Positive Realization of Continuous Linear Systems with Order Bound

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Abstract: This paper discusses the realization problem of a class of linear-invariant system, in which state variables, input and output are restricted to be nonnegative to reflect physical constraints. This paper presents an efficient and general algorithm of positive realization for positive continuous-time linear systems in the case of transfer function with (multiple) real or complex poles. The solution of the corresponding problem for continuous-time positive is deduced from the discrete-time case by a transformation. We deal with the positive realization problem through convex cone analysis. We provide a simple general and unified construction method for the positive realization of the transfer function, which has multiple poles, upper-bound and a sparse realization matrix. We consider a sufficient condition of positive realization.

1 INTRODUCTION

This paper discuss the realization problem of a class of linear-invariant system, in which state variables, input and output are restricted to be nonnegative to reflect physical constraints. The nonnegative constraints can be encountered in engineering, medicine and economics (Brown, 1980) (Gersho and Gopinath, 1979), and (Benvenuti and Farina, 2001).

In the cases of discrete time, the powerful constructive tools of proper generators for general transfer functions have been introduced a lot in the last decade (Nagy and Matolcsi, 2003)(Nagy et al., 2007)(Hadjicostis, 1999)(Nagy and Matolcsi, 2005). Constructive efficient general methods to solve the positive realization in close to minimal dimension have mainly focused on the problems of discrete systems (Anderson et al., 1996)(Benvenuti et al., 1999)(Nagy et al., 2007). However, the positive realization problem of the continuous time case have been studied less than that in the discrete time. We propose a constructive efficient algorithm to solve the positive realization for some given positive system with (possibly multiple) complex poles in continuous time domain. First, we solve the general problem for the positive realization of transfer function with complex poles in the continuous time linear system. The positive realization problem can be derived by finding an appropriate generator of a polyhedral cone intervening reachability and observability. Because the positive realization is not unique, we can choose a proper sparse realization matrices by selecting spanning vectors from the cone generator. We also handle the positive realization of the transfer function with multiple complex poles. The sufficient conditions for the positive realization of the transfer function with multiple complex poles are given and analyzed. The format of the paper is as follow. In Section 2, we introduce the preliminary concepts for the analysis of the continuous positive linear system. The positive realization problems of the transfer function with simple poles are discussed in Section. We consider the generalized positive realization problem of the transfer function with multiple complex or real poles in Section 4.

2 PRELIMINARY

The convex cone $\mathcal{X} = cone(X)$ denotes the smallest convex cone of a set X, which consists of all finite nonnegative linear combinations of elements of the set X. The dual cone, $\mathcal{X}^*$, of a cone $\mathcal{X}$ is defined by $\mathcal{X}^* = \{y | x^T y \geq 0, \forall x \in \mathcal{X}\}$. A convex cone $\mathcal{X}$ is said to be a polyhedral cone if it is spanned by a finite number of vector set $X = \{x_1, \cdots, x_m\}$ with $x_i \in \mathbb{R}^n$ and X is called by a polyhedra generator of $\mathcal{X}$. From now, X is also denoted by the matrix with columns $x_i \in X$. An extreme point of a convex cone is one which is not a proper positive linear combination of any two points of the set. A finite set X is said to be a frame of the polyhedral cone $\mathcal{X}$ if the points of X are extreme points in $\mathcal{X}$ and X spans $\mathcal{X}$. A polyhedral cone is a closed convex cone (Berman and Plemmons, 1994).

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A matrix $A \in \mathbb{R}^{n\times n}$ is said to be a Metzler matrix if all its off-diagonal elements are in $\mathbb{R}_+$. A matrix A is a Metzler matrix if and only if there exists an $\alpha \in \mathbb{R}$ satisfying $(A+\alpha I) \in \mathbb{R}_+^{n\times n}$ (van den Hof, 1997)(Benvenuti and Farina, 1998). Consider a single-input, single output linear time-invariant system $\dot{x} = Ax + bu$, and $y = cx$ where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$ and $c \in \mathbb{R}^{1\times n}$. The linear system is said to be a positive linear system if for all $x_0 \in \mathbb{R}_+^n$ and for all $u(t) \in \mathbb{R}$, we have $y(t) \in \mathbb{R}$ for all t. A strictly proper rational function is said to be positive realizable if there exist a matrix A with nonnegative off-diagonal elements and nonnegative b, c such that $H(s) = c(sI-A)^{-1}b$. $H(s)$ is the set of strictly proper rational transfer functions. Such a realization (A,b,c) is called the positive realization, since it yields nonnegative state response whenever initial states and inputs are nonnegative.

The necessary and sufficient condition of the positive realization has been introduced in (Ohta et al., 1984). The problem of positive realization of a given transfer function is reduced to finding an appropriate polyhedral cone in the room sandwiched by the reachability and observability cones. We summary the necessary and sufficient condition of the existence of positive in the next theorem.

Theorem 2.1 ((Ohta et al., 1984))**.** *Let (A,g,h) be a minimal realization of $H(s)$. Then $H(s)$ is positive realizable if and only if there exists a generator matrix P such that a polyhedral cone $\mathcal{P} = cone(P)$ satisfies*

1. $\exp(At)\mathcal{P} \subset \mathcal{P}$ *for all* $t \geq 0$,
2. $\mathcal{R} \subset \mathcal{P} \subset \mathcal{S}$.

where $P \in \mathbb{R}^{n\times m}$, $\mathcal{R}$ is a reachable set and $\mathcal{S}$ is an observable set.

Technically, it is difficult to find out a proper polyhedral cone $\mathcal{P}$ satisfying the condition of Theorem 2.1. The explicit construction methods of the polyhedral cone for discrete time case can be applied to the continuous time version.

Lemma 2.1. *Let $\mathcal{P}$ be a polyhedral cone in $\mathbb{R}^n$ and $A \in R^{n\times n}$. Then $\exp(At)\mathcal{P} \subset \mathcal{P}$ for any $t \geq 0$ if and only if $(A+\lambda I)\mathcal{P} \subset \mathcal{P}$ for some $\lambda \geq 0$.*

3 SIMPLE POLE CASE

We consider a simple third-order asymptotically stable positive proper transfer function $H(s)$ with partial fractional form as

$$H(s) = \frac{R}{s-\lambda_0} + \frac{\beta_1}{s-\lambda_1} + \frac{\bar{\beta}_1}{s-\bar{\lambda}_1} \tag{1}$$

where $\lambda_0 < 0$, λ_1 is a complex pole with $real(\lambda_1) \leq \lambda_0$, $\bar{\lambda}_1$ is a complex conjugate of λ_1 and β_1 is complex number. A minimal Jordan realization is given by

$$A = \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & x & y \\ 0 & -y & x \end{bmatrix} \triangleq \lambda_0 \oplus A_1, \quad g = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{2}$$
$$h = \begin{bmatrix} R & c_2 & c_3 \end{bmatrix} \triangleq \begin{bmatrix} R & \hat{c} \end{bmatrix}$$

where λ_0 is a real pole with $\lambda_0 < 0$, a complex pole is $\lambda_1 = x + iy$ with $x < \lambda_0$. Here $R > 0$, c_2 and c_3 are appropriate real values.

Definition 3.1. *Let $\mathfrak{P}_m(\rho)$ for $m \geq 1$ denote the set of points in the complex plane that lie in the interior of the regular polygon with m edges having one vertex at point ρ and including a zero point 0. For $m \geq 3$, the polygon $\mathfrak{P}_m$ is defined by a subset in $\mathbb{R}^2$ through the following inequalities:*

$$\mathfrak{P}_m(\rho) = \left\{ (x,y) | r\cos\left[\frac{(2k+1)\pi}{m} - \varphi\right] \leq \rho\cos\frac{\pi}{m} \right\}, \tag{3}$$

for all k with $0 \leq k \leq m$, where $x = r\cos\varphi$ and $y = r\sin\varphi$. When $\rho = 1$, it is simply denoted by $\mathfrak{P}_m$.

Our constructive method is similar to the results in (Benvenuti et al., 1999)(Nagy et al., 2007). A $2\times m$ matrix V consisting of the vertices of a polygon in the complex plane is defined as

$$V = \begin{bmatrix} 1 & \cos\frac{2\pi}{m} & \cos\frac{4\pi}{m} & \cdots & \cos\frac{2\pi(m-1)}{m} \\ 0 & \sin\frac{2\pi}{m} & \sin\frac{4\pi}{m} & \cdots & \sin\frac{2\pi(m-1)}{m} \end{bmatrix}. \tag{4}$$

for a given m. A cone generator matrix $P \in \mathbb{R}^{3\times m}$ is defined as

$$P = \begin{bmatrix} \mathbf{e} \\ V \end{bmatrix} \triangleq \begin{bmatrix} p_1 & p_2 & \cdots & p_m \end{bmatrix} \tag{5}$$

where $\mathbf{e}$ represents an $1\times m$ vector with all entries equal to 1 and p_i's are extreme points in a polyhedral cone $cone(P)$.

Theorem 3.1. *Assume that the three dimensional transfer function $H(s)$ with a pair of strictly conjugate complex poles $(\lambda_1, \bar{\lambda}_1)$ and a real pole λ_0 with $\lambda_0 > real(\lambda_1)$ has a realization such as (2) and the extreme generator P is constructed as in (5). If we have*

$$-\frac{\pi}{2} + \frac{\pi}{m} \leq \arg(\lambda_1 - \lambda_0) \leq \frac{\pi}{2} - \frac{\pi}{m}, \tag{6}$$

$$|\beta_1| \leq \frac{\beta_0}{2} \tag{7}$$

then there is a polyhedral generator $P \in \mathbb{R}^{3\times m}$ such that $cone(P)$ is $\exp(At)$-invariant for any $t > 0$.

Theorem 3.2. *Assume that the conditions of Theorem 3.1 are satisfied. Then there is a sparse circular Toeplitz matrix A_+ with $3m$ elements such that $(\eta I + A)P = PA_+$ for a proper $\eta > 0$. Thus we get a Metzler matrix $A^* = A_+ - \eta I$.*

4 MULTIPLE POLES CASE

We consider the asymptotically stable transfer function $H(z)$ being a positive linear system in the form as

$$H(s) = \frac{\beta_0}{s-\lambda_0} + \sum_{j=1}^{r}\sum_{i=1}^{n_j} \frac{\beta_j^{(i)}}{(s-\lambda_j)^i} \tag{8}$$

where $\beta_0 > 0$ and $H(s)$ has a non-negative impulse response. An $n_j \times n_j$ Jordan form matrix A_i, an $n_j \times 1$ matrix b_i and an $1 \times n_j$ matrix c_i are defined ,respectively, as follows: $A_j = J(\lambda_j)$,

$$J(\lambda_j) \triangleq \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_j \end{bmatrix}, \quad b_j = \mathbf{e}_{n_j} \triangleq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$
$$c_j = \begin{bmatrix} \beta_j^{(n_j)} & \beta_j^{(n_j-1)} & \cdots & \beta_j^{(1)} \end{bmatrix}$$

for $1 \le j \le r$ and $A_0 = \lambda_0$ where $A \oplus B \triangleq diag(A,B)$ and the basis vector $\mathbf{e}_k$ has a 1 as its k-th component and 0's elsewhere. The transfer function $H(z)$ has a canonical minimal Jordan form realization (A,b,c) such that $A = \bigoplus_{j=0}^{r} A_i$, $b = \begin{bmatrix} 1 & b_1^T & b_2^T & \ldots & b_r^T \end{bmatrix}^T$ and $c = \begin{bmatrix} 1 & c_1 & c_2 & \ldots & c_r \end{bmatrix}$. First, let us consider a real rational transfer function $H_1(s)$ with multiple complex conjugate poles of the form

$$H_1(s) = \sum_{k=1}^{n} \left\{ \frac{\beta_k}{(s-\lambda_1)^k} + \frac{\bar{\beta}_k}{(s-\bar{\lambda}_1)^k} \right\} \tag{9}$$

where the pole λ_1 and coefficients β_i are complex and $\bar{\beta}_i$ is defined as the conjugate of β_i. The transfer function $H_1(s)$ has a Jordan canonical form realization such as $(J(\lambda_1) \oplus J(\bar{\lambda}_1), \begin{bmatrix} \mathbf{e}_n^T & \mathbf{e}_n^T \end{bmatrix}^T, \begin{bmatrix} c_1 & \bar{c}_1 \end{bmatrix})$. Then by using similarity transformation, we can obtain a real Jordan form realization $(J(\lambda_1, w), \hat{b}, \hat{c})$ of $H_1(s)$ such that $J(\lambda_1, w) \in \mathbb{R}^{2n \times 2n}$, $\hat{b} \in \mathbb{R}^{2n}$ and $\hat{c}^T \in \mathbb{R}^{2n}$ are given by

$$J(\lambda, w) \triangleq \begin{bmatrix} C & I & O & \cdots & O \\ O & C & I & \cdots & O \\ O & O & C & \cdots & O \\ \vdots & & & \ddots & \vdots \\ O & O & O & \cdots & C \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}$$
$$\hat{c} = \begin{bmatrix} \tilde{\beta}_{2n} & \tilde{\beta}_{2n-1} & \cdots & \tilde{\beta}_1 \end{bmatrix} \tag{10}$$

for any given number w and $C = C(x,y) \triangleq \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$ where the entries of c_2 are defined by $\tilde{\beta}_{2k} = 2Re(\beta_k)$ and $\tilde{\beta}_{2k-1} = 2Im(\beta_k)$ for each k.

Theorem 4.1. *Assume that a transfer function of the form*

$$H(s) = \frac{\beta_0}{z-\lambda_0} + \sum_{k=1}^{n} \left\{ \frac{\beta_k}{(s-\lambda_1)^k} + \frac{\bar{\beta}_k}{(s-\bar{\lambda}_1)^k} \right\} \tag{11}$$

has a non-negative impulse response function where $\beta_0 > 0$ and real$(\lambda_1) < \lambda_0$ and the maximal order n of the pole is larger than 1. Let us define a function r_m as $r_m(z) = \max\{\hat{r} | (\hat{r}\cos\theta, \hat{r}\sin\theta) \in \mathfrak{P}_m, \hat{r} > 0\}$ with respect to some $z = r\cos\theta + ir\sin\theta$. Set $z_1 = \frac{\eta+\lambda_1}{\eta+\lambda_0}$ and $z_2 = \frac{1}{\eta+\lambda_0}$. For sufficiently large $\eta > 0$, a sufficient condition is given by

$$0 < w \le \left(1 - \frac{|z_1|}{r_m(z_1)}\right) \frac{r_m(z_2)}{|z_2|} \tag{12}$$

$$|\beta_k| \le \frac{\beta_0}{(2w)^{k-n}}. \tag{13}$$

Then, there exists a positive realization (A_+, b_+, c_+) of the transfer function $H(s)$ that has the order mn for all $1 \le k \le n$.

Proof. We try to find a sufficient condition of the existence of a positive realization (A_+, b_+, c_+) for the given transfer function $H(s)$. We obtain a real block Jordan form realization (A, b, c) of the transfer function $H(z)$ in equation (11) as

$$A = \lambda_0 \oplus J(\lambda_1, w), \quad b = \begin{bmatrix} 1 \\ \hat{b} \end{bmatrix} \tag{14}$$
$$c = \begin{bmatrix} 1 & \hat{c} \end{bmatrix}$$

where $x = r\cos\theta$, $y = r\sin\theta$ and $J(\lambda_1, w)$, $\hat{b}$ and $\hat{c}$ are defined in equation (10). We use the fact that there exists an $\eta > 0$ such that $(\eta I + A)$-invariant cone $\mathcal{P}$ with mn edges (i.e., $(\eta I + A)\mathcal{P} \subset \mathcal{P}$). We generalize the concept of the cone generator introduced in (Benvenuti et al., 1999) for the case with multiple complex poles. In order to formulate a polyhedral cone generator with $(\eta I + A)$-invariant property, a block shift matrix $Z \in \mathbb{R}^{2n \times 2n}$ and a matrix $V \in \mathbb{R}^{2n \times m}$ are defined as:

$$Z = \begin{bmatrix} O & O & \cdots & O & O \\ I & O & & O & O \\ O & I & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & I & O \end{bmatrix}, \quad \hat{V} = \begin{bmatrix} V \\ O \\ O \\ \vdots \\ O \end{bmatrix}$$

where I is an identity matrix, O is a zero matrix with proper dimension and a $2 \times m$ matrix V is defined as (4). A cone generator matrix $P \in \mathbb{R}^{(2n+1) \times mn}$ is defined as $P = \begin{bmatrix} P_1 & P_2 \cdots P_n \end{bmatrix}$ where P_k is defined as $P_k = \begin{bmatrix} \mathbf{e}^T \\ \varphi_k \end{bmatrix}$ for $1 \le k \le n-1$, $\mathbf{e}$ represents an $m \times 1$

vector with all entries equal to 1 and $\varphi_k = Z^{k-1}\hat{V}$. The columns of matrix P represent the extreme vertices of a finite generated cone $\mathcal{P}$ in $\mathbb{R}^{2n+1}$ (i.e., $cone(P) = \mathcal{P}$) and are positive independent. The polyhedral cone $\mathcal{P}_k$ is generated by P_k, i.e., $\mathcal{P}_k = cone(P_k)$ for each k. Note that $(A+\eta I)P_k$ for each k has only 1-th, k-th and $k+1$-th block components. By the invariance property, $\frac{(A+\eta I)}{\lambda_0+\eta}P_k \in \mathcal{P}$ is required. Set $\tilde{A} = \frac{(A+\eta I)}{\lambda_0+\eta}$. The matrix $\tilde{A}$ has eigenvalues $\{1, z_1, \bar{z_1}\}$ with $|z_1| < 1$. Then the positive realization problem with respect to $\tilde{A}$ is close related to that of the discrete time domain as in (Benvenuti et al., 1999)(Nagy et al., 2007). Choose $\{W_k, W_{k+1}\}$ such that

$$\frac{(A+\eta I)}{\lambda_0+\eta}P_k = \alpha_1 W_k + \alpha_2 W_{k+1} \tag{15}$$

for each k where α_j's satisfy $\alpha_j \geq 0$, $\alpha_1+\alpha_2 = 1$ and all the entries in the first row of $W_k \in \mathcal{P}_k$ are equal to 1. A sufficient condition for a feasible solution (α_1, α_2) should satisfy two inequalities, $\frac{w|z_2|}{r_m(z_2)} \leq \alpha_2$ and $\frac{|z_1|}{r_m(z_1)} \leq \alpha_1$, and an equality $\alpha_1+\alpha_2 = 1$ for a given w. By rearranging the above conditions, we obtain an inequality (12). From this result, we can see that w and η are tunable parameters to get a positive matrix. The polyhedral cone $\mathcal{P}$ is $\eta I + A$-invariant under the above condition in (12). We can prove that $\mathcal{R} \subset \mathcal{P}$ and $\mathcal{P} \subset \mathcal{S}$ without difficulty. □

Theorem 4.2. *Assume that the conditions of Theorem 4.1 are satisfied. Then there is a sparse circular matrix A_+ with at most $3nm$ non-zero elements such that $(\eta I + A)P = PA_+$ for a proper $\eta > 0$. We note $W_k \in cone(P_k)$ in Eq. (15). The columns of W_k is positively linearly combined by the three vectors chosen from P_k similar to the process of Theorem 3.2. We can verify that we can choose a sparse matrix A_+ such that A_+ is defined by*

$$A_+ = \begin{bmatrix} T_1 & \varepsilon I & 0 & \cdots & 0 \\ 0 & T_2 & \varepsilon I & \cdots & 0 \\ 0 & 0 & T_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & T_2 \end{bmatrix} \tag{16}$$

with $T_1 = T(\vec{\mathbf{t}}_1), T_2 = T(\vec{\mathbf{t}}_2)$ and $T_3 = T(\vec{\mathbf{t}}_3)$ where $\vec{\mathbf{t}}_1$ has at most three nonzero elements and $\vec{\mathbf{t}}_2$ and $\vec{\mathbf{t}}_3$ have at most two nonzero elements. Finally, we get a sparse Metzler matrix $A^* = A_+ - \eta I$.

Some of theorems in paper were only mentioned without detail proof due to page limit.

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