APPLICATION OF THE BANACH FIXED POINT THEOREM ON FUZZY QUASI-METRIC SPACES TO STUDY THE COST OF ALGORITHMS WITH TWO RECURRENCE EQUATIONS

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Abstract: Considering recursiveness as a unifying theory for algorithm related problems, we take advantage of algorithms formulation in terms of recurrence equations to show the existence and uniqueness of solution for the two recurrence equations associated to a kind of algorithms defined as two procedures depending the one on the other by applying the Banach contraction principle in a suitable product of fuzzy quasi-metrics on the domain of words.

1 INTRODUCTION

Our study is motivated, in part, by the following algorithm, considered by M.D. Atkinson in (Atkinson, 1996, p. 16-17), which is defined as two procedures depending the one on the other P and Q, such that, for $n \in \omega$:

```
function P(n)
  if n > 0 then
    Q(n-1); C;
    P(n-1); C;
    Q(n-1)

function Q(n)
  if n > 0 then
    P(n-1); C;
    Q(n-1); C;
    P(n-1); C;
    Q(n-1)
```

The algorithm is shown as a pair of recurrence equations expressed in terms of P and Q procedures.

Concrete examples of this class of algorithms could be extracted from language theory scenarios; such a system of equations may represent a couple of mutually dependent rules of a grammar. Another scenario where many cases can be found is object-oriented design. An algorithm like this one in the object-oriented context might express a situation of highly coupled design; a pair of objects from the system with methods that rely non-interactively the one on the other to fulfill a given and more general task.

In order to demonstrate the existence and uniqueness of solution for these recurrence equations we will apply the Banach fixed point theorem in a suitable product of fuzzy quasi-metrics on the domain of words. This technique was already used in (Romaguera et al., 2007) to prove on the existence and uniqueness of solution for the recurrence equations associated with Quicksort, and Divide and Conquer algorithms.

2 PRELIMINARY CONCEPTS AND RESULTS

Our approach is based on the notion of a fuzzy quasi-metric space, which constitutes a nonsymmetric generalization of the Kramosil-Michalek definition (Kramosil and Michalek, 1975) of a fuzzy metric space. A different approach to this study, based on the theory of complexity spaces of M. Schelleckens (Schellekens, 1995), may be found in (Castro-

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Company et al., 2010).

Definition 1. (Cho et al., 2006; Gregori and Romaguera, 2004). *A fuzzy quasi-metric on a set X is a pair $(M,*)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0,\infty)$ such that for all $x,y,z \in X$:*

(KM1) $M(x,y,0) = 0$.

(KM2) $x = y$ *if and only if* $M(x,y,t) = M(y,x,t) = 1$ *for all* $t > 0$.

(KM3) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$ *for all* $t,s \geq 0$.

(KM4) $M(x,y,\_) : [0,\infty) \to [0,1]$ *is left continuous.*

Definition 2. (Kramosil and Michalek, 1975). *A fuzzy metric on a set X is a fuzzy quasi-metric $(M,*)$ on X such that for each $x,y \in X$:*

(KM5) $M(x,y,t) = M(y,x,t)$ *for all* $t > 0$.

Definition 3. (Cho et al., 2006; Gregori and Romaguera, 2004). *A fuzzy (quasi-)metric space is a triple $(X,M,*)$ such that X is a set and $(M,*)$ is a fuzzy (quasi-)metric on X.*

Each fuzzy (quasi-)metric $(M,*)$ on a set X induces a topology τ_M on X which has as a base the family of open balls $\{B_M(x,\varepsilon,t) : x \in X,\ 0 < \varepsilon < 1,\ t > 0\}$, where $B_M(x,\varepsilon,t) = \{y \in X : M(x,y,t) > 1-\varepsilon\}$.

If $(M,*)$ is a fuzzy quasi-metric on a set X, it is obvious that $(M^{-1},*)$ is also a fuzzy quasi-metric on X, where M^{-1} is the fuzzy set in $X \times X \times [0,\infty)$ defined by

$$M^{-1}(x,y,t) = M(y,x,t).$$

Moreover, if we denote by M^i the fuzzy set in $X \times X \times [0,\infty)$ given by

$$M^i(x,y,t) = \min\{M(x,y,t), M^{-1}(x,y,t)\},$$

then $(M^i,*)$ is, clearly, a fuzzy metric on X.

A fuzzy (quasi-)metric space $(X,M,*)$ such that

$$M(x,z,t) \geq \min\{M(x,y,t), M(y,z,t)\},$$

for all $x,z \in X$ and $t > 0$, is said to be a non-Archimedean fuzzy (quasi-)metric space.

In (Grabiec, 1988), M. Grabiec introduced the following notions in order to obtain a fuzzy version of the classical Banach fixed point theorem (an exhaustive study of fixed point theory on fuzzy metric spaces and related structures may be found in (Hadzic and Pap, 2001)):

A sequence $(x_n)_n$ in a fuzzy metric space $(X,M,*)$ is Cauchy provided that $\lim_{n\to\infty} M(x_n,x_{n+p},t) = 1$ for each $t > 0$ and $p \in \mathbb{N}$.

A fuzzy metric space $(X,M,*)$ is complete provided that every Cauchy sequence in X is convergent. In this case, $(M,*)$ is called a complete fuzzy metric on X.

In the sequel, and according to (Gregori and Sapena, 2002) and (Vasuki and Veeramani, 2003), a Cauchy sequence in Grabiec's sense will be called G-Cauchy and a complete fuzzy metric space in Grabiec's sense will be called G-complete.

On the other hand, following (Sehgal and Bharucha-Reid, 1972), a B-contraction on a fuzzy metric space $(X,M,*)$ is a self-map f on X such that there is a constant $k \in (0,1)$ satisfying

$$M(f(x), f(y), kt) \geq M(x,y,t)$$

for all $x,y \in X$, $t > 0$.

Thus, Grabiec's fixed point theorem can be formulated as follows.

Theorem 1. (Grabiec, 1988). *Let $(X,M,*)$ be a G-complete fuzzy metric space such that $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x,y \in X$. Then every B-contraction on X has a unique fixed point.*

The following quasi-metric generalizations of the notions of B-contraction and G-completeness were introduced in (Romaguera et al., 2007).

Definition 4. *A B-contraction on a fuzzy quasi-metric space $(X,M,*)$ is a self-map f on X such that there is a constant $k \in (0,1)$ satisfying*

$$M(f(x), f(y), kt) \geq M(x,y,t)$$

for all $x,y \in X$, $t > 0$. The number k is then called a contraction constant of f.

Definition 5. *A sequence $(x_n)_n$ in a fuzzy quasi-metric space $(X,M,*)$ is called G-Cauchy if it is a G-Cauchy sequence in the fuzzy metric space $(X,M^i,*)$.*

Definition 6. *A fuzzy quasi-metric space $(X,M,*)$ is called G-bicomplete if the fuzzy metric space $(X,M^i,*)$ is G-complete.*

Then, Grabiec's theorem was generalized to fuzzy quasi-metric spaces in (Romaguera et al., 2007) as follows.

Theorem 2. (Romaguera et al., 2007). *Let $(X,M,*)$ be a G-bicomplete fuzzy quasi-metric space such that $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x,y \in X$. Then every B-contraction on X has a unique fixed point.*

Since G-(bi)completeness is a very strong kind of completeness (see (George and Veeramani, 1994; Vasuki and Veeramani, 2003)), George and Veeramani introduced the following notions:

A sequence $(x_n)_n$ in a fuzzy metric space $(X,N,*)$ is a Cauchy sequence (George and Veeramani, 1994) if for each $\varepsilon \in (0,1)$, $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1-\varepsilon$ for all $n,m \geq n_0$.

A fuzzy metric space is complete provided that every Cauchy sequence in X is convergent.

Definition 7. *A fuzzy quasi-metric space $(X,M,*)$ is called bicomplete if the fuzzy metric space $(X,M^i,*)$ is bicomplete.*

Then we have the following nice and useful fact for our approach.

Theorem 3. (Romaguera et al., 2007). *Each bicomplete non-Archimedean fuzzy quasi-metric space is G-bicomplete.*

Let us recall (Gregori and Romaguera, 2004) that if (X,d) is a (quasi-)metric space, then the pair $(M_d,\wedge)$ is a fuzzy (quasi-)metric on X where M_d is the fuzzy set in $X\times X\times[0,\infty)$ given by $M_d(x,y,0)=0$, and, for $t>0$, by

$$M_d(x,y,t)=\frac{t}{t+d(x,y)}.$$

The triple $(X,M_d,\wedge)$ is called the standard fuzzy (quasi-)metric space.

Furthermore, we have that $(M_d)^{-1}=M_{d^{-1}}$ and $(M_d)^i=M_{d^s}$. In addition, topology τ_d, induced by d, coincides with the topology τ_{M_d} induced by the fuzzy (quasi-)metric $(M_d,\wedge)$.

Next we recall several pertinent facts and results on the domain of words and some non-Archimedean quasi-metric that one can construct on it because they will be used in section 3.

The domain of words Σ^∞ (Kunzi, 1995; Matthews, 1994; Romaguera and Schellekens, 2005; Schellekens, 2004; Smyth, 1988, etc) consists of all finite and infinite sequences ("words") over a nonempty set ("alphabet") Σ, ordered by the so-called information order $\sqsubseteq$ on Σ^∞, i.e., $x\sqsubseteq y \Leftrightarrow x$ is a prefix of y, where we assume that the empty sequence ϕ is an element of Σ^∞.

For each $x,y\in\Sigma^\infty$ denote by $x\sqcap y$ the longest common prefix of $x$and y, and for each $x\in\Sigma^\infty$ denote by $\ell(x)$ the length of x. Thus $\ell(x)\in[1,\infty]$ whenever $x\neq\phi$, and $\ell(\phi)=0$.

Given a nonempty alphabet Σ, Smyth introduced in (Smyth, 1988) a non-Archimedean quasi-metric $d_\sqsubseteq$ on Σ^∞ given by $d_\sqsubseteq(x,y)=0$ if $x\sqsubseteq y$, and $d_\sqsubseteq(x,y)=2^{-\ell(x\sqcap y)}$ otherwise (see also (Kunzi, 1995; Rodríguez-López et al., 2008; Romaguera et al., 2007, etc)).

This quasi-metric has the advantage that its specialization order coincides with the order $\sqsubseteq$, and thus the quasi-metric space $(\Sigma^\infty,d_\sqsubseteq)$ preserves the information provided by $\sqsubseteq$. Moreover, the metric $(d_\sqsubseteq)^s$ is given by $(d_\sqsubseteq)^s(x,y)=0$ if $x=y$, and $(d_\sqsubseteq)^s(x,y)=2^{-\ell(x\sqcap y)}$ otherwise; so that $(d_\sqsubseteq)^s$ is exactly the celebrated Baire metric on Σ^∞. Since the Baire metric is complete, it follows that $d_\sqsubseteq$ is a bicomplete non-Archimedean quasi-metric on Σ^∞.

3 THE BANACH FIXED POINT THEOREM ON FUZZY QUASI-METRIC SPACES APPLIED TO ALGORITHMS COST ANALYSIS

In order to apply techniques of fixed point for obtaining the existence and uniqueness of solution for the two recurrence equations associated to algorithms with two recurrence procedures, we shall combine the above results with some facts on the product of (non-Archimedean) fuzzy quasi-metrics that we present in the sequel.

Similarly to (Cho et al., 2009) the product (fuzzy quasi-metric) space of two fuzzy quasi-metric spaces $(X_1,M_1,*)$ and $(X_2,M_2,*)$ is the fuzzy quasi-metric space $(X_1\times X_2,M_1\times M_2,*)$ such that for each $(x_1,x_2),(y_1,y_2)\in X_1\times X_2$ and each $t\geq 0$,

$$(M_1\times M_2)((x_1,x_2),(y_1,y_2),t)=$$
$$M_1(x_1,y_1,t)*M_2(x_2,y_2,t).$$

In particular, if $(X_1,M_1,\wedge)$ and $(X_2,M_2,\wedge)$ are non-Archimedean, then $(X_1\times X_2,M_1\times M_2,\wedge)$ is non-Archimedean.

Furthermore, it is clear that if $(X_1,M_1,*)$ and $(X_2,M_2,*)$ are bicomplete, then $(X_1\times X_2,M_1\times M_2,*)$ is bicomplete.

By applying the above results to the standard fuzzy quasi-metric space of $(\Sigma^\infty,d_\sqsubseteq)$ when $*=\wedge$, we immediately deduce from Theorems 2 and 3 the following.

Theorem 4. $(\Sigma^\infty\times\Sigma^\infty,M_{d_\sqsubseteq}\times M_{d_\sqsubseteq},\wedge)$ *is a bicomplete non-Archimedean fuzzy quasi-metric space such that* $\lim_{t\to\infty}(M_{d_\sqsubseteq}\times M_{d_\sqsubseteq})((x_1,x_2),(y_1,y_2),t)=1$ *for all* $(x_1,x_2),(y_1,y_2)\in\Sigma^\infty\times\Sigma^\infty$. *Therefore, every B-contraction on this space has a unique fixed point.*

As mentioned in Section 1, following Atkinson (Atkinson, 1996, p. 16-17), consider the two recursive procedure algorithm defined, for two procedures P and Q, and $n\in\omega$, by:

```
function P(n)
  if n > 0 then
    Q(n-1); C;
    P(n-1); C;
    Q(n-1)

function Q(n)
```

```
if n > 0 then
  P(n-1); C;
  Q(n-1); C;
  P(n-1); C;
  Q(n-1)
```

where C denotes any statements taking time independent of n.

Then, the execution times $S(n)$ and $T(n)$ of $P(n)$ and $Q(n)$, satisfy, at least approximately, the recurrences

$$S(n) = S(n-1) + 2T(n-1) + K_1,$$

and

$$T(n) = 2S(n-1) + 2T(n-1) + K_2,$$

for $n \in \mathbb{N}$, and with K_1, K_2, nonnegative constants. (We assume that $S(0) > 0$ and $T(0) > 0$).

We shall deduce the existence and uniqueness of solution for the recurrences S and T by means of a version of the Banach fixed point theorem on a suitable (product) fuzzy quasi-metric space constructed on a certain product of domain of words.

To this end, consider the recurrences A and B given by $A(0) > 0$, $B(0) > 0$, and

$$A(n) = pA(n-1) + qB(n-1) + K_1,$$

and

$$B(n) = rA(n-1) + sB(n-1) + K_2,$$

for all $n \in \mathbb{N}$, where p, q, r, s, K_1, K_2, are nonnegative constants with $p, q, r, s > 0$.

Note that recurrences S and T are a particular case of A and B for $p = 1$, $q = r = s = 2$.

In the rest of this section by Σ^∞ we shall denote the domain of words where the alphabet Σ is the set of nonnegative real numbers.

Recurrences A and B suggest the construction of the functional

$$\Phi : \Sigma^\infty \times \Sigma^\infty \to \Sigma^\infty \times \Sigma^\infty,$$

given for each pair $x^1, x^2 \in \Sigma^\infty$, by

$$\Phi(x^1, x^2) = (u^1, u^2),$$

where

$$(u^1)_0 = A(0), \quad (u^2)_0 = B(0),$$

and

$$(u^1)_n = p(x^1)_{n-1} + q(x^2)_{n-1} + K_1,$$
$$(u^2)_n = r(x^1)_{n-1} + s(x^2)_{n-1} + K_2,$$

for all $n \in \mathbb{N}$ such that $n \le (\ell(x^1) \wedge \ell(x^2)) + 1$.

Note that then $\ell(u^j) \ge (\ell(x^1) \wedge \ell(x^2)) + 1$, for $j = 1, 2$.

Next we prove that for each $(x^1, x^2), (y^1, y^2) \in \Sigma^\infty \times \Sigma^\infty$ and each $t > 0$, one has

$$(M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})(\Phi((x^1, x^2)), \Phi((y^1, y^2)), t/2) \ge$$
$$M_{d_\sqsubseteq}(x^1, y^1, t) \wedge M_{d_\sqsubseteq}(x^2, y^2, t).$$

Indeed, put $\Phi(x^1, x^2) = (u^1, u^2)$ and $\Phi(y^1, y^2) = (v^1, v^2)$ and let $t > 0$.

First observe that if $u^1 \sqsubseteq u^2$ and $v^1 \sqsubseteq v^2$, we obtain

$$(M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})(\Phi((x^1, x^2)), \Phi((y^1, y^2)), t/2) =$$
$$M_{d_\sqsubseteq}(u^1, v^1, t/2) \wedge M_{d_\sqsubseteq}(u^2, v^2, t/2) = 1.$$

Otherwise, we will take into account that, by the construction of u^1, u^2, v^1 and v^2, we have

$$\ell(u^k \sqcap v^k) \ge (\ell(x^1 \sqcap y^1) \wedge \ell(x^2 \sqcap y^2)) + 1,$$

for $k = 1, 2$.

Consequently

$$\begin{aligned}
&(M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})(\Phi((x^1, x^2)), \Phi((y^1, y^2)), t/2) \\
&= M_{d_\sqsubseteq}(u^1, v^1, t/2) \wedge M_{d_\sqsubseteq}(u^2, v^2, t/2) \\
&= \frac{t/2}{t/2 + d_\sqsubseteq(u^1, v^1)} \wedge \frac{t/2}{t/2 + d_\sqsubseteq(u^2, v^2)} \\
&= \frac{t}{t + 2^{-\ell(u^1 \sqcap v^1)+1}} \wedge \frac{t}{t + 2^{-\ell(u^2 \sqcap v^2)+1}} \\
&\ge \frac{t}{t + 2^{-(\ell(x^1 \sqcap y^1) \wedge \ell(x^2 \sqcap y^2))}} \\
&= \frac{t}{t + 2^{-\ell(x^1 \sqcap y^1)}} \wedge \frac{t}{t + 2^{-\ell(x^2 \sqcap y^2)}} \\
&= M_{d_\sqsubseteq}(x^1, y^1, t) \wedge M_{d_\sqsubseteq}(x^2, y^2, t) \\
&= (M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})((x^1, x^2), (y^1, y^2), t).
\end{aligned}$$

4 CONCLUSIONS

We have shown that there exists a B-contraction Φ on the (non-Archimedean) G-bicomplete fuzzy quasi-metric space $(\Sigma^\infty \times \Sigma^\infty, M_{d_\sqsubseteq} \times M_{d_\sqsubseteq}, \wedge)$. By Theorem 4, Φ has a unique fixed point which is obviously the solution of the recurrences A and B.

Finally, we observe that, in practice, one actually works on the set Σ^F of all finite words (over the alphabet $[0, \infty)$), that endowed with the restriction of $(M_{d_\sqsubseteq}, \wedge)$ provides a non-Archimedean fuzzy quasi-metric space which, obviously, is not bicomplete. In fact the product space $(\Sigma^F \times \Sigma^F, M_{d_\sqsubseteq} \times M_{d_\sqsubseteq}, \wedge)$ is also a non-bicomplete non-Archimedean fuzzy quasi-metric space. However, for each pair $x^1, x^2 \in \Sigma^F$, the sequence of iterations $(\Phi^k(x^1, x^2))_k$, is a Cauchy sequence in the complete fuzzy metric space $(\Sigma^\infty \times \Sigma^\infty, (M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})^i, \wedge)$ by the property of B-contraction of Φ stated above, and thus it converges to an element (y^1, y^2), with $\ell(y^1) = \ell(y^2) = \infty$, which is, in fact, the solution for the pair of recurrence equations A and B.

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