A SUB-OPTIMAL KALMAN FILTERING FOR DISCRETE-TIME LTI SYSTEMS WITH LOSS OF DATA

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Abstract: In this paper a sub-optimal Kalman filter estimator is designed for the plants subject to loss of data or insufficient observation. The methodology utilized is based on the closed-loop compensation algorithm which is computed through the so-called Modified Linear Prediction Coefficient (MLPC) observation scheme. The proposed approach is aimed at the artificial observation vector which in fact corrects the prediction cycle when loss of data occurs. A non-trivial mass-spring-dashpot case study is also selected to demonstrate some of the key issues that arise when using the proposed sub-optimal filtering algorithm under missing data.

1 INTRODUCTION

Loss of observation is a non-trivial case of study in both control and communication systems. Such loss may be due to the faulty sensors, limited bandwidth of communication channels, confined memory space, and mismatching of measurement instruments to name but a few. Overcoming the side effects arose from missing data in control and communication systems are remained as open research problems for researchers during the last decade (Allison, 2001).

Perhaps, the best known tool for the linear estimation problem is Kalman filtering (Khan and Gu, 2009b). However, Kalman filter depends heavily on the plant dynamics, information of unmeasured stochastic inputs, and measured data and hence it is prone to fail if e.g., data is unavailable for measurement update step. To overcome such shortcomings, one approach for state estimation is to utilise the so-called Open-Loop Estimation (OLE) when observations are subjected to random loss, see e.g. (Schenato, 2005; Liu and Goldsmith, 2004; Sinopoli and Schenato, 2007; Schenato et al., 2007). They have studied LOOB cases, while running the Kalman filter in a open loop fashion, i.e. whenever observation is lost, the predicted quantities are processed for next iteration, without any update.

More specifically, in OLE the prediction is based on the system model and processed as state estimation without being updated due to the unavailability of the observed data. Nonetheless, in practice this approach may diverge at the presence of longer loss duration and it is likely that error covariance could exceed the limits if the upper and lower bounds of error covariance are provided (Huang and Dey, 2007). Another shortcoming of the OLE is the sharp spike phenomena when the observation is resumed after the loss. This is because the Kalman filter gain is set to zero at the OLE during the loss time. But when observation is resumed, Kalman gain first surges to the very high gains and then tries to approach the steady state values in order to compensate loss impact. This consequently results in a sudden peak to reach to the normal trajectory of the estimated state which is not a desirable behaviour for a reliable estimation algorithm. Detail stability analysis of OLE can be found in (Li Xie, 2007).

Under loss of observations for a longer period of time, there is a requirement for an advanced estimation technique which could provide superior estimation performance under loss of data so as to maintain the error covariance bounded. Our proposed approach in this paper is based on an artificial optimal observation vector which is computed based on the minimum error generated through the so-called Modified Linear Prediction Coefficient (MLPC). Another advantage of the proposed method is that it eliminates the spike of the OLE technique.

(Micheli, 2001) has considered a delay in the data arrival which may also be translated as lost or inaccurate measured data. In (Schenato, 2005), a system is assumed to be subjected to both LOOB and delay of observation at the same time. All the above works have suggested switching to an OLE estimator when

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there is LOOB and a closed loop estimator when the observation arrives at destination. This will aim in fact at designing an estimator which is strongly time-varying and stochastic in nature. In order to avoid random sampling and stochastic behaviour of the designed Kalman filter, (Khan and Gu, 2009b) has proposed a few approaches to compensate the loss of observations in the state estimation through Linear Prediction.

Throughout this paper we shall call the variables in the case of loss of data as "compensated variables", e.g. $P_k^{\{2\}}$ is called the compensated filtered error covariance at time step k with loss of observation. The rest of the paper is organized as follows. The theory of the Linear Prediction Coefficient (LPC) is overviewed in Section II. In Section III we discuss the proposed sub-optimal Kalman filter with loss of data. The mass-spring-dashpot case study is given in Section IV. Simulation results are presented in Section V. Section VI summarizes our conclusions.

2 THEORY OF LINEAR PREDICTION COEFFICIENT

Linear prediction (LP) is an integral part of signal reconstruction e.g. speech recognition. The fundamental idea behind this technique is that the signal can be approximated as a linear combination of past samples, see e.g. (Rabiner and Juang, 1993). Whenever there is the loss of observation, a signal window is selected to approximate the lost-data. The weights assigned to this data are computed by minimizing the mean square error. These weights are termed as Linear Prediction Coefficients. Out of the two leading LPC techniques, (namely Internal and External LPC), we shall develop and employ External LPC for LOOB, which suits to our problem with constraints:

- The signal statistical properties are assumed to vary slowly with time.
- Loss window should not be "sufficiently long", otherwise the prediction performance will be inferior.

In this paper, the LP technique is termed as modified because in conventional LPC there is no defined strategy to account the number of previous data, while have defined several simple-to-implement algorithms to decide that factor. One of it would be explain the subsequent section.

Let us assume that the dynamics of the LTI system is given in discrete time and that the data or observation is lost at time instant k. The LP is performed as:

$$\bar{z}_k = \sum_{i=1}^{n} \alpha_i z_{k-i} \tag{1}$$

where $\bar{z}_k$ is called "compensated observation" and $'\alpha'$'s represent weights of linear prediction coefficients for the previous observations and n denotes the order of the LPC filter. Generally speaking, it depicts the maximum number of previous observations considered for computation of compensated observation vector. Also, n is required to be chosen appropriately - higher value of n does not guaranty an accurate approximation of the signal but rather an optimal value of n decides an efficient approximation and hence prediction, see (Rabiner and Juang, 1993).

3 DESIGN OF SUB-OPTIMAL KF WITH LOSS OF DATA

Let us assume that the process under consideration is to be run by random noise signal whose mean and covariance are independent of time, i.e. wide-sense stationary process, given as

$$x_k = Ax_{k-1} + Bu_{k-1} + L_d\xi_k \tag{2}$$
$$z_k = Cx_k + v_k \tag{3}$$

where A, B and C have appropriate dimensions, and x, u, z, ξ and v are state, input, sensed output, plant disturbance and measurement noise, respectively. The plant noise ξ and sensor noise v are assumed to be zero mean white gaussian noises.

CKF computes the priori state estimation which is solely based on (2). This priori estimation is thereby updated with newly resumed observation at each time instant. In the subsequent section, the performance of CKF is tested and verified in a mass-spring-dashpot system which help illustrate the proposed algorithm. If the observation is not available due to any of the reason mention earlier, the compensated observations are calculated through (1).

The posteriori state estimation using this compensated observation will be

$$\bar{x}_{k|k} = x_{k|k-1} + \bar{K}_k(\bar{z}_k - \hat{z}_k) \tag{4}$$

The corresponding a posterior error for this estimate is

$$\begin{aligned} e_{k|k} &= x_k - \bar{x}_{k|k} = x_k - x_{k|k-1} - \bar{K}_k(\bar{z}_k - \hat{z}_k) \\ &= e_{k|k-1} - \bar{K}_k(\bar{z}_k - \hat{z}_k) \end{aligned} \tag{5}$$

where x_k is the actual state of the system. Conservatively, the cost function of the Kalman filter is obtained based on this a posterior error of the state estimation.

The optimal values of the modified linear prediction coefficients (MLPC) are computed based the residual vector as follows.

$$e_z = \bar{z}_k - \hat{z}_k \tag{6}$$

For the compensated estimation algorithm in MLPC, our goal is to minimize the following cost function:

$$\begin{aligned} J_k &= E\left[e_z^T e_z\right] \\ &= E\left[(\bar{z}_k - \hat{z}_k)^T(\bar{z}_k - \hat{z}_k)\right] \end{aligned} \tag{7}$$

The MLPC are computed provided with the minimum cost function i.e.

$$\frac{\partial J_k}{\partial \alpha_j} = 0 = \frac{\partial J_k}{\partial \bar{z}_k} \cdot \frac{\partial \bar{z}_k}{\partial \alpha_j} \tag{8}$$

Performing simple and straight forward algebra the above equation can be simplified as

$$E[\hat{z}_k z_{k-i}] - \sum_{j=1}^{n} \alpha_i E\{z_{k+j} z_{k-i}\} = 0$$

$$\sum_{j=1}^{n} \alpha_i E\{z_{k+j} z_{k-i}\} = E[\hat{z}_k z_{k-i}] \tag{9}$$

$$\sum_{j=1}^{n} \alpha_i \gamma_k[i,j] = r_k(i) \tag{10}$$

or

$$\begin{aligned} R_k \cdot A_{\alpha.k} &= r_k \\ A_{\alpha.k} &= r_k \cdot R_k^{-1} \end{aligned} \tag{11}$$

where

$$R_k = \begin{bmatrix} \gamma_k(0,0) & \gamma_k(0,1) & \cdots & \gamma_k(0,n-1) \\ \gamma_k(1,0) & \gamma_k(1,1) & \cdots & \gamma_k(1,n-1) \\ \gamma_k(2,0) & \gamma_k(2,1) & \cdots & \gamma_k(2,n-1) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_k(n-1,0) & \gamma_k(n-1,1) & \cdots & \gamma_k(n-1,n-1) \end{bmatrix} \tag{12}$$

$$A_{\alpha.k} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{13}$$

and

$$r_k = \begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(n) \end{bmatrix} \tag{14}$$

where $E(z_{k-i}z_{k-j}) = \gamma_k(i,j)$ and $E(z_k z_{k-j}) = \gamma_k(j)$ is the autocorrelation function, which will be explain shortly. Equation (11) requires inverting the matrix of R_k which may be increasingly difficult due to computational demanding, especially at large orders. To get rid of such burdensome calculations, several attempts have been introduced in the literature. Through Levinson Durbon or Leroux-Gueguen algorithm the so-called "Reflection Coefficients (RCs)" are computed, which represent one-to-one linear prediction coefficients. We shall explore and focus how to calculate the optimal values of α_i and n, when the measurement contains a solid deterministic input along with the unmeasured stochastic inputs. In practice, computing the autocorrelation coefficients need extra attention. Generally, the autocorrelation coefficients are represented as

$$\gamma_m = \frac{C_m}{C_0} \tag{15}$$

where C_m is the auto-covariance of y at lag m which is

$$C_m = \frac{1}{n-m} \sum_{j=1}^{n-m} (z_j - \bar{z})(z_{m+j} - \bar{z}) \tag{16}$$

where $\bar{z} = \frac{1}{n}\sum_{j=1}^{n} z_j$ i.e. mean of the data for the selected window. Without loss of generality, we shall assume that that $E(z_k) = CE(x_k) = D_k$.
A straightforward calculation would lead to the result

$$C_m = \frac{1}{n-m}\sum_{j=1}^{n-m}(D_j D_{m+j}) + \bar{D}^2 - \bar{D}\bar{D}_m - \bar{D}\bar{D}_M \tag{17}$$

and

$$C_0 = \frac{1}{n}\sum_{j=1}^{n}(D_j^2) - \bar{D}^2 + \frac{1}{n}\sum_{j=1}^{n}(v_j)^2 \tag{18}$$

where $m \leq \frac{n}{2}$ and

$$D_j = E(z_j) = CE(x_j) \qquad \bar{D}_M = \frac{1}{n-m}\sum_{j=m+1}^{n} D_j$$

$$\bar{D}_m = \frac{1}{n-m}\sum_{j=1}^{n-m} D_j \qquad \bar{D} = \frac{1}{n}\sum_{j=1}^{n} D_j$$

Clearly, one can observe that $\gamma_0 = \frac{C_0}{C_0} = 1$. And $\gamma_1 = \frac{C_1}{C_0} < 1$. Therefore, we can write

$$\gamma_0 \geq \gamma_1 \geq \gamma_2 \geq \cdots \gamma_m \tag{19}$$

The inequality of (19) is an important equation which helps in deciding the order of the LP filter as shown in Algorithm-2. For better understanding of the descriptive design, the measurement vector is written as

$$z_k = \eta_k(Cx_k) + v_k \tag{20}$$

where η_k is a random variable, such that

$$\eta_k = \begin{cases} 1, & \text{if there is no LOOB at time step k} \\ 0, & \text{otherwise} \end{cases} \tag{21}$$

Therefore, the prediction step for the normal operation is as follows

$$\begin{aligned} x_{k|k-1} &= Ax_{k-1|k-1} + Bu_{k-1} \\ P_{k|k-1} &= AP_{k-1|k-1}A^T + L_d Q_{k-1} L_d^T \end{aligned} \quad (22)$$

The above predicted state and predicted state covariance are achievable and remain unaffected with loss of data. The conventional Kalman filter will update the state and covariance on the arrival of observation vector.This updated state and updated state covariance are valid when there is no loss of data, i.e. system is running in the normal operation. However, in the presence of loss of measured data, the above standard technique is failed. Toward this end, we have proposed the closed-loop base MLPC algorithm, which can also tackle the issues arising from data loss for long period of time.

The Open loop estimator propagates the predicted state and covariance without any update due to the unavailability of the measurements as

$$\begin{aligned} x_k^{\{2\}} &= x_k^{\{1\}} = Ax_{k-1}^{\{2\}} + Bu_{k-1} \\ P_k^{\{2\}} &= P_k^{\{1\}} = AP_{k-1}^{\{2\}}A^T + L_d Q_{k-1} L_d^T \end{aligned} \quad (23)$$

While, in the proposed MLCP, the compensated observations are computed through 1 the modified linear prediction scheme providing minimum error production. The estimation produced by compensated observation is very comprehensive than those of open-loop algorithms discussed earlier. The compensated observation are used to calculate compensated innovation vector. Thereafter, the compensated Kalman gain is computed as follows.

$$\bar{K}_k = \bar{P}_k^{\{1\}} C^T (C\bar{\bar{P}}_k^{\{1\}} C^T + \bar{R}_k)^{-1} \quad (24)$$

Hence, the predicted state and covariance are updated using this gain as

$$\begin{aligned} x_k^{\{2\}} &= x_k^{\{1\}} + \bar{K}_k(\bar{z}_k - Cx_k^{\{1\}}) \\ P_k^{\{2\}} &= P_k^{\{1\}} - \bar{P}_k^{\{1\}} C^T (C\bar{\bar{P}}_k^{\{1\}} C^T + \bar{R}_k)^{-1} C\bar{P}_k^{\{1\}} \end{aligned} \quad (25)$$

The closed loop Kalman filtering algorithm is summarized in Algorithm 1. There are various ways to choose the value of the order of LP filter, n. Alternatively among these methods, we have found Algorithm 2 very practical to be implemented in a number of applications.

4 THE CASE STUDY EXAMPLE

The system under study in this paper is a slightly modified version of a mass-spring-dashpot (MSD) system given in (Fekri et al., 2007) as shown in Fig 1 which is a continues time system with dynmics as follows.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + L\xi(t) \\ y(t) &= Cx(t) + \theta(t) \end{aligned} \quad (26)$$

where the state vector is defined as

$$x^T(t) = [\ x_1(t) \quad x_2(t) \quad \dot{x}_1(t) \quad \dot{x}_2(t)\ ] \quad (27)$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b1}{m1} & \frac{b1}{m1} \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_1} & \frac{b_1}{m_2} & -\frac{b_1+b_2}{m_2} \end{bmatrix} \quad (28)$$

Algorithm 1: The proposed closed-loop estimation algorithm using MLPC.

1: At time step: $k-1$, **Prediction** is carried out as $x_k^{\{1\}} = Ax_{k-1}^{\{2\}} + Bu_{k-1}$, and $P_k^{\{1\}} = AP_{k-1}^{\{2\}}A^T + L_d Q_k L_d^T$
2: **Check:** Status of η_k
   if $\eta_k = 1$
   Run normal Kalman filter (obtain Filtered Response i.e. $x_{k|k}$ and $P_{k|k}$)
   Else Obtain compensated filtered response ($x_k^{\{2\}}$ and $P_k^{\{2\}}$) as mentioned below.
3: **Select** a suitable size for window (n) (No. of previous observations) and LP filter order (m) with the constraint $m < n/2$
4: **Construct** autocorrelation matrix R_k.
5: **Construct** modified residual matrix r_k.
6: **Compute** MLPC through $A_{\alpha.k} = R_k^{-1} \cdot r_k$
7: **Calculate** compensated measurement vector as
$$\bar{z}_k = \sum_{j=1}^{n} \alpha_j z_{k-j}$$
8: **Obtain** compensated residual vector
9: **Calculate** Compensated Kalman gain $\bar{K}_k$
10: Measurement update step is carried out as: $x_k^{\{2\}} = x_k^{\{1\}} + \bar{K}_k(\bar{z}_k - Cx_k^{\{1\}})$: and $P_k^{\{2\}} = P_k^{\{1\}} - P_k^{\{1\}} C^T (CP_k^{\{1\}} C^T + \bar{R}_k)^{-1} CP_k^{\{1\}}$:
11: **Return** to Step 1, i.e. repeat prediction cycle;

Algorithm 2: Selection of LP filter order.

1: Select γ_{th}.
2: **Compute** $\gamma_i = \frac{C_i}{C_0}$ $\quad i = 1, 2, \cdots m$
3: **Check**: Is $\gamma_i < \gamma_{th}$,
   Yes Stop further computation of γ_i
   $m \leftarrow i$ and select order of LP filter as $n = 2m + 1$.
   Else
4: **Compute** $i \longleftarrow i + 1$
5: **Repeat** Step 2

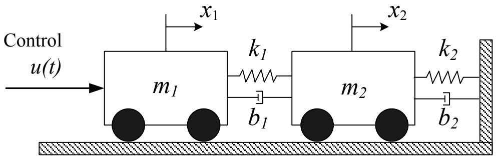

Figure 1: MSD two cart system.

$$B^T = [\ 0 \quad 0 \quad \tfrac{1}{m_1} \quad 0\ ] \tag{29}$$

$$C = [\ 0 \quad 1 \quad 0 \quad 0\ ] \tag{30}$$

$$L = [\ 0 \quad 0 \quad 0 \quad 3\ ] \tag{31}$$

The known parameters are $m_1 = m_2 = 1$, $k_1 = 1$, $k_2 = 0.15$ and $b_1 = b_2 = 0.1$ and the sampling time is $T_s = 1msec$. Plant disturbance and sensor noise dynamics are characterized as

$$E\{\xi(t)\} = 0,\ E\{\xi(t)\xi(\tau)\} = \Xi\delta(t-\tau),\ \Xi = 1 \tag{32}$$

$$E\{\theta(t)\} = 0, \quad E\{\theta(t)\theta(\tau)\} = 10^{-6}\delta(t-\tau) \tag{33}$$

After substituting the above known values the matrices will be as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -0.1 & 0.1 \\ 0.1 & -1.15 & 0.1 & -0.2 \end{bmatrix} \tag{34}$$

and

$$B^T = [\ 0 \quad 0 \quad 1 \quad 0\ ] \tag{35}$$

In subsequent section, we will apply the proposed MLPC algorithm to the above MSD system and show some of the representative results. Many others were also done but are not shown in this paper due to lack of space.

5 SIMULATION RESULTS

Here we implement the above closed-loop MLPC algorithm to the MSD system as discussed in Section IV. For the purpose of our study, the continuous-time dynamics of the MSD system is transformed to an appropriate discrete-time model. Results depict the performance of the Kalman filter when it is running under the open loop i.e. during the period of unavailability of observation, the prediction is not updated and the predicted state and covariance are propagated for the next time instant, see also (Khan and Gu, 2009a). Figure 3 shows the performance of conventional kalman filter via plotting the measured signal; x_2, the position of Mass 2 with no loss of data.

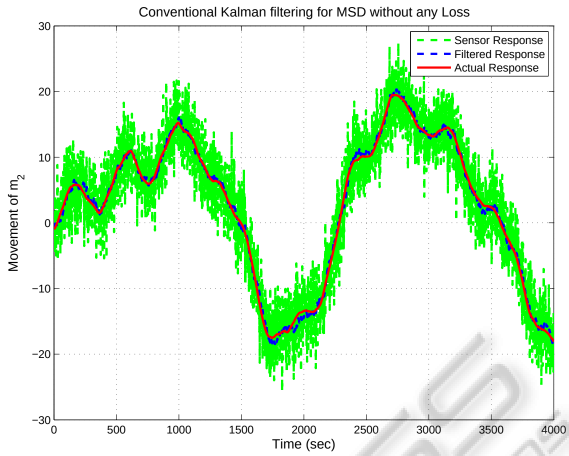

Figure 2: Performance of CKF without data loss.

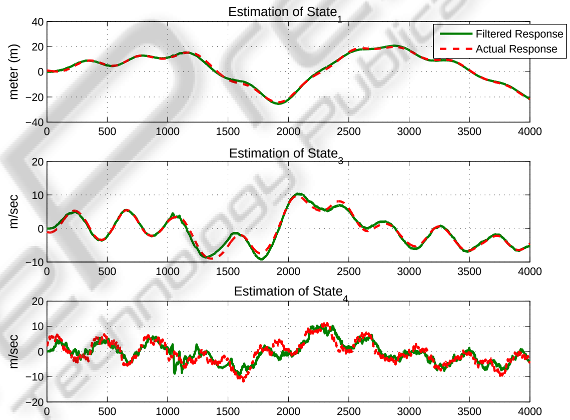

Figure 3: Other plant states.

Figure 4 shows three other states of the MSD plant which again depicts the performance of conventional Kalman filter when it is running normally, i.e. when there is no data loss, for the rest three states (x_1, $x_3 = v_1$ and $x_4 = v_2$), the position of Mass 1, the velocity of Mass 1 and velocity of Mass 1, respectively. Figure 5 shows the comparison analysis of the existing open loop Kalman filtering and the proposed closed loop MLPC algorithm based on compensated observation Kalman filtering. The sensor failure, namely the loss of data, is introduced at 10–15 Secs. Figure 5 shows that the Open-Loop based estimation algorithm diverges shortly and the estimation performance is extremely inferior while the compensated closed-loop observations generate satisfactory results and better estimations. Figure 6 represents the Open-Loop Kalman filtering along with measurements and true sketch. During the loss, the observation value is zero, and the predicted state which is taken as measurement updated state is not following the true state trajectory properly. Also, Figure 7 shows state esti-

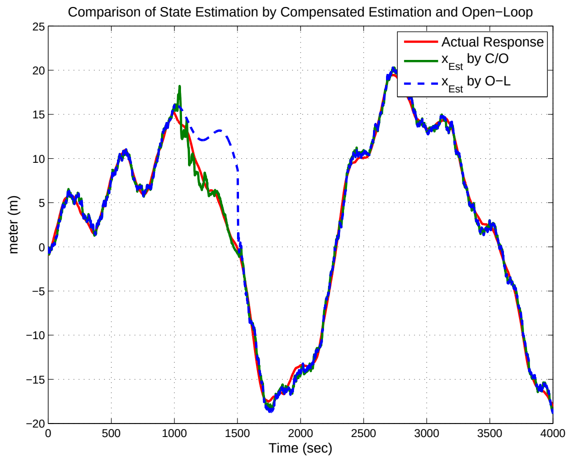

Figure 4: Comparison to two Estimation method.

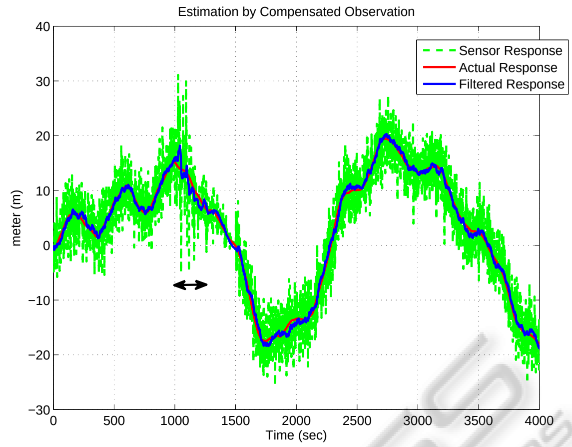

Figure 6: State Estimation through Closed-Loop.

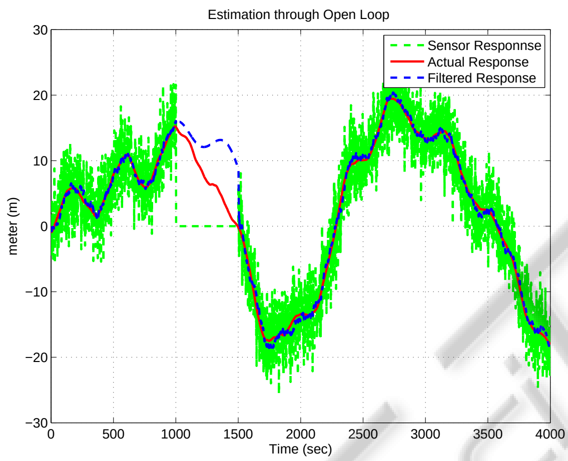

Figure 5: State Estimation through Open-Loop.

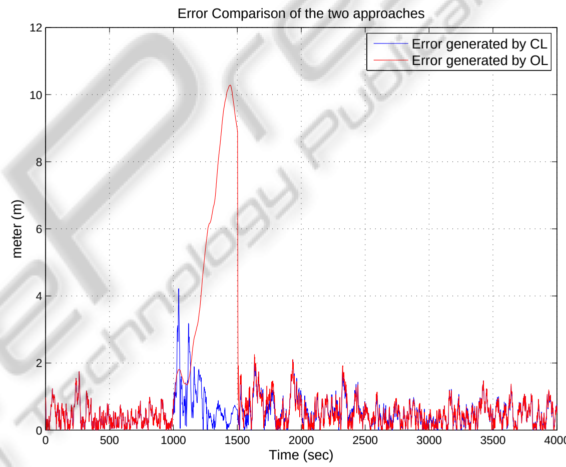

Figure 7: Error Comparison.

mation through the proposed closed-loop. The measurement vector is of higher magnitude but the update state based on this higher value observation are much better and comprehend. As a brief comparison, the absolute error signals are shown in Figure 8. This error plots depict that priority of the proposed closed-loop Kalman filtering MLPC over the previous open-loop Kalman filter with loss of data. It is also true that by providing the upper limit on the error bound, one can notice that the data loss in the open-loop manner will be very conservative than that of the close-loop Kalman filtering.

6 CONCLUSIONS

We have presented a novel approach for state estimation problem in discrete-time LTI systems subject to loss of data. The approach exploits the artificial observation vector which in fact corrects the prediction cycle when loss of data occurs, in order not to allow the estimation error bounds to exceed the desired limits. The resulting closed-loop Kalman filtering also avoids the spike generated in OLE. The performance of the proposed closed-loop Kalman filter approach, when the prediction is updated with compensated observations, was illustrated via a mass-spring-dashpot case study example.

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