering the analogous triangular mask.

Given a mask of the form

$$P_{i,j,k} = \begin{array}{ccccccc} \alpha & & \beta & & \beta & & \alpha \\ & \beta & & \star & & \beta & \\ & & \beta & & \beta & & \\ & & & \alpha & & & \end{array} \quad (9)$$

with $3\alpha + 6\beta = 1$ the triangular patch formed with such a control net is called a triangular permanence patch.

Now, let us come back to rectangular patches and show how the $\alpha = \frac{-1}{4}$ mask was deduced from the Euler-Lagrange PDE $\overrightarrow{x}_{uuvv} = 0$.

The discrete version of $\overrightarrow{x}_{uuvv} = 0$ is given by $\Delta^{2,2} P_{i,j} = 0$, where

$$\begin{aligned}\Delta^{1,0} P_{i,j} &= P_{i+1,j} - P_{i,j} \\ \Delta^{0,1} P_{i,j} &= P_{i,j+1} - P_{i,j}.\end{aligned}$$

Then

$$\begin{aligned}0 = \Delta^{2,2} P_{i,j} &= P_{i+2,j+2} - 2P_{i+2,j+1} - 2P_{i+1,j+2} \\ &+ 4P_{i+1,j+1} - 2P_{i+1,j} - 2P_{i,j+1} + P_{i+2,j} + P_{i,j+2} + P_{i,j}\end{aligned}$$

gives

$$\begin{aligned}P_{i,j} = \frac{-1}{4}\,(&P_{i+1,j+1} - 2P_{i+1,j} - 2P_{i,j+1} - 2P_{i,j-1} \\ &-2P_{i-1,j} + P_{i+1,j-1} + P_{i-1,j+1} + P_{i-1,j-1}\,)\end{aligned} \qquad (10)$$

that is the rectangular mask $\alpha = \frac{-1}{4}$.

This mask could also be deduced as a consequence of the permanence principle. Let us show this. We will determine for which value of α and β, with $4\alpha + 4\beta = 1$, a permanence patch satisfies the permanence principle.

This principle implies that the control point $P_{i,j}$ can be obtained with the discrete Coons formula, Equation (2), from the boundary control points on a $n \times m$ grid or instead one can apply this formula to any 3×3 grid included in the global grid,

$$\begin{array}{ccc} P_{i-1,j-1} & P_{i-1,j} & P_{i-1,j+1} \\ P_{i,j-1} & P_{i,j} & P_{i,j+1} \\ P_{i+1,j-1} & P_{i+1,j} & P_{i+1,j+1} \end{array}.$$

Therefore if we consider that any point in the equation

$$\begin{aligned}P_{i,j} =& \alpha\,(P_{i+1,j+1} + P_{i+1,j-1} + P_{i-1,j-1} + P_{i-1,j+1}) \\ &+ \beta\,(P_{i+1,j} + P_{i,j+1} + P_{i,j-1} + P_{i-1,j}),\end{aligned}$$

can be written in terms of the boundary control points, as we said by means of Equation (2), it leads us to the value $\alpha = \frac{-1}{4}$.

The permanence principle is not verified by triangular Coons patches so the previous reasoning cannot be followed in order to obtain a mask describing the Coons triangle. Anyway we will introduce a mask, which generates a permanence patch, since it is of the kind defined in Equation (9), and which is related to the triangular Coons patch.

We will consider the triangular control net of a triangular Coons patch of degree 3, instead of the general case of degree n,

$$\begin{array}{ccccccc} P_{003} & & P_{012} & & P_{021} & & P_{030} \\ & P_{102} & & P_{111} & & P_{120} & \\ & & P_{201} & & P_{210} & & \\ & & & P_{300} & & & \end{array}$$

The interior control point, P_{111}, is defined, by Equation (3), in terms of the boundary control points of a grid of degree 2. Moreover, the boundary control points on the degree 3 control net are the control points of the degree elevation of degree 2 border curves.

To obtain a triangular mask grenerating a permanence patch we will use the following result that gives us a version of Proposition 3 for the case $n = 3$.

Proposition 4. *A triangular control net of degree 3, $\mathcal{P} = \{P_I\}_{|I|=3}$, is an extremal of the functional, $\mathcal{F}(\mathcal{P})$, among all triangular control nets with a prescribed boundary if and only if*

$$P_{111} = \frac{1}{2}\,(P_{012} - P_{021} + P_{102} + P_{120} - P_{201} + P_{210}).$$

From this condition, given the exterior control points in the case of degree n, we can generate the whole triangular net by solving a linear system where the equations are:

$$\begin{aligned}2P_{i,j,k} =& P_{i-1,j,k+1} - P_{i-1,j+1,k} + P_{i,j-1,k+1} \\ &+ P_{i,j+1,k-1} - P_{i+1,j-1,k} + P_{i+1,j,k-1}\end{aligned}$$

$P_{i,j,k}$ being a interior control point. This equation can be expressed by the following mask:

$$P_{i,j,k} = \frac{1}{2} \quad \times \quad \begin{array}{ccccccc} 0 & & 1 & & -1 & & 0 \\ & 1 & & \star & & 1 & \\ & & -1 & & 1 & & \\ & & & 0 & & & \end{array} \qquad (11)$$

Then if we consider that any interior or border control point in the equation

$$\begin{aligned}P_{111} =& \alpha\,(P_{003} + P_{030} + P_{300}) \\ &+ \beta\,(P_{012} + P_{021} + P_{102} + P_{201} + P_{120} + P_{210})\end{aligned}$$

can be written, thanks to Equation (3), in terms of control points of a degree 2 control net, we find that equality is only attained for the values $\alpha = \frac{-2}{3}$ and $\beta = \frac{1}{2}$.

Therefore the triangular permanence patch for $\alpha = \frac{-2}{3}$ gives the triangular Coons patch of degree 3, although in general a mask cannot be used to obtain a Coons triangle of degree n.

5 GRAPHICS EXAMPLES

Now, let us show some examples of the triangular Bézier surfaces we can obtain, given a boundary, by means of the three different methods we have presented in this work: Coons interpolation, minimization of the functional $\mathcal{F}$ and with the use of the mask defined in Equation (11).

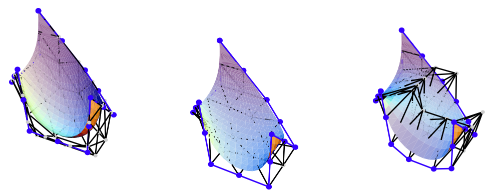

Figure 3: Three Bézier surfaces with the same border. On the left the triangular Coons patch. The one in the middle is a Bézier extremal of the functional $\mathcal{F}$. The figure on the right is obtained by means of the mask in Equation (11).

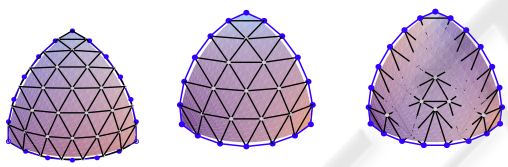

Figure 4: Three more examples of Bézier triangles, the triangular Coons patch, the Bézier extremal of $\mathcal{F}$ in the middle and the Bézier surface built with the mask (11).

From the previous figures it can be seen that the control nets obtained by means of the mask, in Equation (11), derived from the functional $\mathcal{F}$ are quite irregular in comparison with the nets obtained as extremals of the functional.

6 CONCLUSIONS

Here we have conducted a study of one of the most important solutions to the problem of finding a surface interpolating boundary curves: triangular Coons patches in comparison with rectangular Coons patches. We have described three different surface generation methods that start out from prescribed boundary curves.

We have characterized the control net of a triangular Bézier extremal of the functional $\mathcal{F}$. From this characterization we have developed two methods to generate triangular patches given the boundary curves. The first method is to find the extremals of the functional as a solution of a linear system of the control points. The second method makes it possible to build a Bézier triangle by means of a mask deduced from the characterization of cubical extremals.

On the other hand, we have defined the Triangular Discrete Coons patch and we have compared the shapes of the surfaces obtained by these three surface generation methods. We have observed that better results are obtained for the extremals of the functional and for the triangular Coons patch, but the Coons patch implies an increase of degree.

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REFERENCES

Arnal, A., Lluch, A., and Monterde, J. (2003). Triangular bézier surfaces of minimal area. In *ICCSA (3)*, volume 2669 of *Lecture Notes in Computer Science*, pages 366–375. Springer.

Arnal, A., Lluch, A., and Monterde, J. (2008). Triangular bézier approximations to constant mean curvature surfaces. In *ICCS (2)*, volume 5102 of *Lecture Notes in Computer Science*, pages 96–105. Springer.

Coons, S. A. (1967). Surfaces for computer aided design of space forms. Technical Report MAC-TR-41, MIT.

Farin, G. and Hansford, D. (1999). Discrete Coons patches. *Comput. Aided Geom. Design*, 16(7):691–700. Dedicated to Paul de Faget de Casteljau.

Monterde, J. (2003). The plateau-bézier problem. In *IMA Conference on the Mathematics of Surfaces*, volume 2768 of *Lecture Notes in Computer Science*, pages 262–273. Springer.

Monterde, J. (2004). Bézier surfaces of minimal area: the Dirichlet approach. *Comput. Aided Geom. Design*, 21(2):117–136.

Monterde, J. and Ugail, H. (2004). On harmonic and biharmonic Bézier surfaces. *Comput. Aided Geom. Design*, 21(7):697–715.

Monterde, J. and Ugail, H. (2006). A general 4th-order pde method to generate bézier surfaces from the boundary. *Computer Aided Geometric Design*, 23(2):208–225.

Nielson, G. M., Thomas, D., and Wixom, J. (1978). Boundary data interpolation on triangular domains. Technical Report GMR-2834, General Motors Research Laboratories.