AN ORDER HYPERRESOLUTION CALCULUS FOR GÖDEL LOGIC
General First-order Case

Dušan Guller
Department of Applied Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia

Keywords: Gödel Logic : Resolution : Many-valued Logics : Automated Deduction

Abstract: This paper addresses the deduction problem of a formula from a countable theory in the first-order Gödel logic from a perspective of automated deduction. Our approach is based on the translation of a formula to an equivalent satisfiable *CNF* one, which contains literals of the augmented form: either a or $a \rightarrow b$ or $(a \rightarrow b) \rightarrow b$ or $Qx\,c \rightarrow a$ or $a \rightarrow Qx\,c$ where a, c are atoms different from *0* (the false), *1* (the true); b is an atom different from *1*; $Q \in \{\forall, \exists\}$; x is a variable occurring in c. A *CNF* formula is further translated to an equivalent satisfiable finite order clausal theory, which consists of order clauses - finite sets of order literals of the form: either $a \eqcirc b$ or $Qx\,c \eqcirc a$ or $a \eqcirc Qx\,c$ or $a \prec b$ or $Qx\,c \prec a$ or $a \prec Qx\,c$ where a, b, c are atoms; $Q \in \{\forall, \exists\}$; x is a variable occurring in c. $\eqcirc$ and $\prec$ are interpreted by the equality and strict linear order on $[0,1]$, respectively. For an input theory, the proposed translation produces a so-called semantically admissible order clausal theory. An order hyperresolution calculus, operating on semantically admissible order clausal theories, is devised. The calculus is proved to be refutation sound and complete for the countable case.

1 INTRODUCTION

Concerning the three fundamental first-order fuzzy logics, the set of logically valid formulae is Π_2-complete for Łukasiewicz logic, Π_2-hard for Product logic, and Σ_1-complete for Gödel logic, as with classical first-order logic. Among these fuzzy logics, only Gödel logic is recursively axiomatisable. Hence, it is all important to provide a proof method suitable for automated deduction, as one has done for classical logic. In contrast to classical logic, we cannot make shifts of quantifiers arbitrarily and translate a formula to an equivalent (satisfiable) prenex form. In (Baaz et al., 2001; Baaz and Fermüller, 2010), the prenex fragment of Gödel logic in presence of the projection operator $\Delta : [0,1] \longrightarrow [0,1]$,

$$\Delta a = \begin{cases} 1 \; if \, a = 1, \\ 0 \; else, \end{cases}$$

is investigated, denoted as the prenex G^{Δ}_{∞}. (Baaz et al., 2001) solves the validity problem (*VAL*). A variant of Herbrand's Theorem for the prenex G^{Δ}_{∞} is proved, which reduces the *VAL* problem of a formula in the prenex G^{Δ}_{∞} to the *VAL* problem of an open formula in G^{Δ}_{∞}. Further, a meta-level logic of order clauses is defined, which is a fragment of classical one. An order clause is a finite set of inequalities of the form either $a < b$ or $a \leq b$ where $<, \leq$ are meta-level binary predicate symbols and a, b are atoms of G^{Δ}_{∞} considered as meta-level terms. The semantics of the meta-level logic of order clauses is given by classical interpretations on $[0,1]$, varying on assigned (truth) values to atoms of G^{Δ}_{∞} (meta-level terms), which are the strict dense linear order with endpoints on $[0,1]$; $<$ is interpreted as the strict dense linear order with endpoints and $\leq$ as its reflexive closure on $[0,1]$. A formula in the prenex G^{Δ}_{∞} is valid if and only if a translation of it to the order clause form is unsatisfiable with respect to the semantics of the meta-level logic. In the prenex G^{Δ}_{∞}, the problem of the unsatisfiability of a formula cannot straightforwardly be reduced to the *VAL* problem. Although the standard Skolemisation can be used for the reduction of the *VAL* problem to the open case, it does not preserve satisfiability. (Baaz and Fermüller, 2010) have shown that any conjunction of formulae can be translated to an equivalent satisfiable universal form via an alternative version of Skolemisation. The ordered chaining calculi (Bachmair and Ganzinger, 1998) may be used for resolution-style deduction over order clauses.

In the paper, we solve the deduction problem of a formula from a countable theory in Gödel logic. Our approach is based on the translation of a formula to

Partially supported by VEGA Grant 1/0979/12.

Guller D..
An Order Hyperresolution Calculus for Gödel Logic - General First-order Case.
DOI: 10.5220/0004104103290342
In *Proceedings of the 4th International Joint Conference on Computational Intelligence* (FCTA-2012), pages 329-342
ISBN: 978-989-8565-33-4

an equivalent satisfiable *CNF* one, which contains literals of the augmented form: either a or $a \rightarrow b$ or $(a \rightarrow b) \rightarrow b$ or $Qx\,c \rightarrow a$ or $a \rightarrow Qx\,c$ where a, c are atoms different from *0*, *1*; b is an atom different from *1*; $Q \in \{\forall, \exists\}$; x is a variable occurring in c; Lemma 3.1, Section 3. A *CNF* formula is further translated to an equivalent satisfiable finite order clausal theory, which consists of order clauses - finite sets of order literals of the form: either $a \eqcirc b$ or $Qx\,c \eqcirc a$ or $a \eqcirc Qx\,c$ or $a \prec b$ or $Qx\,c \prec a$ or $a \prec Qx\,c$ where a, b, c are atoms; $Q \in \{\forall, \exists\}$; x is a variable occurring in c; Lemma 3.1, Section 3. $\eqcirc$ and $\prec$ are interpreted by the equality and strict linear order on $[0,1]$, respectively. They are added to Gödel logic as new binary connectives. The translation is based on so-called interpolation rules given in Tables 2–4, Section 3. For an input theory, the translation produces a so-called semantically admissible order clausal theory, Section 4, Subsection 4.1. Corollary 4.1 states that for an input countable theory T and formula ϕ, there exists a countable semantically admissible order clausal theory S_T^ϕ such that $T \models \phi$ if and only if S_T^ϕ is unsatisfiable. In case of a finite T, $|S_T^\phi| \in O(|T|^2 + |\phi|^2)$ and the time as well as space complexity of the translation is in $O((|T|^2 + |\phi|^2) \cdot \log(|T| + |\phi|))$. An order hyperresolution calculus, operating on semantically admissible order clausal theories, uses order hyperresolution rules introduced in Tables 6 and 7, Section 4, Subsection 4.3. Most of the resolution rules of ordered chaining calculi (Bachmair and Ganzinger, 1998) (e.g. the factorised chaining rule) have non-empty residua in their consequences; i.e. they infer new (in)equalities. Many of them are only transitive consequences, unnecessary for refutational argument. We avoid this inefficiency using the hyperresolution principle; our rules do not infer new (in)equalities being transitive consequences, which confines search space considerably. The calculus is proved to be refutation sound and complete for the countable case, Theorem 4.4, Section 4, Subsection 4.3.

The paper is organised as follows. Section 2 concerns Gödel logic. Section 3 deals with the translation to order clausal form. Section 4 proposes the order hyperresolution calculus. Section 5 brings conclusions.

2 GÖDEL LOGIC

Throughout the paper, we shall use the common notions and notation of first-order logic. By $\mathcal{L}$ we denote a first-order language. $Var_\mathcal{L}$ | $Func_\mathcal{L}$ | $Pred_\mathcal{L}$ | $Term_\mathcal{L}$ | $GTerm_\mathcal{L}$ | $Atom_\mathcal{L}$ | $GAtom_\mathcal{L}$ denotes the set of all variables | function symbols | predicate symbols | terms | ground terms | atoms | ground atoms of $\mathcal{L}$. $ar_\mathcal{L} : Func_\mathcal{L} \cup Pred_\mathcal{L} \longrightarrow \mathbb{N}$ denotes the mapping assigning an arity to every function and predicate symbol. We assume nullary predicate symbols $0, 1 \in Pred_\mathcal{L}$, $ar_\mathcal{L}(0) = ar_\mathcal{L}(1) = 0$; *0* denotes the false and *1* the true in $\mathcal{L}$. By $Form_\mathcal{L}$ we designate the set of all formulae of $\mathcal{L}$ built up from $Atom_\mathcal{L}$ and $Var_\mathcal{L}$ using the connectives: $\neg$, negation, $\wedge$, conjunction, $\vee$, disjunction, $\rightarrow$, implication, and the quantifiers: $\forall$, the universal quantifier, $\exists$, the existential one. In addition, we introduce new binary connectives $\eqcirc$, equality, and $\prec$, strict order. By $OrdForm_\mathcal{L}$ we designate the set of all so-called order formulae of $\mathcal{L}$ built up from $Atom_\mathcal{L}$ and $Var_\mathcal{L}$ using the connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\eqcirc$, $\prec$, and the quantifiers: $\forall$, $\exists$.[1] In the paper, we shall assume that $\mathcal{L}$ is a countable first-order language; hence, all the above mentioned sets of symbols and expressions are countable. Let ε, ε_i, $1 \le i \le m$, υ_i, $1 \le i \le n$, be either an expression or a set of expressions or a set of sets of expressions, in general. By $vars(\varepsilon_1, \dots, \varepsilon_m) \subseteq Var_\mathcal{L}$ | $freevars(\varepsilon_1, \dots, \varepsilon_m) \subseteq Var_\mathcal{L}$ | $boundvars(\varepsilon_1, \dots, \varepsilon_m) \subseteq Var_\mathcal{L}$ | $preds(\varepsilon_1, \dots, \varepsilon_m) \subseteq Pred_\mathcal{L}$ | $atoms(\varepsilon_1, \dots, \varepsilon_m) \subseteq Atom_\mathcal{L}$ we denote the set of all variables | free variables | bound variables | predicate symbols | atoms of $\mathcal{L}$ occurring in $\varepsilon_1, \dots, \varepsilon_m$. ε is closed iff $freevars(\varepsilon) = \emptyset$. By ℓ we denote the empty sequence. By $|\varepsilon_1, \dots, \varepsilon_m| = m$ we denote the length of the sequence $\varepsilon_1, \dots, \varepsilon_m$. We define the concatenation of the sequences $\varepsilon_1, \dots, \varepsilon_m$ and $\upsilon_1, \dots, \upsilon_n$ as $(\varepsilon_1, \dots, \varepsilon_m), (\upsilon_1, \dots, \upsilon_n) = \varepsilon_1, \dots, \varepsilon_m, \upsilon_1, \dots, \upsilon_n$.

Let X, Y, Z be sets, $Z \subseteq X$; $f : X \longrightarrow Y$ be a mapping. By $\|X\|$ we denote the set-theoretic cardinality of X. X being a finite subset of Y is denoted as $X \subseteq_\mathcal{F} Y$. We designate $f[Z] = \{f(z) \,|\, z \in Z\}$; $f[Z]$ is the image of Z under f; and $f|_Z = \{(z, f(z)) \,|\, z \in Z\}$; $f|_Z$ is the restriction of f onto Z. Let $\gamma \le \omega$. A sequence δ of X is a bijection $\delta : \gamma \longrightarrow X$. X is countable if and only if there exists a sequence of X. Let X be a set of non-empty sets. A selector $\mathcal{S}$ over X is a mapping $\mathcal{S} : X \longrightarrow \bigcup X$ such that for all $x \in X$, $\mathcal{S}(x) \in x$. We denote $\mathcal{S}el(X) = \{\mathcal{S} \,|\, \mathcal{S} \textit{ is a selector over } X\}$. Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}_0^+$. f is of the order of g, in symbols $f \in O(g)$, iff there exist n_0 and $c^* \in \mathbb{R}_0^+$ such that for all $n \ge n_0$, $f(n) \le c^* \cdot g(n)$. Let $t \in Term_\mathcal{L}$, $\phi \in OrdForm_\mathcal{L}$, $T \subseteq_\mathcal{F} OrdForm_\mathcal{L}$. The size of t | ϕ, in symbols $|t| \in \mathbb{N}$ | $|\phi| \in \mathbb{N}$, is defined as the number of nodes of its standard tree representation. We define the size of T as $|T| = \sum_{\phi \in T} |\phi| \in \mathbb{N}$. By

[1] We assume a decreasing connective and quantifier precedence: $\forall$, $\exists$, $\neg$, $\wedge$, $\rightarrow$, $\eqcirc$, $\prec$, $\vee$.

$varseq(\phi)$, $vars(varseq(\phi)) \subseteq Var_{\mathcal{L}}$, we denote the sequence of all variables of $\mathcal{L}$ occurring in ϕ which is built up via the left-right preorder traversal of ϕ. For example, $varseq(\exists w\, (\forall x\, p(x,x,z) \vee \exists y\, q(x,y,z))) = w,x,x,x,z,y,x,y,z$ and $|w,x,x,x,z,y,x,y,z| = 9$. A sequence of variables will often be denoted as $\bar{x}$, $\bar{y}$, $\bar{z}$, etc. Let $Q \in \{\forall, \exists\}$ and $\bar{x} = x_1, \dots, x_n$ be a sequence of variables of $\mathcal{L}$. By $Q\bar{x}\phi$ we denote $Qx_1 \dots Qx_n\, \phi$.

Gödel logic is interpreted by the standard G-algebra augmented by binary operators $\boldsymbol{\eqcirc}$ and $\boldsymbol{\prec}$ for $\eqcirc$ and $\prec$, respectively.

$$G = ([0,1], \leq, \vee, \wedge, \Rightarrow, \overline{}, \boldsymbol{\eqcirc}, \boldsymbol{\prec}, 0, 1)$$

where $\vee$ | $\wedge$ denotes the supremum | infimum operator on $[0,1]$;

$$a \Rightarrow b = \begin{cases} 1 \text{ if } a \leq b, \\ b \text{ else}; \end{cases} \qquad \overline{a} = \begin{cases} 1 \text{ if } a = 0, \\ 0 \text{ else}; \end{cases}$$

$$a \boldsymbol{\eqcirc} b = \begin{cases} 1 \text{ if } a =_{[0,1]} b, \\ 0 \text{ else}; \end{cases} \qquad a \boldsymbol{\prec} b = \begin{cases} 1 \text{ if } a <_{[0,1]} b, \\ 0 \text{ else} \end{cases}$$

where $=_{[0,1]}$ | $<_{[0,1]}$ is the equality | strict order on $[0,1]$. We recall that G is a complete linearly ordered lattice algebra; the residuum operator $\Rightarrow$ of $\wedge$ satisfies the condition of residuation:

$$\text{for all } a,b,c \in G,\ a \wedge b \leq c \Longleftrightarrow a \leq b \Rightarrow c; \qquad (1)$$

Gödel negation $\overline{}$ satisfies the condition:

$$\text{for all } a \in G,\ \overline{a} = a \Rightarrow 0; \qquad (2)$$

the following properties, which will be exploited later, hold:[2]

for all $a,b,c \in G$,

$$a \vee b \wedge c = (a \vee b) \wedge (a \vee c), \quad \text{(distributivity of } \vee \text{ over } \wedge\text{)} \qquad (3)$$

$$a \wedge (b \vee c) = a \wedge b \vee a \wedge c, \quad \text{(distributivity of } \wedge \text{ over } \vee\text{)} \qquad (4)$$

$$a \Rightarrow (b \vee c) = a \Rightarrow b \vee a \Rightarrow c, \qquad (5)$$

$$a \Rightarrow b \wedge c = (a \Rightarrow b) \wedge (a \Rightarrow c), \qquad (6)$$

$$(a \vee b) \Rightarrow c = (a \Rightarrow c) \wedge (b \Rightarrow c), \qquad (7)$$

$$a \wedge b \Rightarrow c = a \Rightarrow c \vee b \Rightarrow c, \qquad (8)$$

$$a \Rightarrow (b \Rightarrow c) = a \wedge b \Rightarrow c, \qquad (9)$$

$$((a \Rightarrow b) \Rightarrow b) \Rightarrow b = a \Rightarrow b, \qquad (10)$$

$$(a \Rightarrow b) \Rightarrow c = ((a \Rightarrow b) \Rightarrow b) \wedge (b \Rightarrow c) \vee c, \qquad (11)$$

$$(a \Rightarrow b) \Rightarrow 0 = ((a \Rightarrow 0) \Rightarrow 0) \wedge (b \Rightarrow 0). \qquad (12)$$

[2]We assume a decreasing operator precedence: $\overline{}$, $\wedge$, $\Rightarrow$, $\boldsymbol{\eqcirc}$, $\boldsymbol{\prec}$, $\vee$.

An interpretation I for $\mathcal{L}$ is a triple

$$\left(\mathcal{U}_I, \{f^I \mid f \in Func_{\mathcal{L}}\}, \{p^I \mid p \in Pred_{\mathcal{L}}\}\right)$$

defined as usual. A variable assignment in I is a mapping $Var_{\mathcal{L}} \longrightarrow \mathcal{U}_I$. We denote the set of all variable assignments in I as $\mathcal{S}_I$. Let $t \in Term_{\mathcal{L}}$; $\bar{v}$ be a sequence of variables of $\mathcal{L}$; $\phi \in OrdForm_{\mathcal{L}}$; $e \in \mathcal{S}_I$. In I with respect to e, we define the value $\|t\|_e^I \in \mathcal{U}_I$ of t by recursion on the structure of t, the value $\|\bar{v}\|_e^I \in \mathcal{U}_I^{|\bar{v}|}$ of $\bar{v}$, the truth value $\|\phi\|_e^I \in [0,1]$ of ϕ by recursion on the structure of ϕ, as usual. A theory of $\mathcal{L}$ is a set of formulae of $\mathcal{L}$. An order theory of $\mathcal{L}$ is a set of order formulae of $\mathcal{L}$. Let $\phi, \phi' \in OrdForm_{\mathcal{L}}$ and $T, T' \subseteq OrdForm_{\mathcal{L}}$. ϕ is equivalent to ϕ', in symbols $\phi \equiv \phi'$, iff, for every interpretation I for $\mathcal{L}$ and $e \in \mathcal{S}_I$, $\|\phi\|_e^I = \|\phi'\|_e^I$. ϕ | T is equisatisfiable to ϕ' | T' iff ϕ | T is satisfiable if and only if ϕ' | T' is satisfiable.

3 TRANSLATION TO ORDER CLAUSAL FORM

At first, we introduce conjunctive normal form (*CNF*) in Gödel logic. In contrast to two-valued logic, we have to consider an augmented set of literals appearing in *CNF* formulae. Let $l, \phi \in Form_{\mathcal{L}}$. l is a literal of $\mathcal{L}$ iff either $l = a$ or $l = a \rightarrow b$ or $l = (a \rightarrow b) \rightarrow b$ or $l = Qx\,c \rightarrow a$ or $l = a \rightarrow Qx\,c$ where $a, c \in Atom_{\mathcal{L}} - \{0, 1\}$, $b \in Atom_{\mathcal{L}} - \{1\}$, $x \in vars(c)$. ϕ is a conjunctive | disjunctive normal form of $\mathcal{L}$, in symbols *CNF* | *DNF*, iff either $\phi = 0$ or $\phi = 1$ or $\phi = \bigwedge_{i \leq n} \bigvee_{j \leq m_i} l_j^i$ | $\phi = \bigvee_{i \leq n} \bigwedge_{j \leq m_i} l_j^i$ where l_j^i is a literal of $\mathcal{L}$. Let $D = l_0 \vee \dots \vee l_n \in Form_{\mathcal{L}}$, l_i is a literal of $\mathcal{L}$. We denote $lits(D) = \{l_0, \dots, l_n\} \subseteq Form_{\mathcal{L}}$. D is a factor iff, for all $i < i' \leq n$, $l_i \neq l_{i'}$. We now describe some generalisation of the translation in (Guller, 2010; Guller, 2012) to the first-order case. A similar approach exploiting the renaming subformulae technique can be found in (Plaisted and Greenbaum, 1986; de la Tour, 1992; Nonnengart et al., 1998; Sheridan, 2004). Let $l \in OrdForm_{\mathcal{L}}$. l is an order literal of $\mathcal{L}$ iff either $l = a \eqcirc b$ or $l = Qx\,c \eqcirc a$ or $l = a \eqcirc Qx\,c$ or $l = Qx\,c \eqcirc Q'y\,d$ or $l = a \prec b$ or $l = Qx\,c \prec a$ or $l = a \prec Qx\,c$ or $l = Qx\,c \prec Q'y\,d$ where $a, b, c, d \in Atom_{\mathcal{L}}$, $x \in vars(c)$, $y \in vars(d)$. An order clause of $\mathcal{L}$ is a finite set of order literals of $\mathcal{L}$; since $=_{[0,1]}$ is commutative, we identify the order literals $\varepsilon_1 \eqcirc \varepsilon_2$ and $\varepsilon_2 \eqcirc \varepsilon_1$ with respect to order clauses. An order clause $\{l_1, \dots, l_n\}$ is written in the form $l_1 \vee \dots \vee l_n$. The order clause $\emptyset$ is called the empty order clause and denoted as $\square$. An order clause $\{l\}$ is called a unit order clause and denoted as l; if it does not cause the ambiguity with the denotation of the single order literal l in given context. We designate the set of all order clauses of $\mathcal{L}$ as $OrdCl_{\mathcal{L}}$. Let

$l, l_0, \ldots, l_n$ be order literals of $\mathcal{L}$ and $C, C' \in OrdCl_{\mathcal{L}}$. We define the size of C as $|C| = \sum_{l \in C} |l| \in \mathbb{N}$. By $l_0 \vee \cdots \vee l_n \vee C$ we denote $\{l_0\} \cup \cdots \cup \{l_n\} \cup C$ where, for all $i, i' \leq n$, $i \neq i'$, $l_i \notin C$ and $l_i \neq l_{i'}$. By $C \vee C'$ we denote $C \cup C'$. C is a subclause of C', in symbols $C \sqsubseteq C'$, iff $C \subseteq C'$. An order clausal theory of $\mathcal{L}$ is a set of order clauses of $\mathcal{L}$. A unit order clausal theory is a set of unit order clauses. Let I be an interpretation for $\mathcal{L}$ and $e \in \mathcal{S}_I$. C is true in I with respect to e, written as $I \models_e C$, iff there exists $l^* \in C$ such that $I \models_e l^*$. I is a model of C, in symbols $I \models C$, iff, for all $e \in \mathcal{S}_I$, $I \models_e C$. Let $S, S' \subseteq OrdCl_{\mathcal{L}}$. I is a model of S, in symbols $I \models S$, iff, for all $C \in S$, $I \models C$. C is a logical consequence of S, in symbols $S \models C$, iff, for every model I of S for $\mathcal{L}$, $I \models C$. S' is a logical consequence of S, in symbols $S \models S'$, iff, for every model I of S for $\mathcal{L}$, $I \models S'$. $C \mid S$ is satisfiable iff there exists a model of $C \mid S$ for $\mathcal{L}$. Let $\phi, \phi' \in OrdForm_{\mathcal{L}}$ and $T, T' \subseteq OrdForm_{\mathcal{L}}$. $\phi \mid T \mid C \mid S$ is equisatisfiable to $\phi' \mid T' \mid C' \mid S'$ iff $\phi \mid T \mid C \mid S$ is satisfiable if and only if $\phi' \mid T' \mid C' \mid S'$ is satisfiable. Let $S \subseteq_{\mathcal{F}} OrdCl_{\mathcal{L}}$. We define the size of S as $|S| = \sum_{C \in S} |C| \in \mathbb{N}$. Let $\mathbb{I} = \mathbb{N} \times \mathbb{N}$; $\mathbb{I}$ is an infinite countable set of indices. Let $\tilde{P} = \{\tilde{p}_{\mathbb{i}} \mid \mathbb{i} \in \mathbb{I}\}$ such that $\tilde{P} \cap Pred_{\mathcal{L}} = \emptyset$; $\tilde{P}$ is an infinite countable set of new predicate symbols. From a computational point of view, the worst case time and space complexity will be estimated using the logarithmic cost measurement. Let $\mathcal{A}$ be an algorithm. $\#O \in \mathbb{N}$ denotes the number of all basic operations executed by $\mathcal{A}$. The translation to order clausal form is based on the following lemma.

Lemma 3.1. *Let $\phi \in Form_{\mathcal{L}}$; $T \subseteq Form_{\mathcal{L}}$ be countable; $F \subseteq \mathbb{I}$ such that there exists n_0 and $F \cap \{(i,j) \mid i \geq n_0\} = \emptyset$; $n_\phi \geq n_0$.*

(i) *There exist either $J_\phi = \emptyset$ or $J_\phi = \{(n_\phi, j) \mid j \leq n_{J_\phi}\}$, $J_\phi \subseteq_{\mathcal{F}} \mathbb{I}$, $J_\phi \cap F = \emptyset$; a CNF $\psi \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}}$; $S_\phi \subseteq_{\mathcal{F}} OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}}$ such that*
   - (a) *$\|J_\phi\| \leq 2 \cdot |\phi|$;*
   - (b) *either $J_\phi = S_\phi = \emptyset$, or $J_\phi = \emptyset$, $S_\phi = \{\square\}$, or $J_\phi \neq \emptyset$, $\square \notin S_\phi \neq \emptyset$;*
   - (c) *there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models \phi \in Form_{\mathcal{L}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}$ and $\mathfrak{A}' \models \psi \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}}$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;*
   - (d) *there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models \phi \in Form_{\mathcal{L}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}$ and $\mathfrak{A}' \models S_\phi \subseteq OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}}$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;*
   - (e) *$|\psi| \in O(|\phi|^2)$; the number of all basic operations of the translation of ϕ to ψ is in $O(|\phi|^2)$; the time and space complexity of the translation of ϕ to ψ is in $O(|\phi|^2 \cdot \log |\phi|)$;*
   - (f) *$|S_\phi| \in O(|\phi|^2)$; the number of all basic operations of the translation of ϕ to S_ϕ is in $O(|\phi|^2)$; the time and space complexity of the translation of ϕ to S_ϕ is in $O(|\phi|^2 \cdot \log |\phi|)$;*
   - (g) *if $\psi \neq 0$ and $\psi \neq 1$, then $\psi = \bigwedge_{i \leq n_\psi} D_i$, D_i is a factor; $J_\phi \neq \emptyset$; for all $i \leq n_\psi$, $\emptyset \neq preds(D_i) \cap \tilde{P} \subseteq \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}$; for all $i < i' \leq n_\psi$, $lits(D_i) \neq lits(D_{i'})$;*
   - (h) *if $S_\phi \neq \emptyset$ and $S_\phi \neq \{\square\}$, then $J_\phi \neq \emptyset$; for all $C \in S_\phi$, $\emptyset \neq preds(C) \cap \tilde{P} \subseteq \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_\phi\}$.*

(ii) *There exist $J_T \subseteq \mathbb{I}$, $J_T \cap F = \emptyset$, and $S_T \subseteq OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}}$ being countable such that*
   - (a) *either $J_T = S_T = \emptyset$, or $J_T = \emptyset$, $S_T = \{\square\}$, or $J_T \neq \emptyset$, $\square \notin S_T \neq \emptyset$;*
   - (b) *there exists an interpretation $\mathfrak{A}$ for $\mathcal{L}$ and $\mathfrak{A} \models T \subseteq Form_{\mathcal{L}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}$ and $\mathfrak{A}' \models S_T \subseteq OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}}$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L}}$;*
   - (c) *if $T \subseteq_{\mathcal{F}} Form_{\mathcal{L}}$, then $J_T \subseteq_{\mathcal{F}} \mathbb{I}$, $\|J_T\| \leq 2 \cdot |T|$; $S_T \subseteq_{\mathcal{F}} OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}}$, $|S_T| \in O(|T|^2)$; the number of all basic operations of the translation of T to S_T is in $O(|T|^2)$; the time and space complexity of the translation of T to S_T is in $O(|T|^2 \cdot \log(1 + |T|))$;*
   - (d) *if $S_T \neq \emptyset$ and $S_T \neq \{\square\}$, then $J_T \neq \emptyset$; for all $C \in S_T$, $\emptyset \neq preds(C) \cap \tilde{P} \subseteq \{\tilde{p}_{\mathbb{j}} \mid \mathbb{j} \in J_T\}$.*

Proof. Technical using interpolation. Let $l \in C \in S_\phi \mid S_T$. Then either $l = a \eqcirc b$ or $l = c \eqcirc a$ or $l = a \eqcirc c$ or $l = a \prec b$ or $l = c \prec a$ or $l = a \prec c$, $a, b \in atoms(S_\phi) \mid atoms(S_T)$, $c \in qatoms(S_\phi) \mid qatoms(S_T)$.

Let $\theta \in Form_{\mathcal{L}}$. There exists $\theta' \in Form_{\mathcal{L}}$ such that (13)

- (a) $\theta' \equiv \theta$;
- (b) $|\theta'| \leq 2 \cdot |\theta|$; θ' can be built up via a postorder traversal of θ with $\#O \in O(|\theta|)$, the time and space complexity in $O(|\theta| \cdot \log |\theta|)$;
- (c) θ' does not contain $\neg$;
- (d) either $\theta' = 0$, or 0 is a subformula of θ' if and only if 0 is a subformula of a subformula of θ' of the form $\vartheta \rightarrow 0$, $\vartheta \neq 0$;
- (e) either $\theta' = 1$ or 1 is not a subformula of θ'.

The proof is by induction on the structure of θ.

In Table 1, for every form of literal, an order clause is assigned so that for every interpretation $\mathfrak{A}$ for $\mathcal{L}$, for all $e \in \mathcal{S}_{\mathfrak{A}}$, $\mathfrak{A} \models_e l$ if and only if $\mathfrak{A} \models_e C$.

Table 1: Translation of l to C.

Case:	l	C	
1	a	$a \eqcirc 1$	$\lvert C\rvert \leq 3 \cdot \lvert l\rvert$
2	$a \to 0$	$a \eqcirc 0$	$\lvert C\rvert \leq 3 \cdot \lvert l\rvert$
3	$a \to b$	$a \prec b \vee a \eqcirc b$	$\lvert C\rvert \leq 3 \cdot \lvert l\rvert$
4	$(a \to 0) \to 0$	$0 \prec a$	$\lvert C\rvert \leq 3 \cdot \lvert l\rvert$
5	$(a \to b) \to b$	$b \prec a \vee b \eqcirc 1$	$\lvert C\rvert \leq 3 \cdot \lvert l\rvert$
6	$Qxc \to a$	$Qxc \prec a \vee Qxc \eqcirc a$	$\lvert C\rvert \leq 3 \cdot \lvert l\rvert$
7	$a \to Qxc$	$a \prec Qxc \vee a \eqcirc Qxc$	$\lvert C\rvert \leq 3 \cdot \lvert l\rvert$

$a, b, c \in Atom_{\mathcal{L}} - \{0, 1\}$, $x \in vars(c)$.

Let $\theta \in Form_{\mathcal{L}} - \{0, 1\}$; (13c–e) hold for θ; $\bar{x}$ be a sequence of variables of $\mathcal{L}$, $vars(\bar{x}) \supseteq vars(\theta)$; $G \subseteq \mathbb{I}$ such that there exists n_1 and $G \cap \{(i, j) \,|\, i \geq n_1\} = \emptyset$; $n_\theta \geq n_1$; $\mathbb{i} = (n_\theta, j_{\mathbb{i}}) \in \mathbb{I}$, $\tilde{p}_{\mathbb{i}} \in \tilde{P}$, $ar(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$, $\{\mathbb{i}\} \cap G \subseteq \{(i, j) \,|\, i \geq n_1\} \cap G = \emptyset$. There exist $n_J \geq j_{\mathbb{i}}$, $J = \{(n_\theta, j) \,|\, j_{\mathbb{i}} + 1 \leq j \leq n_J\} \subseteq_{\mathcal{F}} \mathbb{I}$, $J \cap (G \cup \{\mathbb{i}\}) = \emptyset$; a *CNF* $\psi^s \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$, $S^s \subseteq_{\mathcal{F}} OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$, $s = +, -$, such that for both s, (14)

(a) $\|J\| \leq |\theta| - 1$;

(b) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \to \theta \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models \psi^+ \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$;

(c) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \theta \to \tilde{p}_{\mathbb{i}}(\bar{x}) \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models \psi^- \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$;

(d) for every interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$, $\mathfrak{A} \models \psi^s \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$ if and only if $\mathfrak{A} \models S^s \subseteq OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$;

(e) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \tilde{p}_{\mathbb{i}}(\bar{x}) \to \theta \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models S^+ \subseteq OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$;

(f) there exists an interpretation $\mathfrak{A}$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}$ and $\mathfrak{A} \models \theta \to \tilde{p}_{\mathbb{i}}(\bar{x}) \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$ if and only if there exists an interpretation $\mathfrak{A}'$ for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$ and $\mathfrak{A}' \models S^- \subseteq OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$, satisfying $\mathfrak{A} = \mathfrak{A}'|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\}}$;

Table 3: Unary interpolation rules for $\to$.

Case:		Laws	
$\theta = \theta_1 \to 0$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to (\theta_1 \to 0)}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \to 0 \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to 0) \wedge (\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x}))}$	(9), (8)	(27)
$\lvert\text{Consequent}\rvert = 8 + 2 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \to (\theta_1 \to 0)}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc 0 \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc 0, \theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\}}$		(28)
$\lvert\text{Consequent}\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \to 0) \to \tilde{p}_{\mathbb{i}}(\bar{x})}{((\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to 0) \to 0 \vee \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1)}$	(11)	(29)
$\lvert\text{Consequent}\rvert = 8 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \to 0) \to \tilde{p}_{\mathbb{i}}(\bar{x})}{\{0 \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc 1, \tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\}}$		(30)
$\lvert\text{Consequent}\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \to \theta_1\rvert$			

(g) $|\psi^s| \leq 13 \cdot |\theta| \cdot (1 + |\bar{x}|)$, ψ^s can be built up from θ and $\bar{x}$ via a preorder traversal of θ with $\#O \in O(|\theta| \cdot (1 + |\bar{x}|))$;

(h) $|S^s| \leq 15 \cdot |\theta| \cdot (1 + |\bar{x}|)$, S^s can be built up from θ and $\bar{x}$ via a preorder traversal of θ with $\#O \in O(|\theta| \cdot (1 + |\bar{x}|))$;

(i) $\psi^s = \bigwedge_{i \leq n_{\psi^s}} D_i^s$, $D_i^s \neq \tilde{p}_{\mathbb{i}}(\bar{x})$ is a factor; for all $i \leq n_{\psi^s}$, $\emptyset \neq preds(D_i^s) \cap \tilde{P} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$; for all $i < i' \leq n_{\psi^s}$, $lits(D_i^s) \neq lits(D_{i'}^s)$;

(j) for all $C \in S^s$, $\emptyset \neq preds(C) \cap \tilde{P} \subseteq \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$; $\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc 1, \tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1 \notin S^s$.

The proof is by induction on the structure of θ using the interpolation rules in Tables 2–4.

(i) By (13) for $\phi \in Form_{\mathcal{L}}$, there exists $\phi' \in Form_{\mathcal{L}}$ such that (13a–e) hold for ϕ'. We then distinguish three cases for ϕ'. Case 1: $\phi' = 0$. We put $J_\phi = \emptyset \subseteq_{\mathcal{F}} \mathbb{I}$, $J_\phi \cap F = \emptyset$; $\psi = 0 \in Form_{\mathcal{L}}$; $S_\phi = \{\square\} \subseteq_{\mathcal{F}} OrdCl_{\mathcal{L}}$. Case 2: $\phi' = 1$. We put $J_\phi = \emptyset \subseteq_{\mathcal{F}} \mathbb{I}$, $J_\phi \cap F = \emptyset$; $\psi = 1 \in Form_{\mathcal{L}}$; $S_\phi = \emptyset \subseteq_{\mathcal{F}} OrdCl_{\mathcal{L}}$. Case 3: $\phi' \neq 0$ and $\phi' \neq 1$. Let $\bar{x} = varseq(\phi')$. Let $\mathbb{i} = (n_\phi, 0) \in \mathbb{I}$, $\tilde{p}_{\mathbb{i}} \in \tilde{P}$, $ar(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$. We get by (14) for ϕ', $\bar{x}$, F, n_0, n_ϕ, $\mathbb{i}$, $\tilde{p}_{\mathbb{i}}$ that there exist n_{J^+}, $J^+ = \{(n_\phi, j) \,|\, 1 \leq j \leq n_{J^+}\} \subseteq_{\mathcal{F}} \mathbb{I}$, $J^+ \cap (F \cup \{\mathbb{i}\}) = \emptyset$; a *CNF* $\psi^+ \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J^+\}}$; $S^+ \subseteq_{\mathcal{F}} OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{i}}\} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J^+\}}$; and (14a,b,e,g–j) hold for ϕ', $\bar{x}$, $\tilde{p}_{\mathbb{i}}$, J^+, ψ^+, S^+. We put $n_{J_\phi} = n_{J^+}$, $J_\phi = \{\mathbb{i}\} \cup J^+ \subseteq_{\mathcal{F}} \mathbb{I}$, $J_\phi \cap F = \emptyset$; $\psi = \tilde{p}_{\mathbb{i}}(\bar{x}) \wedge \psi^+ \in Form_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}}$; $S_\phi = \{\tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc 1\} \cup S^+ \subseteq_{\mathcal{F}} OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_\phi\}}$. (ii) straightforwardly follows from (i). $\square$

Table 2: Binary interpolation rules for $\wedge$, $\vee$, $\rightarrow$.

Case:		Laws	
$\theta = \theta_1 \wedge \theta_2$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta_1 \wedge \theta_2}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2)}$	(6)	(15)
$\lvert\text{Consequent}\rvert = 9 + 4 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \theta_1 \wedge \theta_2}{\left\{\begin{array}{l}\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}),\\ \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\end{array}\right\}}$		(16)
$\lvert\text{Consequent}\rvert = 12 + 8 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert$			
Negative interpolation	$\dfrac{\theta_1 \wedge \theta_2 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{(\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x}))}$	(8)	(17)
$\lvert\text{Consequent}\rvert = 9 + 4 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert$			
Negative interpolation	$\dfrac{\theta_1 \wedge \theta_2 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{\left\{\begin{array}{l}\tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}),\\ \theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x}), \theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\end{array}\right\}}$		(18)
$\lvert\text{Consequent}\rvert = 12 + 8 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert$			
$\theta = \theta_1 \vee \theta_2$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow (\theta_1 \vee \theta_2)}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2)}$	(5)	(19)
$\lvert\text{Consequent}\rvert = 9 + 4 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow (\theta_1 \vee \theta_2)}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\}}$		(20)
$\lvert\text{Consequent}\rvert = 12 + 8 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \vee \theta_2) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{(\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x}))}$	(7)	(21)
$\lvert\text{Consequent}\rvert = 9 + 4 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \vee \theta_2) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{\{\tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x}), \theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\}}$		(22)
$\lvert\text{Consequent}\rvert = 12 + 8 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert$			
$\theta = \theta_1 \rightarrow \theta_2, \theta_2 \neq \mathit{0}$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow (\theta_1 \rightarrow \theta_2)}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})) \wedge (\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2)}$	(9), (8)	(23)
$\lvert\text{Consequent}\rvert = 9 + 4 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert$			
Positive interpolation	$\dfrac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow (\theta_1 \rightarrow \theta_2)}{\left\{\begin{array}{l}\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}_2}(\bar{x}),\\ \theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\end{array}\right\}}$		(24)
$\lvert\text{Consequent}\rvert = 12 + 8 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\rvert + \lvert\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \theta_2\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \rightarrow \theta_2) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{\begin{array}{l}((\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})) \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_2}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge\\ (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1) \wedge (\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x}))\end{array}}$	(11), (3), (1)	(25)
$\lvert\text{Consequent}\rvert = 13 + 6 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert$			
Negative interpolation	$\dfrac{(\theta_1 \rightarrow \theta_2) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{\left\{\begin{array}{l}\tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \mathit{1} \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \mathit{1}, \tilde{p}_{\mathbb{i}_2}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \tilde{p}_{\mathbb{i}_2}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}),\\ \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1, \theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\end{array}\right\}}$		(26)
$\lvert\text{Consequent}\rvert = 15 + 8 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\rvert + \lvert\theta_2 \rightarrow \tilde{p}_{\mathbb{i}_2}(\bar{x})\rvert$			

Table 4: Unary interpolation rules for $\forall$ and $\exists$.

Case:

$\forall x\theta_1$

Positive interpolation
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \forall x\theta_1}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1)} \tag{31}$$

$$|\text{Consequent}| = 6+2\cdot|\bar{x}|+|\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1| \le 13\cdot(1+|\bar{x}|)+|\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1|$$

Positive interpolation
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \forall x\theta_1}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\}} \tag{32}$$

$$|\text{Consequent}| = 10+4\cdot|\bar{x}|+|\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1| \le 15\cdot(1+|\bar{x}|)+|\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1|$$

Negative interpolation
$$\frac{\forall x\theta_1 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{(\forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x}))} \tag{33}$$

$$|\text{Consequent}| = 6+2\cdot|\bar{x}|+|\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})| \le 13\cdot(1+|\bar{x}|)+|\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})|$$

Negative interpolation
$$\frac{\forall x\theta_1 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{\{\forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \forall x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\}} \tag{34}$$

$$|\text{Consequent}| = 10+4\cdot|\bar{x}|+|\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})| \le 15\cdot(1+|\bar{x}|)+|\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})|$$

$\exists x\theta_1$

Positive interpolation
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \exists x\theta_1}{(\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1)} \tag{35}$$

$$|\text{Consequent}| = 6+2\cdot|\bar{x}|+|\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1| \le 13\cdot(1+|\bar{x}|)+|\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1|$$

Positive interpolation
$$\frac{\tilde{p}_{\mathbb{i}}(\bar{x}) \rightarrow \exists x\theta_1}{\{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec \exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \vee \tilde{p}_{\mathbb{i}}(\bar{x}) \eqcirc \exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}), \tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1\}} \tag{36}$$

$$|\text{Consequent}| = 10+4\cdot|\bar{x}|+|\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1| \le 15\cdot(1+|\bar{x}|)+|\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \theta_1|$$

Negative interpolation
$$\frac{\exists x\theta_1 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{(\exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})) \wedge (\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x}))} \tag{37}$$

$$|\text{Consequent}| = 6+2\cdot|\bar{x}|+|\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})| \le 13\cdot(1+|\bar{x}|)+|\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})|$$

Negative interpolation
$$\frac{\exists x\theta_1 \rightarrow \tilde{p}_{\mathbb{i}}(\bar{x})}{\{\exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \prec \tilde{p}_{\mathbb{i}}(\bar{x}) \vee \exists x\,\tilde{p}_{\mathbb{i}_1}(\bar{x}) \eqcirc \tilde{p}_{\mathbb{i}}(\bar{x}), \theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})\}} \tag{38}$$

$$|\text{Consequent}| = 10+4\cdot|\bar{x}|+|\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})| \le 15\cdot(1+|\bar{x}|)+|\theta_1 \rightarrow \tilde{p}_{\mathbb{i}_1}(\bar{x})|$$

4 HYPERRESOLUTION OVER ORDER CLAUSES

4.1 Restrictions on Order Clauses

The described translation produces order clausal theories in some restrictive form, which will be utilised in devising an order hyperresolution calculus. Let $\pi \in \mathit{Form}_{\mathcal{L}}$. π is a quantified atom of $\mathcal{L}$ iff $\pi = Qx\,p(t_0,\dots,t_\tau)$ where $p(t_0,\dots,t_\tau) \in \mathit{Atom}_{\mathcal{L}}$, $x \in \mathit{vars}(p(t_0,\dots,t_\tau))$, either $t_i = x$ or $x \not\in \mathit{vars}(t_i)$. $\mathit{QAtom}_{\mathcal{L}} \subseteq \mathit{Form}_{\mathcal{L}}$ denotes the set of all quantified atoms of $\mathcal{L}$. $\mathit{QAtom}_{\mathcal{L}}^{Q} \subseteq \mathit{QAtom}_{\mathcal{L}}$, $Q \in \{\forall,\exists\}$, denotes the set of all quantified atoms of $\mathcal{L}$ of the form $Qx\,a$. Let ε_i, $1 \le i \le n$, be either an expression or a set of expressions or a set of sets of expressions, in general. By $\mathit{qatoms}(\varepsilon_1,\dots,\varepsilon_n) \subseteq \mathit{QAtom}_{\mathcal{L}}$ | $\mathit{qatoms}^{Q}(\varepsilon_1,\dots,\varepsilon_n) \subseteq \mathit{QAtom}_{\mathcal{L}}^{Q}$ we denote the set of all quantified atoms | quantified atoms of the form $Qx\,a$ of $\mathcal{L}$ occurring in $\varepsilon_1,\dots,\varepsilon_n$. Let $Qx\,p(t_0,\dots,t_\tau) \in \mathit{QAtom}_{\mathcal{L}}$, $x \in \mathit{vars}(p(t_0,\dots,t_\tau))$, and $p(t'_0,\dots,t'_\tau) \in \mathit{Atom}_{\mathcal{L}}$. We denote

$$p(t_0,\dots,t_\tau)[i] = t_i, i \le \tau,$$
$$\mathit{boundindset}(Qx\,p(t_0,\dots,t_\tau)) = \{i \,|\, i \le \tau, t_i = x\} \neq \emptyset.$$

Let $I = \{i \,|\, i \le \tau, x \not\in \mathit{vars}(t_i)\}$; and $r_1,\dots,r_k$, $r_i \le \tau$, $k \le \tau$, for all $1 \le i < i' \le k$, $r_i < r_{i'}$, be a sequence such that $\{r_i \,|\, 1 \le i \le k\} = I$. We denote

$$\mathit{freetermseq}(Qx\,p(t_0,\dots,t_\tau)) = t_{r_1},\dots,t_{r_k}, t_{r_i} \in \mathit{Term}_{\mathcal{L}},$$
$$\mathit{freetermseq}(p(t'_0,\dots,t'_\tau)) = t'_0,\dots,t'_\tau, t'_i \in \mathit{Term}_{\mathcal{L}},$$
$$\mathit{freetermseq}(p(t'_0,\dots,t'_\tau)/Qx\,p(t_0,\dots,t_\tau)) = t'_{r_1},\dots,t'_{r_k}, t'_{r_i} \in \mathit{Term}_{\mathcal{L}}.$$

Let l be an order literal of $\mathcal{L}$. l is admissible iff $l = a \diamond b$, $a, b \in \mathit{Atom}_{\mathcal{L}} \cup \mathit{QAtom}_{\mathcal{L}}$. Let $C \in \mathit{OrdCl}_{\mathcal{L}}$. C is admissible iff, for all $l \in C$, l is admissible. Let $S \subseteq \mathit{OrdCl}_{\mathcal{L}}$. S is admissible iff, for all $C \in S$, C is admissible. Let $J \subseteq \mathbb{I}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$. S is semantically admissible iff

(a) S is admissible;

(b) for all $a \in \mathit{qatoms}(S)$, there exists $\mathbb{j}^* \in J$ such that $\mathit{preds}(a) = \{\tilde{p}_{\mathbb{j}^*}\}$;

(c) for all $Qx\,a, Q'x'\,a' \in \mathit{qatoms}(S)$, if $\mathit{preds}(a) = \mathit{preds}(a')$, then $Q = Q'$, $x = x'$, $\mathit{boundindset}(Qx\,a) = \mathit{boundindset}(Q'x'\,a')$.

Corollary 4.1. *Let $T \subseteq \mathit{Form}_{\mathcal{L}}$ be countable; $\phi \in \mathit{Form}_{\mathcal{L}}$; $F \subseteq \mathbb{I}$ such that there exists n_0 and $F \cap \{(i,j) \,|\, i \ge n_0\} = \emptyset$. There exist $J_T^{\phi} \subseteq \mathbb{I}$, $J_T^{\phi} \cap F = \emptyset$, and $S_T^{\phi} \subseteq \mathit{OrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^{\phi}\}}$ being countable such that*

(i) *$T \models \phi$ if and only if S_T^{ϕ} is unsatisfiable;*

(ii) *if $T \subseteq_{\mathcal{F}} \mathit{Form}_{\mathcal{L}}$, then $J_T^{\phi} \subseteq_{\mathcal{F}} \mathbb{I}$, $\|J_T^{\phi}\| \in O(|T|+|\phi|)$; $S_T^{\phi} \subseteq_{\mathcal{F}} \mathit{OrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T^{\phi}\}}$, $|S_T^{\phi}| \in O(|T|^2+|\phi|^2)$; the number of all basic operations of the translation of T and ϕ to S_T^{ϕ} is in $O(|T|^2+|\phi|^2)$; the time and space complexity of the translation of T and ϕ to S_T^{ϕ} is in $O((|T|^2+|\phi|^2)\cdot\log(|T|+|\phi|))$;*

(iii) *S_T^{ϕ} is semantically admissible.*

Proof. (i) We put $J_{n_0} = \{(n_0, j) \,|\, j \in \mathbb{N}\} \subseteq \mathbb{I}$ and $G = F \cup J_{n_0} \subseteq \mathbb{I}$. We get by Lemma 3.1(ii) for T, G, n_0+1 that there exist $J_T \subseteq \mathbb{I}$, $J_T \cap G = \emptyset$; $S_T \subseteq \mathit{OrdCl}_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J_T\}}$ being countable; and 3.1(ii a–d) hold for T, J_T, S_T. By (13) for $\phi \in Form_{\mathcal{L}}$, there exists $\phi' \in Form_{\mathcal{L}}$ such that (13a–e) hold for ϕ'. We then distinguish three cases for ϕ'. Case 1: $\phi' = 0$. We put $J_T^{\phi} = J_T \subseteq \mathbb{I}$, $J_T^{\phi} \cap F = \emptyset$, and $S_T^{\phi} = S_T \subseteq OrdCl_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J_T^{\phi}\}}$ being countable. Case 2: $\phi' = 1$. We put $J_T^{\phi} = \emptyset \subseteq \mathbb{I}$, $J_T^{\phi} \cap F = \emptyset$, and $S_T^{\phi} = \{\square\} \subseteq OrdCl_{\mathcal{L}}$ being countable. Case 3: $\phi' \neq 0$ and $\phi' \neq 1$. Let $\bar{x} = varseq(\phi')$. Let $\mathbb{i} = (n_0, 0) \in \mathbb{I}$, $\tilde{p}_{\mathbb{i}} \in \tilde{P}$, $ar(\tilde{p}_{\mathbb{i}}) = |\bar{x}|$. We get by (14) for $\forall\bar{x}\phi'$, $\bar{x}$, F, n_0, n_0, $\mathbb{i}$, $\tilde{p}_{\mathbb{i}}$ that there exist n_{J^-}, $J^- = \{(n_0, j)\,|\,1 \leq j \leq n_{J^-}\} \subseteq_{\mathcal{F}} \mathbb{I}$, $J^- \cap (F \cup \{\mathbb{i}\}) = \emptyset$; $S^- \subseteq_{\mathcal{F}} OrdCl_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{i}}\}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J^-\}}$; and (14a,f,h,j) hold for $\forall\bar{x}\phi'$, $\bar{x}$, $\tilde{p}_{\mathbb{i}}$, J^-, S^-. We put $J_T^{\phi} = J_T \cup \{\mathbb{i}\} \cup J^- \subseteq \mathbb{I}$, $J_T^{\phi} \cap F = \emptyset$, and $S_T^{\phi} = S_T \cup \{\tilde{p}_{\mathbb{i}}(\bar{x}) \prec 1\} \cup S^- \subseteq OrdCl_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J_T^{\phi}\}}$ being countable. (ii) and (iii) straightforwardly follow from the translation via interpolation. Let $l \in C \in S_T^{\phi}$. Then either $l = a \eqcirc b$ or $l = c \eqcirc a$ or $l = a \eqcirc c$ or $l = a \prec b$ or $l = c \prec a$ or $l = a \prec c$, $a, b \in atoms(S_T^{\phi})$, $c \in qatoms(S_T^{\phi})$. $\square$

4.2 Substitutions

We assume the reader to be familiar with the standard notions of substitutions. Let $X = \{x_1, \dots, x_n\} \subseteq_{\mathcal{F}} Var_{\mathcal{L}}$. A substitution ϑ of $\mathcal{L}$ is a mapping $\vartheta : X \longrightarrow Term_{\mathcal{L}}$. ϑ may be written in the form $x_1/\vartheta(x_1), \dots, x_n/\vartheta(x_n)$. We denote $dom(\vartheta) = X \subseteq_{\mathcal{F}} Var_{\mathcal{L}}$ and $range(\vartheta) = \bigcup_{x\in X} vars(\vartheta(x)) \subseteq_{\mathcal{F}} Var_{\mathcal{L}}$. The set of all substitutions of $\mathcal{L}$ is designated as $Subst_{\mathcal{L}}$. We define $id_{\mathcal{L}} : Var_{\mathcal{L}} \longrightarrow Var_{\mathcal{L}}$, $id_{\mathcal{L}}(x) = x$. Let $\vartheta \in Subst_{\mathcal{L}}$. Let $Qx\,a \in QAtom_{\mathcal{L}}$, $x \in vars(a)$. ϑ is applicable to $Qx\,a$ iff $dom(\vartheta) \supseteq freevars(Qx\,a)$ and $x \notin range(\vartheta|_{freevars(Qx\,a)})$. We define the application of ϑ to $Qx\,a$ as $(Qx\,a)\vartheta = Qx\,a(\vartheta|_{freevars(Qx\,a)} \cup x/x) \in QAtom_{\mathcal{L}}$. Let $\varepsilon_1 \diamond \varepsilon_2$ be an admissible order literal of $\mathcal{L}$. We define the application of ϑ to $\varepsilon_1 \diamond \varepsilon_2$ as $(\varepsilon_1 \diamond \varepsilon_2)\vartheta = \varepsilon_1\vartheta \diamond \varepsilon_2\vartheta$ being an admissible order literal of $\mathcal{L}$. Let ε, ε' be either expressions or sets of expressions of $\mathcal{L}$, in this context. ε' is an instance of ε of $\mathcal{L}$ iff there exists $\vartheta \in Subst_{\mathcal{L}}$ such that $\varepsilon' = \varepsilon\vartheta$. ε' is a variant of ε of $\mathcal{L}$ iff there exists a variable renaming $\rho \in Subst_{\mathcal{L}}$ such that $\varepsilon' = \varepsilon\rho$. Let $C \in OrdCl_{\mathcal{L}}$ be admissible and $S \subseteq OrdCl_{\mathcal{L}}$ be admissible. C is an instance | a variant of S of $\mathcal{L}$ iff there exists $C^* \in S$ such that C is an instance | a variant of C^* of $\mathcal{L}$. We denote $Inst_{\mathcal{L}}(S) = \{C\,|\,C \text{ is an instance of } S \text{ of } \mathcal{L}\} \subseteq OrdCl_{\mathcal{L}}$. Let $\overline{E} = E_0, \dots, E_n$, E_i is a set of expressions of $\mathcal{L}$, in this context. We define the application of ϑ to $\overline{E}$ as $\overline{E}\vartheta = E_0\vartheta, \dots, E_n\vartheta$. ϑ is a unifier of $\mathcal{L}$ for $\overline{E}$ iff, for all $i \leq n$, ϑ is a unifier of $\mathcal{L}$ for E_i. θ is a most general unifier of $\mathcal{L}$ for $\overline{E}$ iff θ is a unifier of $\mathcal{L}$ for $\overline{E}$, and for every unifier ϑ of $\mathcal{L}$ for $\overline{E}$, there exists $\gamma \in Subst_{\mathcal{L}}$ such that $\vartheta|_{freevars(\overline{E})} = \theta|_{freevars(\overline{E})} \circ \gamma$. By $mgu_{\mathcal{L}}(\overline{E}) \subseteq Subst_{\mathcal{L}}$ we denote the set of all most general unifiers of $\mathcal{L}$ for $\overline{E}$.

Theorem 4.2 (Extended Unification Theorem)**.** *Let $\overline{E} = E_0, \dots, E_n$, either $E_i \subseteq_{\mathcal{F}} Term_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} Atom_{\mathcal{L}}$ or $E_i \subseteq_{\mathcal{F}} QAtom_{\mathcal{L}}$ or $E_i \in OrdCl_{\mathcal{L}}$ is admissible. If there exists a unifier of $\mathcal{L}$ for $\overline{E}$, then there exists a most general unifier θ of $\mathcal{L}$ for $\overline{E}$ such that $range(\theta|_{freevars(\overline{E})}) \cap boundvars(\overline{E}) = \emptyset$.*

Proof. Standard. $\square$

4.3 Order Hyperresolution Rules

At first, we introduce some basic notions and notation concerning chains of admissible order literals. Let $\varepsilon_1, \varepsilon_2 \in Atom_{\mathcal{L}} \cup QAtom_{\mathcal{L}}$. $\varepsilon_1 \trianglelefteq \varepsilon_2$ iff either $\varepsilon_1 = \varepsilon_2$ or $\varepsilon_1 = 0$ or $\varepsilon_2 = 1$; or $\varepsilon_1 = \forall x\,a$, $x \in vars(a)$, there exists $t \in Term_{\mathcal{L}}$ and $\varepsilon_2 = a(x/t \cup id_{\mathcal{L}}|_{vars(a)-\{x\}})$; or $\varepsilon_2 = \exists x\,a$, $x \in vars(a)$, there exists $t \in Term_{\mathcal{L}}$ and $\varepsilon_1 = a(x/t \cup id_{\mathcal{L}}|_{vars(a)-\{x\}})$.

Let $J \subseteq \mathbb{I}$ and $S \subseteq OrdCl_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J\}}$ be semantically admissible. For all $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in Atom_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J\}} \cup qatoms(S)$, if $\varepsilon_1 \trianglelefteq \varepsilon_2 \trianglelefteq \varepsilon_3$, then $\varepsilon_1 \trianglelefteq \varepsilon_3$. (39)

The proof. A straightforward consequence of the semantical admissibility of S.

Let $\overline{R_i} = t_1^i, \dots, t_m^i$, $t_j^i \in Term_{\mathcal{L}}$, $i = 1, 2$. We define the union of $\overline{R_1}$ and $\overline{R_2}$ as

$$\overline{R_1} \cup \overline{R_2} = \{t_1^1, t_1^2\}, \dots, \{t_m^1, t_m^2\}, \\ \{t_j^1, t_j^2\} \subseteq_{\mathcal{F}} Term_{\mathcal{L}}.$$

Note that if $m = 0$, then $\overline{R_1} \cup \overline{R_2} = \ell$. Let $J \subseteq \mathbb{I}$ and $S \subseteq OrdCl_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J\}}$ be semantically admissible. Let $\varepsilon_i \in Atom_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J\}} \cup qatoms(S)$, $i = 1, 2$. We define the sequence $\varepsilon_1 \boldsymbol{\trianglelefteq} \varepsilon_2$ of the form either $\emptyset \neq A \subseteq_{\mathcal{F}} Atom_{\mathcal{L}\cup\{\tilde{p}_{\mathbb{j}}\,|\,\mathbb{j}\in J\}}$ or $\emptyset \neq A \subseteq_{\mathcal{F}} qatoms(S)$ or $R_1, \dots, R_n$, $R_i = \{t_1^i, t_2^i\} \subseteq_{\mathcal{F}} Term_{\mathcal{L}}$, in Table 5.

A chain Ξ of $\mathcal{L}$ is a sequence $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$; $\varepsilon_i \diamond_i \upsilon_i$ is an admissible order literal of $\mathcal{L}$. $\varepsilon_0 \in Atom_{\mathcal{L}} \cup QAtom_{\mathcal{L}}$ is the beginning element of Ξ and $\upsilon_n \in Atom_{\mathcal{L}} \cup QAtom_{\mathcal{L}}$ the ending element of Ξ. $\varepsilon_0 \,\Xi\, \upsilon_n$ denotes Ξ together with its respective beginning and ending element. Let $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ be a chain of $\mathcal{L}$. Ξ is an equality chain of $\mathcal{L}$ iff, for all $i \leq n$, $\diamond_i = \eqcirc$, and for all $i < n$, $\upsilon_i = \varepsilon_{i+1}$. Ξ is an increasing chain of $\mathcal{L}$ iff, for all $i < n$, $\upsilon_i \trianglelefteq \varepsilon_{i+1}$. Let $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ be an increasing chain of $\mathcal{L}$. Ξ is a strictly increasing chain of $\mathcal{L}$ iff there exists $i^* \leq n$ such that $\diamond_{i^*} = \prec$. Ξ is an unstrictly increasing chain of $\mathcal{L}$ iff, for all $i \leq n$, $\diamond_i = \eqcirc$. Let Ξ be a chain of $\mathcal{L}$. Ξ is a contradiction of $\mathcal{L}$ iff $\varepsilon\,\Xi\,\upsilon$ is a strictly increasing chain of $\mathcal{L}$ and $\upsilon \trianglelefteq \varepsilon$. Let $S \subseteq OrdCl_{\mathcal{L}}$ be admissible, unit and $\Xi = \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ be a chain |

Table 5: $\varepsilon_1 \trianglelefteq \varepsilon_2$.

$$\varepsilon_1 \trianglelefteq \varepsilon_2 = \begin{cases} \ell & \mathit{if}\ \varepsilon_1 = \mathit{0}; \\ \ell & \mathit{if}\ \varepsilon_2 = \mathit{1}; \\ \{\varepsilon_1, \varepsilon_2\} & \mathit{if\ either}\ \varepsilon_1 \neq \mathit{0}, \varepsilon_2 \neq \mathit{1}, \varepsilon_1, \varepsilon_2 \in \mathit{Atom}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}},\ \mathit{or}\ \varepsilon_1, \varepsilon_2 \in \mathit{qatoms}(S); \\ \mathit{freetermseq}(\varepsilon_1) \cup \mathit{freetermseq}(\varepsilon_2/\varepsilon_1) & \mathit{if}\ \varepsilon_1 \in \mathit{qatoms}(S)^{\forall}, \varepsilon_2 \in \mathit{Atom}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}, \mathit{preds}(\varepsilon_1) = \mathit{preds}(\varepsilon_2); \\ \mathit{freetermseq}(\varepsilon_1/\varepsilon_2) \cup \mathit{freetermseq}(\varepsilon_2) & \mathit{if}\ \varepsilon_1 \in \mathit{Atom}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}, \varepsilon_2 \in \mathit{qatoms}(S)^{\exists}, \mathit{preds}(\varepsilon_1) = \mathit{preds}(\varepsilon_2). \end{cases}$$

an equality chain | an increasing chain | a strictly increasing chain | an unstrictly increasing chain | a contradiction of $\mathcal{L}$. Ξ is a chain | an equality chain | an increasing chain | a strictly increasing chain | an unstrictly increasing chain | a contradiction of S iff, for all $i \leq n$, $\varepsilon_i \diamond_i \upsilon_i \in S$.

Let $\tilde{W} = \{\tilde{w}_\alpha \,|\, ar(\tilde{w}_\alpha) = 0, \alpha < \omega\}$ such that $\tilde{W} \cap \mathit{Func}_{\mathcal{L}} = \emptyset$; $\tilde{W}$ is an infinite countable set of new constant symbols. Let $J \subseteq \mathbb{I}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$ be semantically admissible. A basic order hyperresolution calculus is defined in Table 6. The basic order hyperresolution calculus can be generalised to an order hyperresolution one in Table 7. Let $\mathcal{L}_0 = \mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$ and $S_0 = \{\mathit{0} \prec \mathit{1}\} \cup S \subseteq \mathit{OrdCl}_{\mathcal{L}_0}$. Let $\mathcal{D} = C_0, \dots, C_n$, $C_\kappa \in \mathit{OrdCl}_{\mathcal{L} \cup \tilde{W} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$. $\mathcal{D}$ is a deduction of C_n from S by basic order | basic witnessing order | order hyperresolution iff, for all $\kappa \leq n$, $C_\kappa \in S_0$, or there exist $j_k^* < \kappa$, $k \leq m$, such that C_κ is an order resolvent of $C'_{j_0^*}, \dots, C'_{j_m^*}$ using Rule (40)–(43) | Rule (40)–(45) | Rule (46)–(48) where $C'_{j_k^*}$ is an instance | a variant of $C_{j_k^*}$ of $\mathcal{L}_{\kappa-1}$; $\mathcal{L}_\kappa$ and S_κ are defined by recursion on $1 \leq \kappa \leq n$ as follows:

$$\mathcal{L}_\kappa = \begin{cases} \mathcal{L}_{\kappa-1} \cup \{\tilde{w}_{\alpha^*}\} & \mathit{in\ case\ of\ Rule}\ (44), (45), \\ \mathcal{L}_{\kappa-1} & \mathit{else}; \end{cases}$$

$$S_\kappa = S_{\kappa-1} \cup \{C_\kappa\} \subseteq \mathit{OrdCl}_{\mathcal{L}_\kappa}.$$

$\mathcal{D}$ is a refutation of S iff $C_n = \square$. By $\mathit{clo}^{\mathcal{BH}}(S) \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \tilde{W} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$ | $\mathit{clo}^{\mathcal{BWH}}(S) \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \tilde{W} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$ | $\mathit{clo}^{\mathcal{H}}(S) \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \tilde{W} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$ we denote the closure of S under basic order | basic witnessing order | order hyperresolution.

Lemma 4.3 (Lifting Lemma)**.** *Let $J \subseteq \mathbb{I}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$ be semantically admissible. If $C \in \mathit{clo}^{\mathcal{BH}}(S)$, then there exists $C^* \in \mathit{clo}^{\mathcal{H}}(S)$ such that C is an instance of C^* of $\mathcal{L} \cup \tilde{W} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$.*

Proof. By complete induction on the length of a deduction of C from S by basic order hyperresolution. $\square$

We are in position to prove the refutational soundness and completeness of the order hyperresolution calculus.

Theorem 4.4 (Refutational Soundness and Completeness)**.** *Let $J \subseteq \mathbb{I}$ and $S \subseteq \mathit{OrdCl}_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$ be countable, semantically admissible. $\square \in \mathit{clo}^{\mathcal{H}}(S)$ if and only if S is unsatisfiable.*

Proof. ($\Longrightarrow$) Let $\mathfrak{A}$ be a model of S for $\mathcal{L} \cup \tilde{W} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$ and $C \in \mathit{clo}^{\mathcal{H}}(S)$. Then $\mathfrak{A} \models C$. The proof is by complete induction on the length of a deduction of C from S by order hyperresolution. Let $\square \in \mathit{clo}^{\mathcal{H}}(S)$. Let $\mathfrak{A}$ be a model of S for $\mathcal{L} \cup \tilde{W} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$. We get $\mathfrak{A} \models \square$, which is a contradiction. We conclude that S is unsatisfiable.

($\Longleftarrow$) Let $\mathcal{L}$ contain a constant symbol, $S \neq \emptyset$, $\square \notin \mathit{clo}^{\mathcal{H}}(S)$. We get by Lemma 4.3 for J, S, $\square$ that $\square \notin \mathit{clo}^{\mathcal{BH}}(S)$. It is straightforward to prove that there exist $\mathcal{L}^*$ being an expansion of $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$, a reduction of $\mathcal{L} \cup \tilde{W} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$; and $S^{clo} \subseteq \mathit{OrdCl}_{\mathcal{L}^*}$ being countable, semantically admissible, $S^{clo} \supseteq S$, $\square \notin S^{clo}$, $S^{clo} = \mathit{Inst}_{\mathcal{L}^*}(S^{clo})$, $S^{clo} = \mathit{clo}^{\mathcal{BWH}}(S^{clo})$; the condition of completeness (49) (formulated below) holds. Then $S^{clo} \models S$ and $\mathit{0} \prec \mathit{1} \in S^{clo}$. We put $\mathbb{S} = \{C \,|\, C \in S^{clo}\ \mathit{is\ unit}, \mathit{freevars}(C) = \emptyset\} \subseteq \mathit{OrdCl}_{\mathcal{L}^*}$, $\mathcal{U}_{\mathfrak{A}} = \mathit{GTerm}_{\mathcal{L}^*} \neq \emptyset$, $\mathcal{B} = \mathit{GAtom}_{\mathcal{L}^*} \cup \mathit{qatoms}(\mathbb{S})$. Hence, $\mathit{0}, \mathit{1} \in \mathcal{B}$; $\mathcal{B}$ is countable; there exist $2 \leq \gamma_{\mathcal{B}} \leq \omega$ and a sequence $\delta : \gamma_{\mathcal{B}} \longrightarrow \mathcal{B}$ of $\mathcal{B}$ such that $\delta(0) = \mathit{0}$, $\delta(1) = \mathit{1}$. Let $\varepsilon_1, \varepsilon_2 \in \mathcal{B}$. $\varepsilon_1 \eqcirc \varepsilon_2$ iff $\varepsilon_1 = \varepsilon_2$ or there exists an equality chain $\varepsilon_1 \,\Xi\, \varepsilon_2$ of $\mathbb{S}$. $\varepsilon_1 \lessdot \varepsilon_2$ iff there exists a strictly increasing chain $\upsilon_1 \,\Xi\, \upsilon_2$ of $\mathbb{S}$ and $\varepsilon_1 \trianglelefteq \upsilon_1 \,\Xi\, \upsilon_2 \trianglelefteq \varepsilon_2$. We can formulate the condition of completeness as follows:

$$\text{for all } \varepsilon_1, \varepsilon_2 \in \mathcal{B}, \quad \text{either } \varepsilon_1 \lessdot \varepsilon_2 \text{ or } \varepsilon_1 \eqcirc \varepsilon_2 \text{ or } \varepsilon_2 \lessdot \varepsilon_1. \tag{49}$$

Note that $\mathit{0} \lessdot \mathit{1}$.

$$\mathit{0} \not\eqcirc \mathit{1}; \text{ for all } \varepsilon_1 \in \mathcal{B}, \varepsilon_1 \not\lessdot \mathit{0}, \mathit{1} \not\lessdot \varepsilon_1, \varepsilon_1 \not\lessdot \varepsilon_1. \tag{50}$$

The proof is straightforward.

Let $\{\mathit{0}, \mathit{1}\} \subseteq X \subseteq \mathcal{B}$. A partial valuation $\mathcal{V}$ is a mapping

$\mathcal{V} : X \longrightarrow [0, 1]$ such that $\mathcal{V}(\mathit{0}) = 0$, $\mathcal{V}(\mathit{1}) = 1$.

Table 6: Basic order hyperresolution calculus.

(*Basic order hyperresolution rule*) (40)

$$\frac{l_0 \vee C_0, \dots, l_n \vee C_n \in S_\kappa^I}{\bigvee_{i=0}^{n} C_i \in S_{\kappa+1}};$$

$l_0, \dots, l_n$ *is a contradiction of* $\mathcal{L}_\kappa$.

(*Basic order hyperresolution rule of rank r*) (41)

$$\frac{\bigvee_{j=0}^{m_0} l_j^0 \vee C_0, \dots, \bigvee_{j=0}^{m_n} l_j^n \vee C_n \in S_\kappa^I}{\bigvee_{i=0}^{n} C_i \in S_{\kappa+1}};$$

for all $i \leq n$, $m_i \leq r$;
for all $\mathcal{S} \in \mathcal{S}el(\{m_i + 1 \,|\, i \leq n\})$, *there exists a contradiction of* $\{l^i_{\mathcal{S}(i)} \,|\, i \leq n\}$;
there does not exist $\emptyset \neq I \subset n+1$ *such that for all* $\mathcal{S} \in \mathcal{S}el(\{m_i + 1 \,|\, i \in I\})$, *there exists a contradiction of* $\{l^i_{\mathcal{S}(i)} \,|\, i \in I\}$.

(*Basic order* $\forall$*-saturation rule*) (42)

$$\frac{\varepsilon_0 \diamond_0 \upsilon_0 \vee C_0, \dots, \varepsilon_n \diamond_n \upsilon_n \vee C_n \in S_\kappa^I}{\chi \prec \mu \vee \chi \eqcirc \mu \vee \bigvee_{i=0}^{n} C_i \in S_{\kappa+1}};$$

$\upsilon_n \in atoms(S_\kappa^I), \mu \in qatoms(S_\kappa^I)^\forall, \mu \trianglelefteq \upsilon_n, \upsilon_n[min(boundindset(\mu))] \in Var_\mathcal{L}, \upsilon_n[min(boundindset(\mu))] \notin vars(freetermseq(\upsilon_n/\mu)) \cup \bigcup_{i=0}^n freevars(C_i)$;
$\varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ *is an increasing chain,* $\upsilon_n[min(boundindset(\mu))] \in \bigcap_{i=1}^n freevars(\varepsilon_i) \cap \bigcap_{i=0}^{n-1} freevars(\upsilon_i)$;
$\varepsilon_0 \in atoms(S_\kappa^I) - \{0\} \cup qatoms(S_\kappa^I)$,
if $\upsilon_n[min(boundindset(\mu))] \notin freevars(\varepsilon_0)$, *then* $\chi = \varepsilon_0$, *else* $qatoms(S_\kappa^I)^\forall \ni \chi \trianglelefteq \varepsilon_0$, $\upsilon_n[min(boundindset(\mu))] \notin freevars(\chi)$.

(*Basic order* $\exists$*-saturation rule*) (43)

$$\frac{\varepsilon_0 \diamond_0 \upsilon_0 \vee C_0, \dots, \varepsilon_n \diamond_n \upsilon_n \vee C_n \in S_\kappa^I}{\mu \prec \chi \vee \mu \eqcirc \chi \vee \bigvee_{i=0}^{n} C_i \in S_{\kappa+1}};$$

$\varepsilon_0 \in atoms(S_\kappa^I), \mu \in qatoms(S_\kappa^I)^\exists, \varepsilon_0 \trianglelefteq \mu, \varepsilon_0[min(boundindset(\mu))] \in Var_\mathcal{L}, \varepsilon_0[min(boundindset(\mu))] \notin vars(freetermseq(\varepsilon_0/\mu)) \cup \bigcup_{i=0}^n freevars(C_i)$;
$\varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ *is an increasing chain,* $\varepsilon_0[min(boundindset(\mu))] \in \bigcap_{i=1}^n freevars(\varepsilon_i) \cap \bigcap_{i=0}^{n-1} freevars(\upsilon_i)$;
$\upsilon_n \in atoms(S_\kappa^I) - \{1\} \cup qatoms(S_\kappa^I)$,
if $\varepsilon_0[min(boundindset(\mu))] \notin freevars(\upsilon_n)$, *then* $\chi = \upsilon_n$, *else* $\upsilon_n \trianglelefteq \chi \in qatoms(S_\kappa^I)^\exists$, $\varepsilon_0[min(boundindset(\mu))] \notin freevars(\chi)$.

(*Basic order* $\forall$*-witnessing rule*) (44)

$$\frac{\varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n \in S_\kappa^I}{a\gamma \prec \upsilon_n \in S_{\kappa+1}};$$

$qatoms(S_\kappa^I)^\forall \ni \forall x\, a \trianglelefteq \varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n$ *is a strictly increasing chain such that for all* $i < n$, $\diamond_i = \eqcirc$, $\diamond_n = \prec$, $freevars(\forall x\, a) \cup \bigcup_{i=0}^n freevars(\varepsilon_i \diamond_i \upsilon_i) = \emptyset$;
$\tilde{w}_{\alpha^*} \in \tilde{W}, \tilde{w}_{\alpha^*} \notin Func_{\mathcal{L}_\kappa}$;
$\gamma = x/\tilde{w}_{\alpha^*} \in Subst_{\mathcal{L}_{\kappa+1}}, dom(\gamma) = vars(a)$.

(*Basic order* $\exists$*-witnessing rule*) (45)

$$\frac{\varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n \in S_\kappa^I}{\varepsilon_0 \prec a\gamma \in S_{\kappa+1}};$$

$\varepsilon_0 \diamond_0 \upsilon_0, \dots, \varepsilon_n \diamond_n \upsilon_n \trianglelefteq \exists x\, a \in qatoms(S_\kappa^I)^\exists$ *is a strictly increasing chain such that* $\diamond_0 = \prec$, *for all* $1 \leq i \leq n$, $\diamond_i = \eqcirc$, $\bigcup_{i=0}^n freevars(\varepsilon_i \diamond_i \upsilon_i) \cup freevars(\exists x\, a) = \emptyset$;
$\tilde{w}_{\alpha^*} \in \tilde{W}, \tilde{w}_{\alpha^*} \notin Func_{\mathcal{L}_\kappa}$;
$\gamma = x/\tilde{w}_{\alpha^*} \in Subst_{\mathcal{L}_{\kappa+1}}, dom(\gamma) = vars(a)$.

$$S_\kappa^I = Inst_{\mathcal{L}_\kappa}(S_\kappa) \subseteq OrdCl_{\mathcal{L}_\kappa}.$$

Table 7: Order hyperresolution calculus.

(*Order hyperresolution rule*) (46)

$$\frac{\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0 \vee \bigvee_{j=1}^{m_0} l_j^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n \vee \bigvee_{j=1}^{m_n} l_j^n \in S_\kappa^I}{\Big(\bigvee_{i=0}^{n} \bigvee_{j=1}^{m_i} l_j^i\Big)\theta \in S_{\kappa+1}};$$

$\theta \in mgu_{\mathcal{L}\kappa}\Big(\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0, l_1^0, \dots, l_{m_0}^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n, l_1^n, \dots, l_{m_n}^n, \upsilon_0^0 \trianglelefteq \varepsilon_0^1, \dots, \upsilon_0^{n-1} \trianglelefteq \varepsilon_0^n, \upsilon_0^n \trianglelefteq \varepsilon_0^0\Big)$,
$dom(\theta) = freevars\big(\{\varepsilon_j^i \diamond_j^i \upsilon_j^i \mid j \le k_i, i \le n\}, \{l_j^i \mid 1 \le j \le m_i, i \le n\}\big)$;
there exists $i^* \le n$ *such that* $\diamond_0^{i^*} = \prec$.

(*Order* $\forall$*-saturation rule*) (47)

$$\frac{\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0 \vee \bigvee_{j=1}^{m_0} l_j^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n \vee \bigvee_{j=1}^{m_n} l_j^n \in S_\kappa^I}{\chi \prec \mu \vee \chi \eqcirc \mu \vee \Big(\bigvee_{i=0}^{n} \bigvee_{j=1}^{m_i} l_j^i\Big)\theta \in S_{\kappa+1}};$$

$\theta \in mgu_{\mathcal{L}\kappa}\Big(\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0, l_1^0, \dots, l_{m_0}^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n, l_1^n, \dots, l_{m_n}^n, \upsilon_0^0 \trianglelefteq \varepsilon_0^1, \dots, \upsilon_0^{n-1} \trianglelefteq \varepsilon_0^n\Big)$,
$dom(\theta) = freevars\big(\{\varepsilon_j^i \diamond_j^i \upsilon_j^i \mid j \le k_i, i \le n\}, \{l_j^i \mid 1 \le j \le m_i, i \le n\}\big)$;
$\upsilon_0^n\theta \in atoms(S_\kappa^I), \mu \in qatoms(S_\kappa^I)^\forall, \mu \trianglelefteq \upsilon_0^n\theta$,
$\upsilon_0^n\theta[min(boundindset(\mu))] \in Var_{\mathcal{L}}, \upsilon_0^n\theta[min(boundindset(\mu))] \notin vars(freetermseq(\upsilon_0^n\theta/\mu)) \cup \bigcup_{i=0}^{n} freevars(\{l_j^i\theta \mid 1 \le j \le m_i\})$;
$\upsilon_0^n\theta[min(boundindset(\mu))] \in \bigcap_{i=1}^{n} freevars(\varepsilon_0^i\theta) \cap \bigcap_{i=0}^{n-1} freevars(\upsilon_0^i\theta)$;
$\varepsilon_0^0\theta \in atoms(S_\kappa^I) - \{0\} \cup qatoms(S_\kappa^I)$,
if $\upsilon_0^n\theta[min(boundindset(\mu))] \notin freevars(\varepsilon_0^0\theta)$, *then* $\chi = \varepsilon_0^0\theta$, *else* $qatoms(S_\kappa^I)^\forall \ni \chi \trianglelefteq \varepsilon_0^0\theta$, $\upsilon_0^n\theta[min(boundindset(\mu))] \notin freevars(\chi)$.

(*Order* $\exists$*-saturation rule*) (48)

$$\frac{\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0 \vee \bigvee_{j=1}^{m_0} l_j^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n \vee \bigvee_{j=1}^{m_n} l_j^n \in S_\kappa^I}{\mu \prec \chi \vee \mu \eqcirc \chi \vee \Big(\bigvee_{i=0}^{n} \bigvee_{j=1}^{m_i} l_j^i\Big)\theta \in S_{\kappa+1}};$$

$\theta \in mgu_{\mathcal{L}\kappa}\Big(\bigvee_{j=0}^{k_0} \varepsilon_j^0 \diamond_j^0 \upsilon_j^0, l_1^0, \dots, l_{m_0}^0, \dots, \bigvee_{j=0}^{k_n} \varepsilon_j^n \diamond_j^n \upsilon_j^n, l_1^n, \dots, l_{m_n}^n, \upsilon_0^0 \trianglelefteq \varepsilon_0^1, \dots, \upsilon_0^{n-1} \trianglelefteq \varepsilon_0^n\Big)$,
$dom(\theta) = freevars\big(\{\varepsilon_j^i \diamond_j^i \upsilon_j^i \mid j \le k_i, i \le n\}, \{l_j^i \mid 1 \le j \le m_i, i \le n\}\big)$;
$\varepsilon_0^0\theta \in atoms(S_\kappa^I), \mu \in qatoms(S_\kappa^I)^\exists, \varepsilon_0^0\theta \trianglelefteq \mu$,
$\varepsilon_0^0\theta[min(boundindset(\mu))] \in Var_{\mathcal{L}}, \varepsilon_0^0\theta[min(boundindset(\mu))] \notin vars(freetermseq(\varepsilon_0^0\theta/\mu)) \cup \bigcup_{i=0}^{n} freevars(\{l_j^i\theta \mid 1 \le j \le m_i\})$;
$\varepsilon_0^0\theta[min(boundindset(\mu))] \in \bigcap_{i=1}^{n} freevars(\varepsilon_0^i\theta) \cap \bigcap_{i=0}^{n-1} freevars(\upsilon_0^i\theta)$;
$\upsilon_0^n\theta \in atoms(S_\kappa^I) - \{1\} \cup qatoms(S_\kappa^I)$,
if $\varepsilon_0^0\theta[min(boundindset(\mu))] \notin freevars(\upsilon_0^n\theta)$, *then* $\chi = \upsilon_0^n\theta$, *else* $\upsilon_0^n\theta \trianglelefteq \chi \in qatoms(S_\kappa^I)^\exists$, $\varepsilon_0^0\theta[min(boundindset(\mu))] \notin freevars(\chi)$.

For all $i < i' \le n, freevars(\bigvee_{j=0}^{k_i} \varepsilon_j^i \diamond_j^i \upsilon_j^i \vee \bigvee_{j=1}^{m_i} l_j^i) \cap freevars(\bigvee_{j=0}^{k_{i'}} \varepsilon_j^{i'} \diamond_j^{i'} \upsilon_j^{i'} \vee \bigvee_{j=1}^{m_{i'}} l_j^{i'}) = \emptyset; S_\kappa^I = Inst_{\mathcal{L}_\kappa}(S_\kappa) \subseteq OrdCl_{\mathcal{L}_\kappa}$.

We denote $dom(\mathcal{V}) = X$, $\{0,1\} \subseteq dom(\mathcal{V}) \subseteq \mathcal{B}$. We define a partial valuation $\mathcal{V}_\alpha$ by recursion on $2 \le \alpha \le \gamma_{\mathcal{B}}$ as follows:

$$\mathcal{V}_2 = \{(0,0),(1,1)\};$$

$$\mathcal{V}_\alpha = \mathcal{V}_{\alpha-1} \cup \{(\delta(\alpha-1), \lambda_{\alpha-1})\} \qquad (3 \le \alpha \le \gamma_{\mathcal{B}} \textit{ is a successor ordinal}),$$

$$\mathbb{E}^{\equiv}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, a \equiv \delta(\alpha-1), a \in dom(\mathcal{V}_{\alpha-1})\},$$

$$\mathbb{D}^{\lessdot}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, a \lessdot \delta(\alpha-1), a \in dom(\mathcal{V}_{\alpha-1})\},$$

$$\mathbb{U}^{\lessdot}_{\alpha-1} = \{\mathcal{V}_{\alpha-1}(a) \,|\, \delta(\alpha-1) \lessdot a, a \in dom(\mathcal{V}_{\alpha-1})\},$$

$$\lambda_{\alpha-1} = \begin{cases} \dfrac{\bigvee \mathbb{D}^{\lessdot}_{\alpha-1} + \bigwedge \mathbb{U}^{\lessdot}_{\alpha-1}}{2} & \textit{if } \mathbb{E}^{\equiv}_{\alpha-1} = \emptyset, \\ \bigvee \mathbb{E}^{\equiv}_{\alpha-1} & \textit{else}; \end{cases}$$

$$\mathcal{V}_{\gamma_{\mathcal{B}}} = \bigcup_{\alpha < \gamma_{\mathcal{B}}} \mathcal{V}_\alpha \qquad (\gamma_{\mathcal{B}} \textit{ is a limit ordinal}).$$

For all $2 \le \alpha \le \gamma_{\mathcal{B}}$, $\mathcal{V}_\alpha$ is a partial valuation, (51)
$dom(\mathcal{V}_\alpha) = \delta[\alpha]$; and for all $2 \le \alpha \le \alpha' \le \gamma_{\mathcal{B}}$,
$\mathcal{V}_\alpha \subseteq \mathcal{V}_{\alpha'}$.

The proof is by induction on $2 \le \alpha \le \gamma_{\mathcal{B}}$.

For all $2 \le \alpha \le \gamma_{\mathcal{B}}$, for all $a, a' \in dom(\mathcal{V}_\alpha)$, (52)
if $a \equiv a'$, then $\mathcal{V}_\alpha(a) = \mathcal{V}_\alpha(a')$;
if $a \lessdot a'$, then $\mathcal{V}_\alpha(a) < \mathcal{V}_\alpha(a')$;
if $\mathcal{V}_\alpha(a) = 0$, then $a \equiv 0$;
if $\mathcal{V}_\alpha(a) = 1$, then $a \equiv 1$.

The proof is by induction on $2 \le \alpha \le \gamma_{\mathcal{B}}$.

We put $\mathcal{V} = \mathcal{V}_{\gamma_{\mathcal{B}}}$, $dom(\mathcal{V}) \overset{(51)}{=} \delta[\gamma_{\mathcal{B}}] = \mathcal{B}$;

$$f^{\mathfrak{A}}(u_1, \dots, u_\tau) = f(u_1, \dots, u_\tau), \qquad f \in Func_{\mathcal{L}^*}, u_i \in \mathcal{U}_{\mathfrak{A}};$$

$$p^{\mathfrak{A}}(u_1, \dots, u_\tau) = \mathcal{V}(p(u_1, \dots, u_\tau)), \qquad p \in Pred_{\mathcal{L}^*}, u_i \in \mathcal{U}_{\mathfrak{A}};$$

$$\mathfrak{A} = \big(\mathcal{U}_{\mathfrak{A}}, \{f^{\mathfrak{A}} \,|\, f \in Func_{\mathcal{L}^*}\}, \{p^{\mathfrak{A}} \,|\, p \in Pred_{\mathcal{L}^*}\}\big).$$

We get $\mathfrak{A} \models S^{clo} \models S \subseteq OrdCl_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$. We conclude that $\mathfrak{A}|_{\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}}$ is a model of S for $\mathcal{L} \cup \{\tilde{p}_{\mathbb{j}} \,|\, \mathbb{j} \in J\}$ and S is satisfiable. The theorem is proved. □

In Table 8, we show that $\phi = \forall x (q_1(x) \to q_2) \to (\exists x\, q_1(x) \to q_2) \in Form_{\mathcal{L}}$ is logically valid using the proposed translation to order clausal form and the basic order hyperresolution calculus.

5 CONCLUSIONS

The order hyperresolution calculus is amenable to adding the projection operator Δ[3] to Gödel logic, as a unary connective of $\mathcal{L}$. Henceforward, we suppose that $Form_{\mathcal{L}}$ designates the set of all formulae of $\mathcal{L}$ built up from $Atom_{\mathcal{L}}$ and $Var_{\mathcal{L}}$ using the connectives: $\neg$, Δ, $\wedge$, $\vee$, $\to$, $\equiv$, $\prec$, and the quantifiers: $\forall$, $\exists$; $OrdForm_{\mathcal{L}}$ designates the set of all order formulae of $\mathcal{L}$ built up from $Atom_{\mathcal{L}}$ and $Var_{\mathcal{L}}$ using the connectives: $\neg$, Δ, $\wedge$, $\vee$, $\to$, $\equiv$, $\prec$, and the quantifiers: $\forall$, $\exists$. We slightly modify the definition of literal. Let $l \in Form_{\mathcal{L}}$. l is a literal of $\mathcal{L}$ iff either $l = a$ or $l = a \to b$ or $l = (a \to b) \to b$ or $l = \Delta d \to a$ or $l = a \to \Delta d$ or $l = Qxc \to a$ or $l = a \to Qxc$ where $a, c, d \in Atom_{\mathcal{L}} - \{0,1\}$, $b \in Atom_{\mathcal{L}} - \{1\}$, $x \in vars(c)$. The definition of order literal remains unchanged. We add two rows to Table 1, given in Table 9. We add unary interpolation rules for Δ, Table 10. This way modified Lemma 3.1 will still hold. Thanks to having the definition of order literal unchanged, the rest of the formal treatment remains intact. So, in the countable case, we have proposed a refutation sound and complete hyperresolution proof method over semantically admissible order clausal theories together with an efficient translation of theories in general Gödel logic (with Δ) to such clausal theories, and hence, we have solved the deduction problem of a formula from a theory in the context of automated deduction.

REFERENCES

Baaz, M., Ciabattoni, A., and Fermüller, C. G. (2001). Herbrand's theorem for prenex Gödel logic and its consequences for theorem proving. In Nieuwenhuis, R. and Voronkov, A., editors, *LPAR*, volume 2250 of *Lecture Notes in Computer Science*, pages 201–215. Springer.

Baaz, M., Ciabattoni, A., and Fermüller, C. G. (2003). Hypersequent calculi for Gödel logics - a survey. *J. Log. Comput.*, 13(6):835–861.

Baaz, M. and Fermüller, C. G. (2010). A resolution mechanism for prenex Gödel logic. In Dawar, A. and Veith, H., editors, *CSL*, volume 6247 of *Lecture Notes in Computer Science*, pages 67–79. Springer.

Bachmair, L. and Ganzinger, H. (1994). Rewrite-based equational theorem proving with selection and simplification. *J. Log. Comput.*, 4(3):217–247.

Bachmair, L. and Ganzinger, H. (1998). Ordered chaining calculi for first-order theories of transitive relations. *J. ACM*, 45(6):1007–1049.

de la Tour, T. B. (1992). An optimality result for clause form translation. *J. Symb. Comput.*, 14(4):283–302.

[3] Cf. INTRODUCTION.

Table 8: An example: $\phi = \forall x\,(q_1(x) \rightarrow q_2) \rightarrow (\exists x\, q_1(x) \rightarrow q_2)$.

$$\phi = \forall x\,(q_1(x) \rightarrow q_2) \rightarrow (\exists x\, q_1(x) \rightarrow q_2)$$

$$\{\tilde{p}_0(x) \prec \mathit{1}, \big(\underbrace{\forall x\,(q_1(x) \rightarrow q_2)}_{\tilde{p}_1(x)} \rightarrow (\underbrace{\exists x\, q_1(x) \rightarrow q_2}_{\tilde{p}_2(x)})\big) \rightarrow \tilde{p}_0(x)\} \tag{26}$$

$$\{\tilde{p}_0(x) \prec \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) \eqcirc \mathit{1} \vee \tilde{p}_0(x) \eqcirc \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) \eqcirc \tilde{p}_0(x), \tilde{p}_1(x) \rightarrow \forall x\,(\underbrace{q_1(x) \rightarrow q_2}_{\tilde{p}_3(x)}), (\underbrace{\exists x\, q_1(x)}_{\tilde{p}_4(x)} \rightarrow \underbrace{q_2}_{\tilde{p}_5(x)}) \rightarrow \tilde{p}_2(x)\} \tag{32),(26}$$

$$\{\tilde{p}_0(x) \prec \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) \eqcirc \mathit{1} \vee \tilde{p}_0(x) \eqcirc \mathit{1}, \tilde{p}_2(x) \prec \tilde{p}_0(x) \vee \tilde{p}_2(x) \eqcirc \tilde{p}_0(x), \tilde{p}_1(x) \prec \forall x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) \eqcirc \forall x\, \tilde{p}_3(x), \tilde{p}_3(x) \rightarrow (\underbrace{q_1(x)}_{\tilde{p}_6(x)} \rightarrow \underbrace{q_2}_{\tilde{p}_7(x)}),$$

$$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) \eqcirc \mathit{1} \vee \tilde{p}_2(x) \eqcirc \mathit{1}, \tilde{p}_5(x) \prec \tilde{p}_2(x) \vee \tilde{p}_5(x) \eqcirc \tilde{p}_2(x), \tilde{p}_4(x) \rightarrow \exists x\, \underbrace{q_1(x)}_{\tilde{p}_8(x)}, q_2 \prec \tilde{p}_5(x) \vee q_2 \eqcirc \tilde{p}_5(x)\} \tag{24),(36}$$

$$S^{\phi} = \Big\{ \boxed{\tilde{p}_0(x) \prec \mathit{1}} \qquad [1]$$

$$\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \tilde{p}_2(x) \eqcirc \mathit{1} \vee \boxed{\tilde{p}_0(x) \eqcirc \mathit{1}} \qquad [2]$$

$$\boxed{\tilde{p}_2(x) \prec \tilde{p}_0(x)} \vee \tilde{p}_2(x) \eqcirc \tilde{p}_0(x) \qquad [3]$$

$$\boxed{\tilde{p}_1(x) \prec \forall x\, \tilde{p}_3(x) \vee \tilde{p}_1(x) \eqcirc \forall x\, \tilde{p}_3(x)} \qquad [4]$$

$$\boxed{\tilde{p}_3(x) \prec \tilde{p}_7(x) \vee \tilde{p}_3(x) \eqcirc \tilde{p}_7(x)} \vee \tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) \eqcirc \tilde{p}_7(x) \qquad [5]$$

$$\boxed{q_1(x) \prec \tilde{p}_6(x) \vee q_1(x) \eqcirc \tilde{p}_6(x)} \qquad [6]$$

$$\boxed{\tilde{p}_7(x) \prec q_2 \vee \tilde{p}_7(x) \eqcirc q_2} \qquad [7]$$

$$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \tilde{p}_5(x) \eqcirc \mathit{1} \vee \boxed{\tilde{p}_2(x) \eqcirc \mathit{1}} \qquad [8]$$

$$\boxed{\boxed{\tilde{p}_5(x) \prec \tilde{p}_2(x)} \vee \tilde{p}_5(x) \eqcirc \tilde{p}_2(x)} \qquad [9]$$

$$\boxed{\tilde{p}_4(x) \prec \exists x\, \tilde{p}_8(x) \vee \tilde{p}_4(x) \eqcirc \exists x\, \tilde{p}_8(x)} \qquad [10]$$

$$\boxed{\tilde{p}_8(x) \prec q_1(x) \vee \tilde{p}_8(x) \eqcirc q_1(x)} \qquad [11]$$

$$\boxed{q_2 \prec \tilde{p}_5(x) \vee q_2 \eqcirc \tilde{p}_5(x)} \Big\} \qquad [12]$$

Rule (40) : [1][2] :

$$\tilde{p}_2(x) \prec \tilde{p}_1(x) \vee \boxed{\tilde{p}_2(x) \eqcirc \mathit{1}} \qquad [13]$$

Rule (40) : [3][13] :

$$\boxed{\tilde{p}_2(x) \eqcirc \tilde{p}_0(x)} \vee \tilde{p}_2(x) \prec \tilde{p}_1(x) \qquad [14]$$

Rule (40) : [1][13][14] :

$$\boxed{\tilde{p}_2(x) \prec \tilde{p}_1(x)} \qquad [15]$$

Rule (40) : [8][15] :

$$\tilde{p}_5(x) \prec \tilde{p}_4(x) \vee \boxed{\tilde{p}_5(x) \eqcirc \mathit{1}} \qquad [16]$$

Rule (40) : [9][16] :

$$\boxed{\tilde{p}_5(x) \eqcirc \tilde{p}_2(x)} \vee \tilde{p}_5(x) \prec \tilde{p}_4(x) \qquad [17]$$

Rule (40) : [15][16][17] :

$$\boxed{\tilde{p}_5(x) \prec \tilde{p}_4(x)} \qquad [18]$$

Rule (41) : [4][5][7][9][12][15] :

$$\boxed{\tilde{p}_6(x) \prec \tilde{p}_7(x) \vee \tilde{p}_6(x) \eqcirc \tilde{p}_7(x)} \qquad [19]$$

repeatedly **Rule (43)** : [6][7][11][19] :

$$\vdots \qquad \vdots \qquad \vdots$$

$$\boxed{\exists x\, \tilde{p}_8(x) \prec q_2 \vee \exists x\, \tilde{p}_8(x) \eqcirc q_2} \qquad [20]$$

Rule (41) : [10][12][18][20] :

$$\square \qquad [21]$$

Table 9: Translation of l to C.

Case:	l	C	
8	$\Delta d \to a$	$d \prec 1 \vee a = 1$	$\lvert C\rvert \leq 3 \cdot \lvert l\rvert$
9	$a \to \Delta d$	$a = 0 \vee d = 1$	$\lvert C\rvert \leq 3 \cdot \lvert l\rvert$

$a, d \in Atom_{\mathcal{L}} - \{0, 1\}$.

Table 10: Unary interpolation rules for Δ.

Case:		
$\Delta\theta_1$		
Positive interpolation	$\dfrac{\tilde{p}_{\mathtt{i}}(\bar{x}) \to \Delta\theta_1}{(\tilde{p}_{\mathtt{i}}(\bar{x}) \to \Delta\tilde{p}_{\mathtt{i}_1}(\bar{x})) \wedge (\tilde{p}_{\mathtt{i}_1}(\bar{x}) \to \theta_1)}$	(53)
$\lvert Consequent\rvert = 5 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathtt{i}_1}(\bar{x}) \to \theta_1\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathtt{i}_1}(\bar{x}) \to \theta_1\rvert$		
Positive interpolation	$\dfrac{\tilde{p}_{\mathtt{i}}(\bar{x}) \to \Delta\theta_1}{\{\tilde{p}_{\mathtt{i}}(\bar{x}) = 0 \vee \tilde{p}_{\mathtt{i}_1}(\bar{x}) = 1, \tilde{p}_{\mathtt{i}_1}(\bar{x}) \to \theta_1\}}$	(54)
$\lvert Consequent\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\tilde{p}_{\mathtt{i}_1}(\bar{x}) \to \theta_1\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\tilde{p}_{\mathtt{i}_1}(\bar{x}) \to \theta_1\rvert$		
Negative interpolation	$\dfrac{\Delta\theta_1 \to \tilde{p}_{\mathtt{i}}(\bar{x})}{(\Delta\tilde{p}_{\mathtt{i}_1}(\bar{x}) \to \tilde{p}_{\mathtt{i}}(\bar{x})) \wedge (\theta_1 \to \tilde{p}_{\mathtt{i}_1}(\bar{x}))}$	(55)
$\lvert Consequent\rvert = 5 + 2 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathtt{i}_1}(\bar{x})\rvert \leq 13 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathtt{i}_1}(\bar{x})\rvert$		
Negative interpolation	$\dfrac{\Delta\theta_1 \to \tilde{p}_{\mathtt{i}}(\bar{x})}{\{\tilde{p}_{\mathtt{i}_1}(\bar{x}) \prec 1 \vee \tilde{p}_{\mathtt{i}}(\bar{x}) = 1, \theta_1 \to \tilde{p}_{\mathtt{i}_1}(\bar{x})\}}$	(56)
$\lvert Consequent\rvert = 6 + 2 \cdot \lvert\bar{x}\rvert + \lvert\theta_1 \to \tilde{p}_{\mathtt{i}_1}(\bar{x})\rvert \leq 15 \cdot (1 + \lvert\bar{x}\rvert) + \lvert\theta_1 \to \tilde{p}_{\mathtt{i}_1}(\bar{x})\rvert$		

Guller, D. (2010). A DPLL procedure for the propositional Gödel logic. In Filipe, J. and Kacprzyk, J., editors, *IJCCI (ICFC-ICNC)*, pages 31–42. SciTePress.

Guller, D. (2012). On the satisfiability and validity problems in the propositional Gödel logic. In Madani, K., Dourado Correia, A., Rosa, A., and Filipe, J., editors, *Computational Intelligence*, volume 399 of *Studies in Computational Intelligence*, pages 211–227. Springer Berlin / Heidelberg.

Marchioni, E. and Metcalfe, G. (2010). Interpolation properties for uninorm based logics. In *ISMVL*, pages 205–210. IEEE Computer Society.

Nonnengart, A., Rock, G., and Weidenbach, C. (1998). On generating small clause normal forms. In Kirchner, C. and Kirchner, H., editors, *CADE*, volume 1421 of *Lecture Notes in Computer Science*, pages 397–411. Springer.

Plaisted, D. A. and Greenbaum, S. (1986). A structure-preserving clause form translation. *J. Symb. Comput.*, 2(3):293–304.

Sheridan, D. (2004). The optimality of a fast CNF conversion and its use with SAT. In *SAT*.