RECENT ADVANCES AND APPLICATIONS OF THE THEORY OF STOCHASTIC CONVEXITY. APPLICATION TO COMPLEX BIO-INSPIRED AND EVOLUTION MODELS

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Abstract: The theory of stochastic convexity is widely recognised as a framework to analyze the stochastic behaviour of parameterized models by different notions in both univariate and multivariate settings. These properties have been applied in areas as diverse as engineering, biotechnology, and actuarial science. Consider a family of parameterized univariate or multivariate random variables $\{X(\theta)|\theta \in T\}$ over a probability space $(\Omega, \Im, Pr)$, where the parameter θ usually represents some distribution moments. Regular, sample-path, and strong stochastic convexity notions have been defined to intuitively describe how the random objects $X(\theta)$ grow convexly (or concavely) concerning their parameters. These notions were extended to the multivariate case by means of directionally convex functions, yielding the concepts of stochastic directional convexity for multivariate random vectors and multivariate parameters. We aim to explain some of the basic concepts of stochastic convexity, to discuss how this theory has been used into the stochastic analysis, both theoretically and in practice, and to provide some of the recent and of the historically relevant literature on the topic. Finally, we describe some applications to computing/communication systems based on bio-inspired models.

1 INTRODUCTION

The theory of stochastic convexity is widely recognised as a framework to analyze the stochastic behaviour of parameterized models by different notions in both univariate and multivariate settings. These properties have been applied in areas as diverse as engineering, biotechnology, hydrology, actuarial science, and medicine, and they have been useful to analyze queueing systems, total times to complete jobs, wireless communication networks, total claim amount distributions of insurance portfolios, and residence times of a substance in the human body, among others. Consider a family of parameterized univariate or multivariate random variables $\{X(\theta)|\theta \in T\}$ over a probability space $(\Omega, \Im, Pr)$, with $T \subset \mathbb{R}^n$ with $n \geq 1$, being some sublattice, where the parameter θ usually represents the mean or the variance, or both them for the random variable $X(\theta)$, although it also may represent another rate influencing the probability distribution of $X(\theta)$. For practical instances, θ is sometimes assumed to reflect some biological, physical, exposure, environmental, or economical conditions, that determine a scenario where the random object $X(\theta)$ is analyzed. Let $g(\theta) = E[\phi(X(\theta))]$ be the expected value of the functional ϕ for a univariate random variable $X(\theta)$. Let $\leq$ denote the componentwise ordering for vectors of scalars. The stochastic increasing monotonicity of $X(\theta)$ (or $g(\theta)$) with respect to its parameter (in the sense that $\theta_1 \leq \theta_2$ implies $X(\theta_1) \leq_{st} X(\theta_2)$ or equivalently, $Pr(X(\theta_1) > t) \leq Pr(X(\theta_2) > t)$ for all t) (see also Shaked and Shanthikumar, 2007) constitutes a useful property in a wide range of areas as those aforementioned. Numerous examples of functionals of random variables having such property can be found in the literature, for instance, an increasing net profit (or an increasing expected net profit) is a common assumption in several financial settings. Additionally, different second-order properties, as the convexity, the concavity and the supermodularity of the parameterized random variable $X(\theta)$ with respect to its parameter, have been described by different concepts in the literature. Regular, sample-path, and strong stochastic convexity notions have been defined to intuitively describe how the random objects $X(\theta)$ grow convexly (or concavely) concerning their pa-

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rameters, and they have played a prominent role into the stochastic analysis in different fields. The purpose of this paper is to explain some of the basic concepts of stochastic convexity, and to discuss how this theory has been used into the stochastic analysis, both theoretically and in practice. We will omit the technical details of the results that the reader can check in the corresponding articles. A second purpose is to provide the reader to some of the recent and historically relevant literature on this topic. Finally, we focus on several applications to bio-inspired models.

For a univariate random variable $X(\theta)$ with respect to one real or integer-valued scalar θ, next, we recall several stochastic convexity notions, where we will use $\{X(\theta)\}$ as a shorthand of $\{X(\theta)|\theta \in T\}$. First, $\{X(\theta)\}$ is *stochastic increasing and convex in the sample-path sense*, denoted by $SICX-sp$, if for any four parameter values θ_i, $i=1,2,3,4$, that fulfil $\theta_1+\theta_4=\theta_2+\theta_3$ and $\theta_4 \geq \max\{\theta_2,\theta_3\}$, there exists four random variables $\widehat{X}_i$, $i=1,2,3,4$ on a common probability space $(\Omega,\Im,Pr)$, such that $\widehat{X}_i =_{st} X(\theta_i)$ for any i; and $\forall w \in \Omega$

$$\widehat{X}_4(w) \geq \max\{\widehat{X}_2(w),\widehat{X}_3(w)\} \tag{1}$$

$$\widehat{X}_1(w)+\widehat{X}_4(w) \geq \widehat{X}_2(w)+\widehat{X}_3(w). \tag{2}$$

This concept was developed in Shaked and Shanthikumar, 1988a, 1990a, 1990b. From (Shanthikumar and Yao, 1991), we have the following stronger notion. $\{X(\theta)\}$ is *strong stochastic increasing and convex*, denoted by $SSICX$, if $X(\theta) =_{st} \phi(\varepsilon,\theta)$, for any ϕ being an increasing function on θ and ε being a random variable whose distribution function is independent of θ. The regular notion, also called the functional notion (Shaked and Shanthikumar, 1990a) is defined as follows. $\{X(\theta)\}$ is *stochastic increasing and convex*, denoted by $SICX$, if $E[\phi(X(\theta))]$ is increasing and convex in θ for any increasing and convex function ϕ. Another related concept introduced by (Shaked and Shanthikumar, 1990b) is given next. $\{X(\theta)\}$ is *stochastic increasing and convex in stochastic sense*, denoted by $SICX-st$, if $E[\phi(X(\theta))]$ is increasing and convex in θ for any increasing function ϕ. Notice that the convexity of ϕ is not required for this definition. The $SICX-st$ property is equivalent to the fact that the distribution function of $X(\theta)$, denoted by $F(x,\theta)$, is decreasing and concave in θ for any x. The following implications hold for the previous notions:

$$SSICX \Rightarrow SICX-sp \Rightarrow SICX \text{ and } SICX-st \Rightarrow SICX-sp. \tag{3}$$

However, $SICX-st$ and $SSICX$ do not imply each other. Many well known parametric families of distributions, as the Normal, Gamma, Lognormal, Poisson, Binomial and Geometric, fulfil some of these properties. We refer to the earlier references for the definitions of the stochastic concavity notions. Nice reviews of these concepts, their basic properties and applications can be found in (Shaked and Shanthikumar, 1991); (Chang et al., 1994) and (Shaked and Shanthikumar, 2007). The earlier notions are extended to the multivariate case by means of directionally convex functions (see Marinacci and Montrucchio, 2005, for the definition, main properties and background of these functions, that were defined by Wright, 1954). A useful characterization of the increasing directionally convex functions was given in (Shaked and Shanthikumar, 1990a). A real-valued function ϕ defined on $\mathbb{R}^n$ is increasing and directionally convex if and only if ϕ is increasing, componentwise convex, and supermodular. Also, ϕ is said to be supermodular (see Marshall and Olkin, 1979) if $\phi(x \vee y)+\phi(x \wedge y) \geq \phi(x)+\phi(y)$, for all x, $y \in \mathbb{R}^n$ with the operators $+$, $\vee$ and $\wedge$ denoting, respectively, the componentwise sum, maximum and minimum. Stochastic directional convexity for multivariate random vectors with respect to multivariate parameters was introduced by (Shaked and Shanthikumar, 1990a), when the parameters have on values over convex subsets of the real line. These concepts were also studied by (Meester and Shanthikumar, 1993), who used this theory to obtain the regularity of some stochastic processes, including Markov chains, and to derive applications in queueing systems; and extended to a general space by (Meester and Shanthikumar, 1999). Stochastic convexity in the sense of the usual stochastic order, introduced by (Shaked and Shanthikumar, 1990b) in the univariate case, was extended to the multivariate case by (Shanthikumar and Yao, 1991), where is called strong stochastic convexity. The functional notion of stochastic increasingness and directional convexity is denoted by $SI-DCX$. Other notions of stochastic convexity in a generalized sense are defined and studied in (Denuit et al., 1999), and notions involving partial sums in (Chao and Luh, 2004).

2 DISCUSSION

In this section we discuss how the theory of the stochastic convexity has been used into the stochastic analysis, both theoretically and in practice, and we provide some historically relevant and recent bibliographic references and remarks on their main developments and applications. Generally, the sample-path notions of stochastic convexity are useful in the mathematical proofs to state stochastic convexity of random objects, while the functional definitions are good for applications, for example, when ϕ is the functional of a performance measure of a system in engineering to be bounded or the objective function of a mathematical model to be optimized. In addition, the functional notions allow one to apply the compositional rules in (Meester and Shanthikumar, 1999) to stand the stochastic increasingness and directional convexity (concavity) by using some transformation of the parameters. The fact that this theory does not rely on closed-form distribution functions allows one to analyze stochastic models which are unwieldy analytically. Several techniques of probabilistic risk assessment, statistics and operational research have been connected with the stochastic convexity and the stochastic directional convexity. They lead to solve some decision-making problems in several contexts. First, looking to a dynamical perspective, these structural properties allow one to characterize the spatio and the temporal behaviour of some stochastic processes, being relevant for the probabilistic risk asessment concerning some maintenance policies in reliability (see e.g., (Shanthikumar and Yao, 1992, and Meester and Shanthikumar, 1993). Secondly, in many optimization problems, the monotonicity and the convexity assumptions play an important role. In deterministic dynamic programming problems, for example, if the reward function $g(s,x)$ is strictly concave in s and x and increasing in s, the feasible action correspondence is an increasing and convex graph; and the transition function is continuous, bounded and concave, then the value function is continuous, bounded, increasing and concave (see Stokey et al., 1989). In stochastic dynamic programming, the problem is more difficult due to the monotonicity and the convexity for a transition probability, and some conditions for the differentiability and the concavity of the value function are stated by using notions similar to the stochastic convexity in stochastic sense in (Atakan, 2003). The stochastic convexity also arises to solve optimization problems involving other operational research models, for allocation or manufacturing as in (Gallego et al., 1993) and (Kim and Park, 1999); and other problems involving pairwise interchange arguments based on the interplay of this theory with the stochastic majorization of vectors, for scheduling as in (Shanthikumar, 1987), and (Chang and Yao, 1990) (we refer to the book Marshall and Olkin, 1979, for majorization concepts). Thirdly, the stochastic convexity has been a bridge property into the analysis of the variability of mixture models with uncertain parameters. The statistical modelling of the uncertainty constitutes one of the major goals in experimental studies. Theoretically, a distinction between the random uncertainty (due to natural conditions) and the epistemic uncertainty (due to the lack of knowledge of the data) is given in the literature. This distinction is not so easy in practice. Parameter uncertainty involves both issues, since when forecastings are performed by a model often there is a random component in which some of the conditions given in the inputs are not controlled and specified (see e.g., Ang and Tang, 2007). Mixture models with random parameters, having arbitrary probability distributions of the marginals, and an arbitrary joint distribution function to model the dependence structure of the random vector, provide a non-parametric setting to account for the uncertainty in a broad sense. We notice that the $SI-DCX$ property of the mixture model $X(\theta)$ when the parameters are held fixed implies the increasingness and directional convexity of the expected value of any increasing and convex function of the parameterized random variable $X(\theta)$. This property leads to variability comparisons of the mixture models $X(\Theta)$ for two random vectors of parameters Θ and Θ' describing two scenarios, whose mixing distributions are ordered by the increasing directionally convex order, see (Shaked and Shanthikumar, 2007) for details on the directionally convex orders. Recall that the variability ordering, also known as increasing convex ordering, (see Muller and Stoyan, 2002), denoted by $X(\Theta) \leq_{icx} X(\Theta')$ means that $E[\phi(X(\Theta))] \leq E[\phi(X(\Theta'))]$, for every increasing convex real-valued function ϕ for which the expectations exist. In particular, if the mixture model $X(\Theta)$ fulfills the $SI-DCX$ property, then the following relationship holds:

$$\Theta \leq_{idcx} \Theta' \Rightarrow X(\Theta) \leq_{icx} X(\Theta') \tag{4}$$

where $\Theta \leq_{idcx} \Theta'$ denotes the increasing directionally convex ordering, that allows one to compare the marginal distributions in increasing convex sense, jointly with the increasingness in positive dependence of the joint distribution of the random vector (see Muller and Scarsini, 2001). The decreasing directionally convex order, denoted by $\Theta \leq_{ddcx} \Theta'$, is used to combine less valued with more volatility components. They belong to the class of multivariate integral orders that are defined by the corresponding property of the expected value of a functional of the random vector. Other properties and results on the connection of the directionally convex orders and the variability of mixture models can be found in (Ruschendorf, 2005). A first consequence of the relationship (4) is the construction of distributional bounds for the mixture $X(\Theta)$ by fixing some values of the parameter Θ' or of the random variable $X(\Theta')$. We notice that the distribution moments are increasing convex functionals of the random variables and can be bounded by the previous stochastic comparison. Accordingly, when $X \leq_{icx} Y$ and the random variables have equal means, then it is written $X \leq_{cx} Y$ and it implies that $Var(X) \leq Var(Y)$, where *Var* denotes the variance of the random variable. Hence, the convex ordering provides an instrument to compare and to quantitatively evaluate the magnitude and the dispersion of a performance measure of a system, under different mixing distributions of the parameters. Convex risk measures of the random variable are preserved by this ordering. Other convex functions, as the maximal value, for a sequence of independent and identically distributed random variables can be bounded by using the previous relationship (4). Bounds for the mixture models $X(\Theta)$ from Equation (4) can be derived also by fixing the dependence structure of the parameter vector Θ. Some dependence concepts allow one to understand the relationship between the stochastic convexity and the stochastic dependence of random vectors of parameters by means of directionally convex orders (see (Joe, 1997) for a general treatment of the stochastic dependence). *Positive quadrant dependence* (denoted by *PQD*) and *positive supermodular dependence* notions involve integral comparisons with a given random vector with fixed marginals and independent components. Some examples of bivariate distributions fulfilling the *PQD* property for some values of their parameters are given by the Kibbles bivariate Gamma, the Moran-Dowton exponential, the Marshall-Olkin, the F-G-M bivariate exponential and the Lomax distributions. Nonparametric statistical tests to check the earlier dependence notion can be found in (Denuit and Scaillet, 2004) and (Scaillet, 2006).

The first applications of this topic can be found in the queueing and the stochastic processes theories: (Shaked and Shanthikumar 1988a,1988b,1990a,1990b), (Chang et al., 1991), (Meester and Shanthikumar, 1990,1993). Recent applications in queueing systems are given in (Chao and Luh, 2004), (Miyoshi and Rolski, 2004), (Shioda and Ishi, 2004), (Rolski, 2005), and (Ortega, 2011). The last article considers routing in queueing networks, to develop some applications in communication networks, that we will discuss later. The issue of stochastic convexity arises recently into the study of stochastic processes, especially point processes. At the best of our knowledge, recent results can be seen in (Miyoshi, 2004), and (Fernandez-Ponce, et al, 2008a,b). They are involved in the multivariate stochastic comparisons of some stochastic processes (see Kulik and Szekli, 2005). The following monographs include an exhaustive development of the theory, and/or some detailed applications for selected operational research problems. Chang et al., 1994 provided applications to the analysis of a random yield model, a joint setup problem, a manufacturing process with trial runs, a production network with constant work-in-process, and a scheduling model in tandem production lines and parallel assembly systems. The convexity and the concavity in the stochastic sense of other measures associated with production-inventory systems are studied in (Yao and Zheng, 2002), that includes applications of this theory for quality control with warranty, process control with inspection of machines for batch production, and inventory control with substitution, among others. Yao (1994,1996) provided surveys with manufacturing and production applications. Recent developments to operational research models can be found in (Ahn et al., 2005), (Ott and Shanthikumar, 2006), and (Ortega and Alonso, 2011). Stochastic directional convexity of random sums of random variables has been dealt with by (Fernandez-Ponce, et al., 2008a), (Escudero et al., 2010), and (Ortega and Escudero, 2010), among others. Observe that general results in the direction of Equation (4), for random sums, can be found in the last articles. This is a classic problem for sums and univariate parameters, as can be check in Makowski and Philips, 1992. Applications have been depicted for the sums and the random sums in different fields, as for instance, the total claim amounts under the Sparre-Andersen risk model in actuarial science, and the times of systems under shock models in reliability, etc. The stochastic convexity of some special linear combinations defined by indicator functions is applied in (Escudero and Ortega, 2008) for the analysis of reinsurance models for large claims based on truncation with random retention levels. Other linear combinations of random variables are studied in (Ortega, 2010).

3 APPLICATION BASED ON BIO-INSPIRED MODELS

Current research trends are exploiting the similar features between biological systems and complex communication systems to develop techniques in these fields based on biological schemes. Nature has been shown to be able to adapt rapidly to environmental changes, and assemble simple structures into complex operations. Next, we discuss some models that are biologically inspired, where the stochastic convexity properties play an important role.

Multiplicative processes arise in biology and ecology to describe the growth of organisms and populations of species, as protein families, that evolve in a multiplicative manner, see e.g., (Reed and Hughes,

2002) from a probabilistic point of view. The main idea is that the random growth of an organism is expressed as a percentage of its current actual size. Assuming that we start with an organism of size X_0, then at each step j, the organism may grow or shrink, according to a random variable Y_j, such that the size of an organism at step j is given by $X_j(\theta) = X_{j-1}(\theta)Y_j$. In financial mathematics, the Black-Scholes option pricing model, which is a specific application of Itos lemma, describes the price of a security that moves in discrete time steps according to the earlier equality, where Y_j is lognormally distributed usually (see e.g., Hull, 2002). Some bio-inspired models and networks have been used to model evolution in computing and communication systems. Some of these systems are characterized by strongly dynamic environments and heterogeneous nodes, among others. Huberman and Adamic have applied the multiplicative processes to describe the growth of sites on the web, as well as the growth of user traffic on web sites (see Huberman and Adamic, 1999,2000). Mitzenmacher, 2004 used a similar model to explain the behaviour of both tails of the distribution of the size of computer files. There is a vast recent literature on the use of the random products in the study of computer systems. Other applications of multiplicative processes can be found in hydrology, geology, and chemistry (Crow and Shimizu, 1988 and Escudero et al., 2010).

Ortega, 2010 has provided the conditions under which the multiplicative processes are $SI-DCX$. Applications are given there to the analysis of the size of traffic on web sites. The stochastic convexity of the size of the traffic $X_j(\theta)$ with respect to its environmental parameter means that the expected value of any convex increasing function of the size increases and behaves componentwise convex and supermodular with respect to its parameter. The main statistical consequences of this property are described above from the relationship (4), especially the construction of distributional bounds. For communication systems, bounds of these measures provide valuable insight into the primary factors affecting the performance of systems, as those influencing the system bottleneck; and they become useful to eliminate non appropriate options at an early stage of the analysis. Similar conclusions may be reached for other bio-inspired adaptive models. Specifically, the tree networks constitute one of the most useful network topologies, with applications in computer science, taxonomy, location, molecular biology, evolution, and ecology. Ortega, 2011 considers rooted tree networks and rooted butterfly networks that are modelled by queueing systems, where the items enter at the root and proceed away until they reach their destination and exit the system, the arcs of the network are represented by FIFO servers, and the service times have arbitrary probability distributions and they correspond to the times needed for an item to cross edges. The $SI-DCX$ of the exit time of the item k from the queue j in the network is the key property to develop variability comparisons of departure times and delay times of the system with a discrete probability distribution for the routing policy and with correlated interarrival times and service times. Bounds for the previous performance measures are derived as above.

4 CONCLUSIONS

To finish, we emphasize on different advantages of the concepts of stochastic convexity. Many measures of the performance of a system are modelled as mixtures defined by composition of arithmetic functionals of parameterized non-negative random variables, and these functionals fulfil monotonic directional convexity properties. Nevertheless, by appropriate assumptions for the random components in the model for fixed parameters; and by using Theorem 3.2 in (Meester and Shanthikumar, 1999), we can state that the parameterized model is SI -DCX. We recall that the stochastic directional convexity allows one to analyze the mixtures without requiring an analytical expression of the distribution function of the random variable, and to obtain bounds from the relationship (4). This makes this theory to become an attractive alternative to evaluate or to explore the probabilistic behaviour of the performance measure, when there is lack of information about the marginal distribution of the parameters and the correlations among them (provided that modelling the dependence structure from empirical data is a rather complex task; and the same happens for selection of prior distributions to account for covariates). Also, the definitions of the stochastic convexity properties are set for parameters taking on values over general spaces in (Meester and Shanthikumar, 1999), and this allows versatile scenarios of parameters. Some related optimization problems, e.g., to find the maximal value of a performance measure, can be addressed directly from stochastic convexity properties, by using notions of dependence as the comonotonicity (see Escudero et al., 2010).

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