Illustrating the Difficulties of Zimmermann Method for Solving the Fuzzy Linear Programming by the Geometric Approach

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Abstract: In this paper we first recall Zimmermann method and the Geometric approach for solving fuzzy linear programming problem. We show, by the geometric approach, Zimmerman method has some difficulties. Numerical examples are provided for illustrating the difficulties. Finally, the IZM algorithm for improving Zimmermann method is recalled.

1 INTRODUCTION

Following "*Decision Making in Fuzzy Environment*" proposed by (Bellman and Zadeh, 1970) and "*On Fuzzy Mathematical Programming*" proposed by (Tanaka et al., 1974), Zimmermann, (1976) first introduced FLP as a conventional LP.

Since then, FLP has been developed in a number of directions with many wide applications. Among the others, the approach of (Verdegay, 1982) and (Chanas, 1983) which presents a parametric programming method for solving FLP, is the most often used. Guu and Wu (1999) developed a two-phase approach for solving the problem, which concentrates on *the fuzzy efficiency of solutions*. Safi et al., (2007) showed some difficulties in ZM by algebraic approach. They proposed an algorithm (IZM algorithm), that eliminates these difficulties.

The majority of studies for handling FLP problems focus on developing different algebraic methods. Safi et al., (2007) used the fuzzy geometry proposed by (Rosenfeld, 1994) and presented a geometric approach for solving FLP problems.

In this note we illustrate the difficulties of Zimmermann method (ZM) by the geometric approach.

2 THE ZIMMERMANN METHOD

Consider the following general form of the FLP problem:

$$\begin{aligned} &\widetilde{max} \quad z = \sum_{j=1}^{n} c_j x_j \\ &s.t. \quad \sum_{j=1}^{n} a_{ij} x_j \lesssim b_i, \quad i =, \dots, m \\ &\qquad x_j \geq 0 \end{aligned} \tag{2.1}$$

where, $\widetilde{max}$ and $\lesssim$ denote the relaxed or fuzzy versions of the ordinary *max* and $\leq$ symbols, respectively. For representing the fuzzy goal, let us assume that the objective function must be essentially greater than or equal to an aspiration level b_0 that has been chosen by the decision maker (DM). Then we consider the following problem:

$$\begin{aligned} &find \quad \mathbf{x} = (x_1, \dots, x_n) \\ &s.t. \quad \sum_{j=1}^{n} c_j x_j \gtrsim b_0 \\ &\sum_{j=1}^{n} a_{ij} x_j \lesssim b_i, \ i = 1, \dots, m \\ &\qquad x_j \geq 0 \end{aligned} \tag{2.2}$$

The above fuzzy inequalities can be interpreted as the fuzzy subsets $\tilde{C}^i, i = 0,1, \dots, m$ of $\Re^n$such that $\tilde{C}^i = \{(\mathbf{x}, \tilde{C}^i(\mathbf{x})), \mathbf{x} \geq 0\}, i = 0,1, \dots, m$, and

$$\tilde{C}^0(\mathbf{x}) = \begin{cases} 1 & if \quad \sum_{j=1}^{n} c_j x_j \geq b_0 \\ 1 - \frac{b_0 - \sum_{j=1}^{n} c_j x_j}{p_0} & if \quad b_0 - p_0 \leq \sum_{j=1}^{n} c_j x_j < b_0 \qquad i = 1,2,\dots,m \\ 0 & if \quad \sum_{j=1}^{n} c_j x_j, < b_0 - p_0 \end{cases} \tag{2.3}$$

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$$\tilde{C}(\mathbf{x})=\begin{cases}1 & \text{if} \quad \sum_{j=1}^{n} a_{ij}x_j \le b_i \\ 1-\frac{\sum_{j=1}^{n} a_{ij}x_j - b_i}{p_i} & \text{if} \quad b_i < \sum_{j=1}^{n} a_{ij}x_j \le b_i + p_i \quad i=1,2,\dots,m \\ 0 & \text{if} \quad b_i + p_i < \sum_{j=1}^{n} a_{ij}x_j,\end{cases} \tag{2.4}$$

where p_0 and p_i, $i = 1,2,\ldots$m are positive constants subjectively assigned by the DM expressing the limitation of admissible violation for the fuzzy goal and the ith fuzzy constraint, respectively. In order to find the best decision for Problem (2.2) Zimmermann solves the following problem

$$\begin{aligned} \max \quad & \lambda \\ s.t. \quad & \sum_{g=1}^{n} c_j x_j \ge b_0 - (1-\lambda)p_0 \\ & \sum_{j=1}^{n} a_{ij}x_j \le b_i + (1-\lambda)p_i \qquad i=1,2,\ldots,\mathrm{m} \\ & x_j \ge 0, \quad j=1,\ldots,n, \quad \lambda\in[0,1]. \end{aligned} \tag{2.5}$$

3 THE GEOMETRIC APPROACH

Safi et al., (2007) studied FLP from a geometric viewpoint. In this section we recall some definitions and theorems from the geometric approach.

3.1 Fuzzy Geometric Preliminaries

Definition 3.1.1. (Rosenfeld, 1994): A fuzzy subset $\widetilde{C}$ of the plane is called a *fuzzy half plane* in direction θif the value of its membership function, $\widetilde{C}(x_\theta,y_\theta)$, depends only on x_θ. In this case, the membership function should be a monotonically non-decreasing function.

Theorem 3.1.2. (Safi et al., 2007): *Let$\widetilde{C}$ be a fuzzy subset of the plane such that its membership function in the (x,y) coordinate system is in the form of Equations (2.3) or (2.4) for $n=2$.Then there exists a directionθ such that $\widetilde{C}$ is a fuzzy half plane in this direction.*

Definition 3.1.3. (Rosenfeld, 1994): Let $\widetilde{C}^1,\widetilde{C}^2,\ldots,\widetilde{C}^n$ be fuzzy half planes in directions $\theta_1,\theta_2,\ldots,\theta_n$, respectively. Then $\widetilde{S}=\bigcap_{i=1}^{n}\widetilde{C}^i$ is called a *fuzzy polygon*.

Theorem 3.1.4. (Rosenfeld, 1994)*: If $\widetilde{S}$ is a fuzzy polygon then $\widetilde{S}_\alpha$is a crisp polygon for all* $\alpha\in[0,1]$.

3.2 Feasibility and Optimality

The following definitions and theorems are from Safi et al., (2007).

Definition 3.2.1. Consider FLP problem (2.2). $\tilde{S}=\bigcap_{i=1}^{m}\tilde{C}^i$is called the *fuzzy feasible space*, and,$\tilde{D}=\tilde{S}\cap\tilde{C}^0$is called the *fuzzy decision space*. Here we use the min-operator for intersection.

Definition 3.2.2.A point $\mathbf{x}\in\Re^n$is called a λ-*feasible* point of $\widetilde{S}$. if $\widetilde{S}(\mathbf{x})\ge\lambda$.

Theorem 3.2.3.*Every convex combination oftwo λ-feasible points of $\widetilde{S}$. is again a λ-feasible point of $\widetilde{S}$.*

Definition 3.2.4. For $\widetilde{A}\neq\phi$, set $\alpha^*=\sup\{\alpha \mid \widetilde{A}_\alpha\neq\phi\}$. Then $\widetilde{A}_{\alpha^*}$ is called the *nonempty supremum cut* (NSC) of the fuzzy set $\widetilde{A}$ and denoted by NSC($\widetilde{A}$).

Definition 3.2.5. Let $\widetilde{D}$ be the fuzzy decision space for the problem (2.1). For $\widetilde{D}\neq\phi$, $\mathrm{NSC}(\widetilde{D})=\widetilde{D}_{\alpha^*}$ is called the set of optimal solutions with the optimal value α^*. If $\widetilde{D}=\phi$, we say that the problem does not have any optimal solution.

Safi, et.al (2007) discussed the optimal solution and the optimal objective value in Definition 3.2.5 which are completely consistent with those in ZM.

4 ILLUSTRATING THE DIFFICULTIES

Safi et al., (2007) has investigated some difficulties in ZM from the algebraic viewpoint. In this section we study the difficulties by means of the geometric approach.

Example 4.1. Consider the following problem:

$$\begin{array}{llccccc} \tilde{\max} & z = & x_1 & + & x_2 & & \\ s.t. & & x_1 & + & 2x_2 & \lessapprox & 10 \\ & & -2x_1 & + & x_2 & \lessapprox & 3 \\ & & 2x_1 & + & x_2 & \lessapprox & 12 \\ & & x_1 & , & x_2 & \ge & 0. \end{array} \tag{4.1}$$

Let$b_0=3,\ p_\circ=1,\ \ p_1=2,\ \ p_2=3,\ \ p_3=3$. The Zimmermann algebraic method solves the associated problem (2.5) and obtains the alternative optimal solutions: $\mathbf{x}_A^*=(3,\ 0)$, $\mathbf{x}_B^*=(0,\ 3)$, $\mathbf{x}_C^*=(0.8,\ 4.6)$, $\mathbf{x}_D^*=(6,\ 0)$ and $\mathbf{x}_E^*=(4.6667,\ 2.6667)$ with $\lambda^*=1$.The geometric approach provides Figures 4-1 and 4-2 as $\widetilde{D}$ and its contour plot, respectively. The set of optimal solutions, $NSC(\widetilde{D})$, is the innermost (white) 5-gone in Figure 4-2. In this figure the alternative basic optimal solutions are the extreme points: $\mathbf{x}^*_A$, $\mathbf{x}^*_B$, $\mathbf{x}^*_C$, $\mathbf{x}^*_D$ and $\mathbf{x}^*_E$. Also $NSC(\widetilde{D})$ $=\widetilde{D}_1$, therefore $\lambda^* = 1$.

The line segment between $\mathbf{x}^*_A$ and $\mathbf{x}^*_B$ is the objective function with the value 3, i.e., $x_1 + x_2 = 3$. Clearly $z = x_1 + x_2$ attains the maximum value in $\mathbf{x}^*_E$.

Since the purpose of ZM is to obtain the best value for λ, it does not prefer one of the AOS to the others. Therefore, unless we check the value of z for all AOS, it is possible to introduce (for example) $\mathbf{x}^*_A$ to the DM as the optimal solution, ignoring the fact that the best value for z occurs at $\mathbf{x}^*_E$.

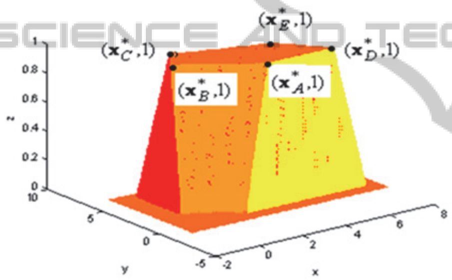

Figure 4.1: The decision space of Example 4.1.

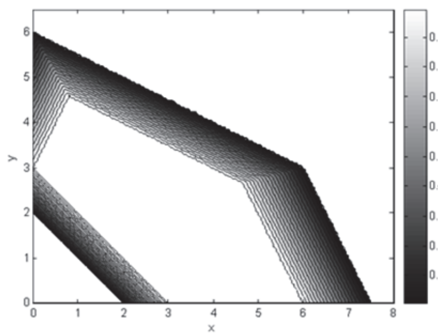

Figure 4.2: The contour plot of Figure 4-1.

When we solve this problem by WinQSB, a cycling happens between $\mathbf{x}^*_A$ and $\mathbf{x}^*_B$, hence only these two solutions is shown, as the alternative basic optimal solutions. Thus, the other three alternative basic optimal solutions, those give better values for z, have been lost.

In the final example, ZM obtains an optimal solution with a finite value for z, whereas the optimal value of z is unbounded.

Example 4.2. Consider the following problem:

$$\begin{array}{lccccc} \tilde{\max} & z = & x_1 & + & x_2 & \\ s.t. & & 2x_1 & - & 5x_2 & \lesssim 10 \\ & & 5x_1 & - & 2x_2 & \lesssim 30 \\ & & x_1 & , & x_2 & \geq 0 \end{array}$$

Let $b_0 = 6$, $p_0 = 1$, $p_1 = 2$, $p_2 = 3$. The geometric approach gives Figures 4-3 and 4-4 as $\widetilde{D}$ and its contour plot, respectively. The set of optimal solutions, $NSC(\widetilde{D})$, is the above region in figure 4-3. Clearly the objective function $z = x_1 + x_2$ can be increased in the white region to infinity. ZM does not distinguish this case and presents a solution with finite value. That is because the alternative optimal basic solutions, are $\mathbf{x}^*_A = (5.7143,\ 0.2857)$, $\mathbf{x}^*_B = (6.1905,\ 0.4762)$ and $\mathbf{x}^*_C = (0,\ 6.0)$, which none of them gives the best value for z.

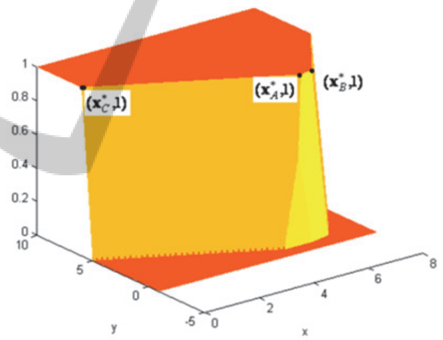

Figure 4.3: The decision space of Example 4.2.

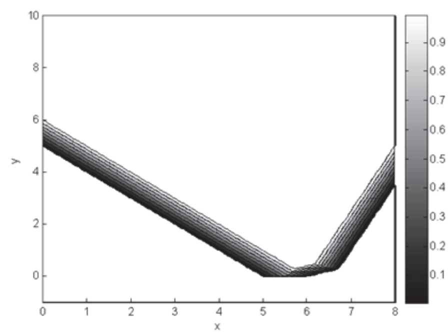

Figure 4.4: The contour plot of Figure 4-3.

5 THE IZM ALGORITHM

Safi et al., (2007) proposed the following algorithm for improving ZM and called it "*Improved*

Zimmermann Method" (IZM):

Step 1. for solving (2.1), take values b_0 and p_i; $i = 0,1,...,m$, from the DM.

Step 2. Solve (2.5) for obtaining the optimal ($\mathbf{x}^*,\lambda^*$).

Step 3. If problem (2.5) does not have any feasible solution; Stop. If it has AOS, then go to step 4. Else, $z^* = \mathbf{cx}^*$ is the best value for z. Stop.

Step 4. Solve the following LP problem:

$$\begin{aligned} \max \quad & z = \mathbf{cx} \\ s.t. \quad & \mathbf{cx} \ \geq b_\circ - (1-\lambda^*)p_\circ \\ & (\mathbf{Ax})_i \leq b_i + (1-\lambda^*)p_i \qquad i = 1,\ldots,m \\ & \mathbf{x} \ \geq 0. \end{aligned} \tag{5.1}$$

If problem (4.2) is unbounded, stop. Let $\mathbf{x}^{**}$ be the optimal solution of (4.2). If the set of all AOS is not singleton go to Step 5. Else, Stop.

Step 5. (Efficiency, Guu and Wu, 1999) Solve:

$$\begin{aligned} \max \quad & \sum_{i=0}^{m} \lambda_i \\ s.t. \quad & \tilde{A}^i(\mathbf{x}) \geq \lambda_i \geq \tilde{A}^i(\mathbf{x}^{**}) \quad i = 0,1,2,...,m \\ & \mathbf{cx} = \mathbf{cx}^{**} \\ & \mathbf{x} \geq 0. \end{aligned}$$

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