SOLVING FUZZY LINEAR SYSTEMS IN THE STOCHASTIC ARITHMETIC BY APPLYING CADNA LIBRARY

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Keywords: Iterative methods, Fuzzy Linear Systems (FLS), Stochastic arithmetic, CESTAC method, CADNA library, Hausdorff distance, Fuzzy number.

Abstract: In this paper, a fuzzy linear system with crisp coefficient matrix is considered in order to solve in the stochastic arithmetic. The fuzzy CESTAC method is applied in order to validate the computed results. The Gauss-Seidel and Jacobi iterative methods are used for solving a given fuzzy linear system. In order to implement the proposed algorithm, the CADNA library is applied to find the optimal number of iterations. Finally, two numerical examples are solved based on the given algorithm in the stochastic arithmetic.

1 INTRODUCTION

A general model for solving a FLS whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number was first proposed by Friedman, Ming and Kandel, (1998). They used the embedding method and replaced the original fuzzy linear system by a crisp linear system with a nonnegative coefficients matrix. In the sequel, solving this kind of fuzzy linear system based on the numerical and iterative methods were proposed by others. Some of these works were presented by Abbasbandy et al. (2006), Allahviranloo (2004)(2005), Dehghan and Hashemi (2006). Since, the results of the iterative methods are obtained in the floating-point arithmetic, the termination criterion depends on a positive number like ε. So, the final results may not be accurate or the number of iterations may increase without increasing the accuracy of the results. Therefore, the validation of the computed results is important. In this case, because of the round-off error propagation, the computer may not able to improve the accuracy of the computed solution. By using the stochastic arithmetic in place of the traditional floating-point arithmetic, one can rely the results and estimate the accuracy of them (Abbasbandy and Fariborzi Araghi, 2004; Chesneaux, 1992; Fariborzi Araghi, 2008; Vignes, 1993). CESTAC (Controle et Estimation Stochastique des Arrondis de Calculs) method is an efficient method in order to estimate the accuracy of the results and find the optimal number of iterations (Vignes,1993). CADNA (Control of Accuracy and Debugging for Numerical Applications) library is a tool to implement the stochastic arithmetic automatically. The first goal of this software is the estimation of the accuracy of each computed result. CADNA detects numerical instabilities (informatical zero) during the run of the program. CADNA works on Fortran or C++ codes. When a result is a stochastic zero (i.e. is insignificant), the symbol @.0 is printed. CADNA detects numerical instabilities during the run of the program (Jezequel and Chesneaux, 2008). For more details about this library we refer the reader to "http://www-pequan.lip6.fr/cadna".

By using the CESTAC method, N runs of the computer program take place in parallel. In this way, one runs every arithmetical operation N times synchronously before running the next operation. In this method, by running the program N times, for each result of any floating-point arithmetic operation, a set of N computed results X_i; $i = 1,2,...N$, is obtained. N can be chosen any natural number like $2,3,5,7$, but in order to decrease operations cost, usually $N = 3$ is considered. This method is able to estimate the round-off error on each result and determine the accuracy of it.

Let F be the set of all values represented on the computer. Thus, any real value x is represented in the form of $X \in F$ on the computer. It has been mentioned in (Vignes, 1993) that in a binary floating-point arithmetic with P mantissa bits, the rounding error stems from assignment operator is

Fariborzi Araghi M. and Fattahi H..
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DOI: 10.5220/0003675804460450
In *Proceedings of the International Conference on Evolutionary Computation Theory and Applications* (FCTA-2011), pages 446-450
ISBN: 978-989-8425-83-6
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$$X = x - \varepsilon 2^{E-P} a, \tag{1}$$

where ε is the sign of x, $2^{-P}a$ is the lost part of the mantissa due to round-off error and E is the binary exponent of the result. In single precision case, $P = 24$. If the floating-point arithmetic is as rounding to $+\infty$ or $-\infty$, then $-1 < a < 1$.

According to (1) if we want to perturb the last mantissa bit (or previous bits if necessary) of the value x, it is sufficient to change a in the interval $[-1,1]$. We consider a as a random variable uniformly distributed on $[-1,1]$. Thus X, the calculated result, is a random variable and its precision depends on its mean (μ) and its standard deviation (σ).

The idea of CESTAC method is to consider that every result $X \in F$ of a floating-point operation corresponds to two informatical results one rounded off from below (X^-) , the second rounded off from above (X^+), each of them representing the exact arithmetical result x, with equal validity.

In this paper, we apply the stochastic arithmetic to solve a fuzzy linear system by applying the fuzzy CESTAC method for the Jacobi and Gauss-Seidel iterative methods. For this purpose, the CADNA library is used over the Linux operating system. The programs have been provided by C++. The preliminaries are given in Section 2. The fuzzy CESTAC method and the algorithm of solving FLS are introduced in Section 3. In Section 4, two examples are solved by using of the stochastic arithmetic and CADNA library and compare the results with the results of the floating-point arithmetic.

2 PRELIMANERIES

Definition 1. An arbitrary fuzzy number X in parametric form is represented by an ordered pair functions $(\underline{X}(r), \overline{X}(r))$, $0 \le r \le 1$, where $\underline{X}(r)$ is a bounded left-continuous non- decreasing function over $[0,1]$, and $\overline{X}(r)$ is a bounded left-continuous non-increasing function over $[0,1]$. Also, $\underline{X}(r) \le \overline{X}(r), 0 \le r \le 1$. We denote the set of all fuzzy numbers by E^1.

Definition 2. Let $\tilde{X} = (\underline{X}(r), \overline{X}(r))$ and $\tilde{Y} = (\underline{Y}(r), \overline{Y}(r))$ be arbitrary fuzzy numbers then the Hausdorff distance of these numbers is defined by:

$$D(\tilde{X},\tilde{Y}) =$$

$$sup_{0 \le r \le 1} \max\{|\underline{X}(r) - \underline{Y}(r)|, |\overline{X}(r) - \overline{Y}(r)|\}. \tag{2}$$

Definition 3. Consider the $n \times n$ linear system of equations:

$$AX = \mathcal{Y}, \tag{3}$$

where the coefficients matrix $A = [a_{ij}]$ is a crisp matrix and $y_i \in E^1, i = 1,2,...,n$. The fuzzy system (3) is called a fuzzy linear system (FLS). This system can be converted to a crisp $2n \times 2n$ linear system as follows (Abbasbandy et al., 2006; Allahviranloo, 2004; Dehghan and Hashemi, 2006):

$$SX = Y, \qquad S = \begin{pmatrix} B & C \\ C & B \end{pmatrix} \tag{4}$$

where, B contains the positive entries of A, and C contains the absolute values of the negative entries of A and $A = B - C$.

We suppose that $S_{ii} > 0, i = 1,2,...,2n$. Let $B = D_1 + L_1 + U_1$ where D_1, L_1 and U_1 are the diagonal, the strict lower and the strict upper triangular matrices respectively. So, the elements of $X^{(k+1)} = (\underline{X}^{(k+1)}, \overline{X}^{(k+1)}), k = 0,1,2,...,$ in the Jacobi iterative technique are obtained as follows (Allahviranloo,2004):

$$\underline{X}^{(k+1)} = -D_1^{-1}(L_1 + U_1)\underline{X}^{(k)} + D_1^{-1}C\overline{X}^{(k)} + D_1^{-1}\underline{Y}, \tag{5}$$

$$\overline{X}^{(k+1)} = -D_1^{-1}(L_1 + U_1)\overline{X}^{(k)} + D_1^{-1}C\underline{X}^{(k)} + D_1^{-1}\overline{Y}. \tag{6}$$

Also, the elements of the vector $X^{(k+1)}$ for the Gauss-Seidel iterative method is (Allahviranloo, 2004):

$$\underline{X}^{(k+1)} = -(D_1 + L_1)^{-1}U_1\underline{X}^{(k)} -$$

$$(D_1 + L_1)^{-1}C\overline{X}^{(k)} + (D_1 + L_1)^{-1}\underline{Y}, \tag{7}$$

$$\overline{X}^{(k+1)} = -(D_1 + L_1)^{-1}U_1\overline{X}^{(k)} -$$

$$(D_1 + L_1)^{-1}C\underline{X}^{(k)} + (D_1 + L_1)^{-1}\overline{Y}. \tag{8}$$

3 INTRODUCING FUZZY CESTAC METHOD

Let $\tilde{x} = (\underline{x}(r), \overline{x}(r))$ be a fuzzy number in E^1. Then, $\tilde{x}$ is represented as $\tilde{X} = (\underline{X}(r), \overline{X}(r)), 0 \le r \le 1$ in the computer. It can be shown that:

$$\underline{X}(r) = \underline{x}(r) - \varepsilon_1 2^{E_1 - P}\underline{\alpha}, \tag{9}$$

$$\overline{X}(r) = \overline{x}(r) - \varepsilon_2 2^{E_2 - P}\overline{\alpha}, \tag{10}$$

where, ε_1 and ε_2 are the signs of $\underline{x}(r)$ and $\overline{x}(r)$ respectively and $2^{-P}\underline{\alpha}$ and $2^{-P}\overline{\alpha}$ are the lost part of the mantissa due to round-off error and E_1 and E_2 are the binary exponents of the results. In single precision case, $P = 24$ and $-1 \le \underline{\alpha}, \overline{\alpha} \le 1$. In the CESTAC method, $\underline{\alpha}$ and $\overline{\alpha}$ in (9) and (10) are considered as random variables uniformly distributed on $[-1,1]$. In order to find samples for the obtained random variables, we perturb the last mantissa bit (or previous

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bits) of the values $\underline{X}(r)$ and $\overline{X}(r), 0 \le r \le 1$ (Vignes, 1993). The algorithm of fuzzy CESTAC method is as follows where d is a small positive value like 10^{-2} and τ_β is the value of T distribution with $N-1$ degree of freedom and confidence interval $1-\beta$. If $N=3$ and $\beta = 0.05$ then $\tau_\beta = 4.303$.

Algorithm 1:

For $r = 0(d)1$ do the following steps:
1) Find N samples for $\underline{X}(r)$ and $\overline{X}(r)$ as

$$\underline{X}_1(r), \underline{X}_2(r), \ldots, \underline{X}_N(r)$$

and

$$\overline{X}_1(r), \overline{X}_2(r), \ldots, \overline{X}_N(r),$$

by means of the perturbation of the last bit of the mantissa,

2) Compute

$$\underline{X}_{ave}(r) = \frac{\sum_{i=1}^N \underline{X}_i(r)}{N}$$

and

$$\overline{X}_{ave}(r) = \frac{\sum_{i=1}^N \overline{X}_i(r)}{N}$$

3) Compute

$$\underline{S}^2(r) = \frac{1}{N-1} \sum_{i=1}^N (\underline{X}_i(r) - \underline{X}_{ave}(r))^2$$

and

$$\overline{S}^2(r) = \frac{1}{N-1} \sum_{i=1}^N (\overline{X}_i(r) - \overline{X}_{ave}(r))^2,$$

4) Compute

$$C_{\underline{X}_{ave}(r), \underline{X}(r)} = log_{10} \frac{\sqrt{N} |\underline{X}_{ave}(r)|}{\tau_\beta . \underline{S}(r)}$$

and

$$C_{\overline{X}_{ave}(r), \overline{X}(r)} = log_{10} \frac{\sqrt{N} |\overline{X}_{ave}(r)|}{\tau_\beta . \overline{S}(r)}$$

as the common significant digits between the exact values $\underline{X}(r)$ and $\overline{X}(r)$ and the approximate values $\underline{X}_{ave}(r)$ and $\overline{X}_{ave}(r)$ respectively,

5) If

$$C_{\underline{X}_{ave}(r), \underline{X}(r)} \le 0 \text{ or } \underline{X}_{ave}(r) = 0$$

and

$$C_{\overline{X}_{ave}(r), \overline{X}(r)} \le 0 \text{ or } \overline{X}_{ave}(r) = 0$$

then write $\tilde{X} = @.0$

For solving fuzzy linear systems in the stochastic arithmetic we use the following algorithm by applying Jacobi and Gauss-Seidel iterative methods with the initial vector $X^{(0)} = (\underline{X}^{(0)}, \overline{X}^{(0)})$. The programs have been written in C++ and executed on a Linux machine using CADNA library. For the termination criterion, we consider the Hausdorff distance to be an informatical zero (@.0). In the algorithm, the value d is a small positive number like $d = 0.1$.

Algorithm 2:

1- Let $k = 0$ and $i = 1$,
2- For $r = 0(d)1$ do the following steps based on the fuzzy CESTAC method:
2.a) Find $\underline{X}^{(k+1)}$ and $\overline{X}^{(k+1)}$ by using Jacobi or Gauss-Seidel iterative method mentioned in (5)-(8),
2.b) Let

$$\underline{dif} = |\underline{X}^{(k+1)} - \underline{X}^{(k)}|$$

and

$$\overline{dif} = |\overline{X}^{(k+1)} - \overline{X}^{(k)}|,$$

and put the maximum value of them as $maxdif[i]$ in the array $maxdif$,
2.c) $i = i+1$,
3- Find the maximum element of the array $maxdif$ and call it "$Hmax$" which is the approximation of Hausdorff distance in the stochastic arithmetic. If $Hmax = @.0$ then go to step 4 else put $k = k+1$ and go to step 2,
4- Print k as the optimal iteration and $\underline{X}^{(k)} \simeq \underline{X}$ and $\overline{X}^{(k)} \simeq \overline{X}$ as the approximate solution of the linear system (4).

Now, from the procedure mentioned by Khojasteh Salkuyeh and Toutounian (2006), we can prove the following theorem for computing of the common significant digits of each corresponding components of the computed solution and exact solution for a linear system using an iterative method. In this theorem, the notation $C_{X,X'}$ means the common significant digits between two distinct real vectors X and X' in $\mathbb{R}^n$ which is defined as $C_{X,X'} = log_{10} \frac{||X+X'||_2}{2\sqrt{n}||X-X'||_2}$.

Theorem 3.1. Let $\underline{X}^{(k+1)} = P_1 \underline{X}^{(k)} + Q_1$ and $\overline{X}^{(k+1)} = P_2 \overline{X}^{(k)} + Q_2, k \ge 0$, be convergence iterative method to the exact solution $X = (\underline{X}, \overline{X})$ of the system (4) with $Q_1, Q_2 \ne 0$. Then, for sufficiently large value of k, we have

$$log_{10}|1 - ||P_1||_2| \le C_{\underline{X}^{(k)}, \underline{X}} - C_{\underline{X}^{(k)}, \underline{X}^{(k+1)}} \le log_{10}(1 + ||P_1||_2).$$

$$log_{10}|1 - ||P_2||_2| \le C_{\overline{X}^{(k)}, \overline{X}} - C_{\overline{X}^{(k)}, \overline{X}^{(k+1)}} \le log_{10}(1 + ||P_2||_2).$$

Since the above iterative procedure is convergent if $||P_1||_2 < 1$ and $||P_2||_2 < 1$, hence according to the theorem 3.1, when $||P_1||_2 << 1$ and $||P_2||_2 << 1$ then,

$$C_{\underline{X}^{(k)}, \underline{X}} \simeq C_{\underline{X}^{(k)}, \underline{X}^{(k+1)}}.$$

$$C_{\overline{X}^{(k)}, \overline{X}} \simeq C_{\overline{X}^{(k)}, \overline{X}^{(k+1)}}.$$

4 NUMERICAL EXAMPLES

In this section, we solve two examples by using the above algorithm based on the stochastic arithmetic. The programs have been implemented by CADNA library. In each examples, the number of iterations in both arithmetic are shown in single precision case. In the floating-point arithmetic we use the termination criterion $Hmax < \varepsilon$ where ε is a given positive number.

Example 4.1. Consider the following fuzzy linear system (Freidman et al.,1998)

$$\begin{cases} x_1 - x_2 = (r, 2-r), \\ x_1 + 3x_2 = (4+r, 7-2r), \end{cases}$$

Table 1: The number of iterations (example 4.1).

	Jacobi	Gauss-Seidel	ε
Stochastic	26	9	—
Floating-point	34	10	10^{-7}

As we observe in the table 1, for Jacobi method the algorithm is stopped at the iteration $k = 34$ in the floating-point arithmetic with $\varepsilon = 10^{-7}$. But, in the stochastic arithmetic the optimal number of iterations is $k = 26$. It means that after this number the continuation of the iterations is useless and the accuracy of the solution does not increase. Also, we can compare the results of the Gauss-Seidel method. The optimal solution of Jacobi method in the stochastic arithmetic in the iteration $k_{opt} = 26$ is:

$$x_1 = (0.1375001E+01+0.6249994E+00r, \; 0.2875000E+01-0.87499957E+00r)$$

and

$$x_2 = (0.8750008E+00+0.1249997E+00r, \; 0.1375000E+01-0.3749999E+00r).$$

Also, the optimal solution of Gauss-Seidel method in the stochastic arithmetic in the iteration $k_{opt} = 9$ is:

$$x_1 = (0.1374999E+01+0.6250000E+00r, \; 0.2874999E+01-0.8750000E+00r)$$

and

$$x_2 = (0.8750008E+00+0.1250000E+00r, \; 0.1375000E+01-0.3750000E+00r).$$

The approximate result in the floating-point arithmetic $k = 34$ is:

$$x_1 = (1.375+0.625r, 2.875-0.875r)$$

and

$$x_2 = (0.875+0.125r, 1.375-0.375r).$$

If we solve the system by Maple 8 directly and calculate Hausdorff distance between the exact solution and the result of iteration $k = 26$ of Jacobi method based on (2), we have $D(\tilde{X}, \tilde{X}^{(26)}) = 9.5364 \times 10^{-7}$ which means the computed result in the iteration 26 is very near to the exact solution.

Example 4.2. Consider the following fuzzy linear system (Dehghan and Hashemi, 2006)

$$\begin{cases} 8x_1 + 2x_2 + x_3 - 3x_5 = (r, 2-r), \\ -2x_1 + 5x_2 + x_3 - x_4 + x_5 = (4+r, 7-2r), \\ x_1 - x_2 + 5x_3 + x_4 + x_5 = (1+2r, 6-3r), \\ -x_3 + 4x_4 + 2x_5 = (1+r, 3-r), \\ x_1 - 2x_2 + 3x_5 = (3r, 6-3r), \end{cases}$$

Table 2: The number of iterations (example 4.2).

	Jacobi	Gauss-Seidel	ε
Stochastic	108	19	—
Floating-point	$> 10^4$	60	10^{-7}

The optimal solution of Jacobi method in the stochastic arithmetic in the iteration $k_{opt} = 108$ is:

$$x_1 = (0.728699E+00-0.3305729E+00r, \; 0.441342E-01+0.353992E+00r),$$

$$x_2 = (0.6141832E+00+0.166261E+00, \; 0.107726E+01-0.296823E+00r),$$

$$x_3 = (0.126275E+00+0.2905860E+00r, \; 0.9182206E+00-0.5013588E+00r),$$

$$x_4 = (0.2419159E+00-0.331494E+00r, \; -0.415803E+00+0.3262251E+00r),$$

and

$$x_5 = (0.475281E+00+0.9123067E+00r, \; 0.2394742E+01-0.100715E+01r).$$

Also, the optimal solution of Gauss-Seidel method in the stochastic arithmetic in the iteration $k_{opt} = 19$ is:

$$x_1 = (0.7286988E+00-0.330572E+00r, \; 0.441349E-01+0.3539914E+00r),$$

$$x_2 = (0.6141824E+00+0.166262E+00r, \; 0.107726E+01-0.2968246E+00r),$$

$$x_3 = (0.126275E+00+0.290586E+00r, \; 0.9182215E+00-0.5013596E+00r),$$

$$x_4 = (0.241915E+00-0.331493E+00r, \; -0.4158026E+00+0.326224E+00r),$$

and

$$x_5 = (0.475279E+00+0.9123077E+00r, \; 0.2394743E+01-0.100715E+01r),$$

The exact solution of the system by using Maple 8 is:

$$x_1 = (0.7287-0.3306r, 0.04413+0.3540r),$$

$$x_2 = (0.6142+0.1663r, 1.077-0.2968r),$$

$$x_3 = (0.1263+0.2906r, 0.9182-0.5014r),$$

$$x_4 = (0.2419-0.3315r, -0.4158+0.3262r),$$

and

$$x_5 = (0.4753+0.9123r, 2.395-1.007r).$$

In Jacobi method the optimal number of iterations is $k_{opt} = 108$ in the stochastic arithmetic, but in the floating point arithmetic the number of iterations exceed

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$k = 10000$ with criterion $Hmax < \varepsilon = 10^{-7}$. These iterations are useless because after the iteration $k = 108$ the accuracy does not increase but the floating-point arithmetic is not able to recognize it. In this case, $D(\tilde{X}, \tilde{X}^{(108)}) = 1.18017 \times 10^{-5}$.

5 CONCLUSIONS

In this work, we proposed an algorithm in order to approximate the solution of a FLS in the stochastic arithmetic. In this case, we are able to find the accuracy of results and validate the results of the algorithm. Also, we can find the optimal number if iterations in the iterative methods for solving fuzzy linear systems such as Jacobi and Gauss-Seidel methods. In order to estimate the number of the significant digits we used the CESTAC method and in order to implement the stochastic arithmetic we applied the CADNA library. Consequently, the stochastic arithmetic can play an important role to rely the numerical solution of FLS.

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ACKNOWLEDGEMENTS

The authors would like to thank the research center of IAUCTB for their financial supports.

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