Generalized Hesitant Fuzzy Sets

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Abstract: The hesitant fuzzy set (HFS) is useful to deal with the situation that decision makers (DMs) assign several possible values to a fixed set. It is convenient to collect and deal with DMs' preferences in group decision making. However, HFSs have the information loss problem and cannot tell DMs from each other in group decision making. In order to deal with these problems, we develop a generalized hesitant fuzzy set (GHFS) in this paper, which is an extension of the HFS.

1 INTRODUCTION

Zadeh (1965) introduced the idea of fuzzy sets (FSs) as a powerful tool to address fuzziness, then several famous extensions have been developed, such as intuitionistic fuzzy sets (IFSs; Atanassov, 1986), type-2 fuzzy sets (T2FSs); (Zadeh, 1975); (Mizumoto and Tanaka, 1976); (Dubois and Prade, 1980), fuzzy multisets (FMSs); (Yager, 1986), interval-valued fuzzy sets (IVFSs); (Zadeh, 1975), interval-valued intuitionistic fuzzy sets (IVIFSs); (Atanassov and Gargov, 1989), and hesitant fuzzy sets (HFSs); (Torra, 2010).

Atanassov (1986) introduced the notion of IFSs, whose basic elements are intuitionistic fuzzy numbers (IFNs) (Xu and Yager, 2006); (Xu, 2007). Each IFN is characterized by a membership degree and a non-membership degree, which satisfies the condition that their sum is smaller or equal to 1. The IFN can be used to depict uncertainty and vagueness of an object, and thus it is a basic tool to express data information under fuzzy environments (Li et al., 2009); (Liu, 2009); (Ye, 2010).

FMSs are another generalization of FSs that permit multiple occurrences of an element, and correspond to the case where the membership degrees to the multisets are not Boolean but fuzzy. FMSs are effective for the application to information retrieval on the world wide web, where a search engine retrieves multiple occurrences of the same subjects with possible different degrees of relevance (Miyamoto, 2003). However, the basic operations of FMSs are not applied to FSs and IFSs.

T2FSs, described by membership functions that are characterized by more parameters, permit the fuzzy membership as a fuzzy set improving the modeling capability than the original one. FSs, IFSs and FMSs all can be considered as particular cases of T2FSs. Many studies have been conducted on T2FSs due to their remarkable modeling capability (Doctor and Hagras, 2005); (Hagras, 2004), at the same time, T2FSs have some difficulties in establishing the secondary membership functions, and difficulties in manipulation (Greenfield et al., 2009); (Karnik and Mendel, 2001); (Rickard et al., 2009).

Among existed fuzzy sets, HFSs, originally introduced by Torra (2010), have close relationships with IFSs and FMSs, can also be considered as a particular case of T2FSs. The motivation to propose the HFSs is that when defining the membership of an element, the difficulty of establishing the membership degree is not a margin of error (as in IFSs), or some possibility distribution (as in T2FSs) on the possible values, but a set of possible values. Torra (2010) gave an example to illustrate this situation: two DMs discuss the membership of x into A, one wants to assign 0.5 and the other 0.6, which can be denoted by a hesitant fuzzy element (HFE), $h = \{0.5, 0.6\}$. In such a case, two values given by two DMs can be collected into a HFE, which means that HFEs can be used to represent several preferences provided by different DMs in a single HFE. This advantage of HFSs contributes to the preference collection in group decision making. We also can use FMSs to model this situation, but the operations of FMSs do not apply correctly to HFSs (Torra, 2010).

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Compared with IFSs, HFSs are a tool to represent uncertainty by several discrete possible values, which is convenient to be used to collect discrete data from the mathematical point of view. However, as the situation that two DMs discuss the membership of x into A mentioned above, if they both assign 0.5 to x, denoted by a HFE $h' = \{0.5\}$, we can only save one value, and loss the other one. In this situation, if the two DMs give their evaluation values anonymously, we can save one value reasonably; if the two DMs have different importance, we have to loss some information. DMs are of vital importance in group decision making, we often need to consider their difference in practice, a leading DM for example. As a significant problem, it's common to consider the different importance of DMs in group decision making, lots of studies concentrate on the determination of the weighting vector of DMs (Yager, 1988; 2004); (Yager and Xu, 2006); (Wu et al., 2009); (Zhou and Chen, 2011); (Chen and Zhou, 2011).

Naturally, the loss of information provided by important DMs may lend to an ineffective result. To overcome this limitation, we develop a generalized hesitant fuzzy set (GHFS) which saves all information associated with different DMs. And as an extension of HFSs, GHFEs have close relationships with existed FSs.

We organize the paper as follows. Section 2 reviews some basic knowledge of IFSs and HFSs. Sections 3 presents the concept of GHFSs, discusses some properties of GHFSs, and studies the relationships among GHFSs, HFSs and IFSs. Section 4 gives the concluding remarks.

2 MANUSCRIPT PREPARATION

Atanassov (1986) originally introduced the concept of the intuitionistic fuzzy set (IFS) below.

Definition 1 (Atanassov, 1986)**.** Let X be a fixed set, an intuitionistic fuzzy set (IFS) A on X is represented in terms of two functions μ : $X \to [0,1]$ and ν : $X \to [0,1]$, with the condition, $0 \le \mu(x) + \nu(x) \le 1$, $\forall x \in X$, where μ represents the degree of membership and ν the degree of nonmembership of x into the set A. IFSs are often represented as $< x, \mu_A, \nu_A >$, for all $x \in X$. For convenience, Xu and Yager (2006) called $\alpha = < \mu_A, \nu_A >$ an intuitionistic fuzzy number (IFN).

Atanassov (1986) gave some basic operations on IFSs, which ensure that the operational results are also IFSs.

Definition 2 (Atanassov, 1986)**.** Let a set X be fixed, and let A, A_1 and A_2 be three IFSs, represented by the functions μ_A and ν_A, μ_{A_1} and ν_{A_1}, μ_{A_2} and ν_{A_2}, respectively. Then the following operations are valid:

1) Complement: $\overline{A} = \{< x, \nu_A(x), \mu_A(x) >\}$;

2) Union: $A_1 \cap A_2 = \begin{Bmatrix} < x, \min\{\mu_{A_1}(x), \mu_{A_2}(x)\}, \\ \max\{\nu_{A_1}(x), \nu_{A_2}(x)\} > \end{Bmatrix}$;

3) Intersection:

$A_1 \cup A_2 = \begin{Bmatrix} < x, \max\{\mu_{A_1}(x), \mu_{A_2}(x)\}, \\ \min\{\nu_{A_1}(x), \nu_{A_2}(x)\} > \end{Bmatrix}$;

4) $\oplus$-union:

$A_1 \oplus A_2 = \begin{Bmatrix} < x, \mu_{A_1}(x) + \mu_{A_2}(x) - \mu_{A_1}(x)\mu_{A_2}(x), \\ \nu_{A_1}(x)\nu_{A_2}(x) > \mid x \in X \end{Bmatrix}$;

5) $\otimes$-intersection:

$A_1 \otimes A_2 = \begin{Bmatrix} < x, \mu_{A_1}(x)\mu_{A_2}(x), \nu_{A_1}(x) + \nu_{A_2}(x) \\ -\nu_{A_1}(x)\nu_{A_2}(x) \mid x \in X > \end{Bmatrix}$.

Torra (2010) defined the hesitant fuzzy set (HFS) in terms of a function that returns a set of membership values for each element in the domain as follows:

Definition 3 (Torra, 2010)**.** Let X be a fixed set, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of $[0,1]$, which can be represented as the following mathematical symbol:

$$E = \{< x, h_E(x) > \mid x \in X\} \quad (1)$$

where $h_E(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, we call $h_E(x)$ a hesitant fuzzy element (HFE) and H the set of all the HFEs.

Given three HFEs h, h_1 and h_2, Torra (2010) defined some operations listed below.

1) $h^c = \bigcup_{\gamma \in h} \{1 - \gamma\}$;

2) $h_1 \cup h_2 = \{h \in (h_1 \cup h_2 \mid h \ge \max(h_1^-, h_2^-))\}$;

3) $h_1 \cap h_2 = \{h \in (h_1 \cap h_2 \mid h \le \min(h_1^+, h_2^+))\}$.

Xia and Xu (2010) developed some new operations as follows:

1) $h^{\lambda} = \bigcup_{\gamma \in h} \{\gamma^{\lambda}\}$, $\lambda > 0$;

2) $\lambda h = \bigcup_{\gamma \in h} \{1-(1-\gamma)^{\lambda}\}$, $\lambda > 0$;

3) $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;

4) $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\}$.

Torra (2010) gave a definition below that corresponds to the envelope of a HFE.

Definition 4 (Torra, 2010)**.** Given a hesitant fuzzy element (HFE) h, an intuitionistic fuzzy number (IFN) $A_{env(h)}$ is defined as the envelope of h. This number, which will be denoted by $A_{env}(h)$, is represented by (μ, ν) with μ and ν defined as $\mu = h^-$, $\nu = 1 - h^+$, where $h^+ = \max\{\gamma \mid \gamma \in h\}$ and $h^- = \min\{\gamma \mid \gamma \in h\}$.

Furthermore, Torra (2010) studied some properties of $A_{env}(h)$:

1) $A_{env}(h^c) = (A_{env}(h))^c$;

2) $A_{env}(h_1 \cup h_2) = A_{env}(h_1) \cup A_{env}(h_2)$;

3) $A_{env}(h_1 \cap h_2) = A_{env}(h_1) \cap A_{env}(h_2)$.

3 GENERALIZED HESITANT FUZZY SET AND SOME PROPERTIES

3.1 Generalized Hesitant Fuzzy Set

Given several HFSs, we propose a Cartesian product of HFSs to construct a generalized hesitant fuzzy set (GHFS). The definition is as follows:

Definition 5. Let X be a fixed set, $h_D(x) = \bigcup_{\gamma_D \in h_D(x)} \{\gamma_D\}$ $(D = 1, \ldots, m)$ be hesitant fuzzy sets (HFSs) on X. Then, a generalized hesitant fuzzy set (GHFS), that is H_{h_D}, is defined as

$$H_{h_D}(x) = h_1(x) \times \ldots \times h_m(x)$$
$$= \bigcup_{\gamma_1 \in h_1(x), \ldots, \gamma_m \in h_m(x)} \left\{ \begin{array}{l} < x, (\gamma_1(x), \ldots, \gamma_m(x)) > \\ \mid x \in X \end{array} \right\} \quad (2)$$

For convenience, we call

$$H_{h_D} = h_1 \times \ldots \times h_m = \bigcup_{\gamma_1 \in h_1, \ldots, \gamma_m \in h_m} \{(\gamma_1, \ldots, \gamma_m)\}$$

a generalized hesitant fuzzy element (GHFE). Let $u = (\gamma_1, \ldots, \gamma_m)$, we call u a membership unit (MU). Each MU corresponds to the selection of one membership each in every one of those given HFSs. Thus, we can save all the information associated the all the DMs. Based on u, a GHFE, H, can also be indicated by

$$H = \bigcup_{u \in H} \{u\} = \bigcup_{\gamma_1, \ldots, \gamma_m \in u} \{(\gamma_1, \ldots, \gamma_m) \mid u \in H\} \quad (3)$$

In group decision making, assume m decision makers (DMs), provided m HFSs to a fixed set, then we can construct a GHFS, which saves all the information associated with all DMs.

Given a GHFE, H, we define its “reduced GHFE” as

$$h_H = \bigcup_{\gamma_1, \ldots, \gamma_m \in u} \{\gamma_1, \ldots, \gamma_m \mid u \in H\} = \bigcup_{\gamma \in H} \{\gamma\} \quad (4)$$

Obviously, the reduced GHFE is a HFE. For example, given a GHFE, $H = \{(0.2, 0.3), (0.3, 0.3)\}$, then the “reduced GHFE” is $h_H = \{0.2, 0.3\}$, that is a HFE and it is an unique result. And the H is constructed by two HFEs, $h_1 = \{0.2, 0.3\}$, $h_2 = \{0.3\}$. Furthermore, if there is only one MU in H, then the GHFE is equivalent to a HFE. Assume a GHFE, $H = \{(\gamma_1, \ldots, \gamma_m)\}$ $(\gamma_1 \neq \ldots \neq \gamma_m)$, consequently, $H = \{\gamma_1, \ldots, \gamma_m\}$, that is a HFE. In such a case, the HFS is a particular case of the GHFS, which is stated below.

Proposition 1. HFSs are a particular case of GHFSs.

Consequently, we also have the following proposition:

Proposition 2. Several HFSs can construct a GHFS.

Consider that IFSs are a particular case of HFSs, where HFSs are nonempty closed intervals (Torra, 2010). According to Proposition 1, IFSs are also can be considered as a particular case of GHFSs. We state this below.

Proposition 3. IFSs are a particular case of GHFSs.

Consequently, we have

Proposition 4. Several IFSs can construct a GHFS.

Given a GHFE, we can get an unique reduced HFE, we state this as follows:

Proposition 5. Each GHFS has an unique reduced HFS.

3.2 Basic Operations and Properties

The definition of the complement of a GHFE is defined as follows:

Definition 6. Given a generalized hesitant fuzzy element (GHFE) H, we define its complement as

$$H^c = \bigcup_{\gamma_1 \in h_1, \ldots, \gamma_m \in h_m} \{(1-\gamma_1, \ldots, 1-\gamma_m)\} = \bigcup_{\gamma_1, \ldots, \gamma_m \in u} \{(1-\gamma_1, \ldots, 1-\gamma_m) \mid u \in H\} \quad (5)$$

Obviously, the complement of complement of the GHFE is itself, which can be concluded as below.

Proposition 6. The complement of the GHFE is involutive, and it can be represented as $(H^c)^c = H$.

Given a GHFE H, we define the minimum and maximum memberships of H as

1) The minimum membership of H:

$$\gamma_H^- = \min\{\gamma \mid \gamma \in H\};$$

2) The maximum membership of H:

$$\gamma_H^+ = \max\{\gamma \mid \gamma \in H\}.$$

For example, we assume a GHFE, $H = \{(0.2, 0.3), (0.2, 0.4)\}$, then

$$\gamma_H^- = \min\{0.2, 0.3, 0.4\} = 0.2,$$

$$\gamma_H^+ = \max\{0.2, 0.3, 0.4\} = 0.4.$$

For two GHFEs, H_1 and H_2, we now define their union and intersection below.

Definition 7. Assume two generalized hesitant fuzzy elements (GHFEs) H_1 and H_2, the union of them is defined as

$$H_1 \bigcup H_2 = \bigcup_{u \in (H_1 \cup H_2)} \{u \mid u \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\} \quad (6)$$

or, equivalently

$$H_1 \bigcup H_2 = \bigcup_{\substack{u_1 \in H_1, \\ u_2 \in H_2}} \{u_1, u_2 \mid u_1, u_2 \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\}$$

where $\gamma_{H_1}^-$ and $\gamma_{H_2}^-$ are the minimum memberships in H_1 and H_2 respectively.

The intersection of GHFE is defined as

$$H_1 \bigcap H_2 = \bigcup_{u \in (H_1 \cap H_2)} \{u \mid u \leq \min(\gamma_{H_1}^+, \gamma_{H_2}^+)\} \quad (7)$$

or, equivalently

$$H_1 \bigcap H_2 = \bigcup_{\substack{u_1 \in H_1, \\ u_2 \in H_2}} \{u_1, u_2 \mid u_1, u_2 \leq \min(\gamma_{H_1}^+, \gamma_{H_2}^+)\}$$

where $\gamma_{H_1}^+$ and $\gamma_{H_2}^+$ are the maximum memberships in H_1 and H_2 respectively.

Example 1. Let $H_1 = \{(0.2, 0.3), (0.2, 0.4)\}$ and $H_2 = \{(0.3, 0.4)\}$ be two GHFEs, we have $\gamma_{H_1}^- = 0.2$, $\gamma_{H_2}^- = 0.3$, $\gamma_{H_1}^+ = 0.4$ and $\gamma_{H_2}^+ = 0.4$. By Definition 7, we can get

$$H_\alpha = H_1 \bigcup H_2 = \{(0.3, 0.4)\},$$

$$H_\beta = H_1 \bigcap H_2 = \{(0.2, 0.3), (0.2, 0.4), (0.3, 0.4)\}.$$

The operations between GHFEs and HFEs have close relationship.

Proposition 7. Assume two GHFEs, H_1 and H_2, let h_{H_1} and h_{H_2} be the two reduced GHFEs of H_1 and H_2, the following are valid:

1) $h_{(H_1 \cup H_2)} = h_{H_1} \bigcup h_{H_2}$;

2) $h_{(H_1 \cap H_2)} = h_{H_1} \bigcap h_{H_2}$.

Proof. 1) For any two GHFEs, H_1 and H_2, by the operations of HFEs, and Eq. (4), we can get

$$h_{H_1} \bigcup h_{H_2} = \bigcup_{\gamma^1 \in H_1, \gamma^2 \in H_2} \left\{ \begin{array}{l} \gamma^1, \gamma^2 \mid \gamma^1, \gamma^2 \geq \\ \max(\gamma_{H_1}^-, \gamma_{H_2}^-) \end{array} \right\} \quad (8)$$

By Eq. (4), it can be shown that

$$h_{(H_1 \cup H_2)} = \bigcup_{\gamma \in (H_1 \cup H_2)} \{\gamma\} \quad (9)$$

Since

$$H_1 \bigcup H_2 = \bigcup_{u \in (H_1 \cup H_2)} \{u \mid u \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\} \quad (10)$$

We have

$$\begin{aligned} h_{(H_1 \cup H_2)} &= \bigcup_{\gamma \in u} \{\gamma \mid u \in (H_1 \bigcup H_2), u \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-)\} \\ &= \bigcup_{\gamma^1 \in H_1, \gamma^2 \in H_2} \left\{ \begin{array}{l} \gamma^1, \gamma^2 \mid \gamma^1, \gamma^2 \in u, u \in (H_1 \bigcup H_2), \\ u \geq \max(\gamma_{H_1}^-, \gamma_{H_2}^-) \end{array} \right\} \\ &= \bigcup_{\gamma^1 \in H_1, \gamma^2 \in H_2} \left\{ \begin{array}{l} \gamma^1, \gamma^2 \mid \gamma_1, \gamma_2 \geq \\ \max(\gamma_{H_1}^-, \gamma_{H_2}^-) \end{array} \right\} \\ &= h_{H_1} \bigcup h_{H_2} \end{aligned} \quad (11)$$

which completes the proof.

The proof of the intersection of GHFEs is similar to the proof of union above, which is not listed here.

Example 2 (Example 1 continuation). Since

$$H_\alpha = H_1 \bigcup H_2 = \{(0.3, 0.4)\},$$

$$H_\beta = H_1 \bigcap H_2 = \{(0.2, 0.3), (0.2, 0.4), (0.3, 0.4)\},$$

we have

$$h_{(H_1 \cup H_2)} = \{0.3, 0.4\}, h_{(H_1 \cap H_2)} = \{0.2, 0.3, 0.4\}.$$

According to Eq.(4), we have

$$h_{H_1} = \{0.2, 0.3, 0.4)\},\ h_{H_2} = \{0.3, 0.4)\},$$

and according to the operations of HFEs, we have

$$h_{H_1} \cup h_{H_2} = \{0.3, 0.4\} = h_{(H_1 \cup H_2)},$$

$$h_{H_1} \cap h_{H_2} == \{0.2, 0.3, 0.4\} = h_{(H_1 \cap H_2)}.$$

We now develop some operations of GHFEs further.

Definition 8. Given three GHFEs, $H = \bigcup_{u \in H} \{u\}$, $H_1 = \bigcup_{u_1 \in H_1} \{u_1\}$, $H_2 = \bigcup_{u_2 \in H_2} \{u_2\}$, $\lambda > 0$, since membership units u, u_1 and u_2 can be considered as three hesitant fuzzy elements (HFEs), we have the following operations:

1) $H^{\lambda} = \bigcup_{u \in H} \{u^{\lambda}\}$;
2) $\lambda H = \bigcup_{u \in H} \{\lambda u\}$;
3) $H_1 \oplus H_2 = \bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1 \oplus u_2\}$;
4) $H_1 \otimes H_2 = \bigcup_{u_1 \in H_1, u_2 \in H_2} \{u_1 \otimes u_2\}$.

Example 3. Suppose two GHFEs,

$$H_1 = \{(0.1, 0, 2), (0.1, 0.3)\},\ H_2 = \{(0.2, 0.3)\},$$

let $\lambda = 2$, we have

$$H_1^2 = \bigcup_{u_1 \in H} \{u_1^2\} = \{(0.01, 0.04), (0.01, 0.09)\},$$

$$2H_1 = \bigcup_{u_1 \in H_1} \{2u_1\} = \{(0.19, 0.36), (0.19, 0.51)\},$$

$$H_1 \oplus H_2 = \begin{Bmatrix} (0.298, 0.397, 0.396, 0.496), \\ (0.298, 0.397, 0.494, 0.591) \end{Bmatrix},$$

$$H_1 \otimes H_2 = \begin{Bmatrix} (0.02, 0.03, 0.04, 0.06), \\ (0.02, 0.03, 0.06, 0.09) \end{Bmatrix}.$$

On the basis of the relationships between GHFEs and HFEs, we can further develop the following proposition:

Proposition 8. For any three GHFEs H, H_1 and H_2, and their reduced GHFEs h_H, h_{H_1} and h_{H_2}, $\lambda > 0$, the following are valid:

1) $h_{H^{\lambda}} = (h_H)^{\lambda}$;
2) $h_{\lambda H} = \lambda(h_H)$;
3) $h_{(H_1 \oplus H_2)} = h_{H_1} \oplus h_{H_2}$;
4) $h_{(H_1 \otimes H_2)} = h_{H_1} \otimes h_{H_2}$.

Proof. For any three GHFEs H, H_1 and H_2, and their reduced GHFEs h_H, h_{H_1} and h_{H_2}, $\lambda > 0$, we have

1) $h_{H^{\lambda}} = \bigcup_{\gamma \in u} \{\gamma^{\lambda} \mid u \in H\} = \bigcup_{\gamma \in H} \{\gamma^{\lambda}\} = (h_H)^{\lambda}$;

2) $h_{\lambda H} = \bigcup_{\gamma \in u} \{1 - (1-\gamma)^{\lambda} \mid u \in H\}$
$= \bigcup_{\gamma \in H} \{1 - (1-\gamma)^{\lambda}\} = \lambda(h_H)$;

3) $h_{(H_1 \oplus H_2)} = \bigcup_{\substack{\gamma_1 \in u_1, \\ \gamma_2 \in u_2}} \begin{Bmatrix} \gamma_1 + \gamma_2 - \gamma_1\gamma_2 \\ \mid u_1 \in H_1, u_2 \in H_2 \end{Bmatrix}$
$= \bigcup_{\substack{\gamma_1 \in H_1, \\ \gamma_2 \in H_2}} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$
$= h_{H_1} \oplus h_{H_2}$;

4) $h_{(H_1 \otimes H_2)} = \bigcup_{\gamma_1 \in u_1, \gamma_2 \in u_2} \{\gamma_1\gamma_2 \mid u_1 \in H_1, u_2 \in H_2\}$
$= \bigcup_{\gamma_1 \in H_1, \gamma_2 \in H_2} \{\gamma_1\gamma_2\} = h_{H_1} \otimes h_{H_2}$

HFEs and IFNs have a close relationship that HFEs were deemed IFNs when HFEs are nonempty closed intervals. Given an IFN, (μ, ν), we can get a corresponding HFE, h, i.e., $h = [\mu, 1-\nu]$ if $\mu \neq 1-\nu$; given a HFE, h, the envelope of h is a IFN, i.e., $A_{env}(h) = < h^-, 1-h^+ >$. The envelope of GHFEs also has a close connection with IFNs. We now give a definition of the envelope of a GHFE as follows:

Definition 9. Given a generalized hesitant fuzzy element (GHFE) $H = \bigcup_{u \in H} \{u\}$, the envelope of H can be defined as $A_{env}(H)$ represented by $\{(\mu, \nu)\}$ with μ and ν defined as follows:

1) $\mu \in \{u^-\}$;
2) $\nu \in \{1-u^+\}$.

where u^- and u^+ are the minimum and maximum memberships of u, respectively.

It's clear that the envelope of a GHFE is a set of IFNs. In addition, in the particular case that a GHFE, H, is equivalent to a HFE, h, proposed in proposition 1, the envelope of H is equivalent to the envelope of h, i.e., $A_{env}(H) = A_{env}(h)$. Thus, $A_{env}(h)$ is also a particular case of $A_{env}(H)$, which is stated below.

Proposition 9. $A_{env}(h)$ is a particular case of $A_{env}(H)$.

Example 4. Given a GHFE,

$$H=\{(0.2,0.3,0.4),(0.2,0.3,0.5)\},$$

according to Definition 9, we have

$$A_{env}(H)=\{<0.2,0.6>,<0.2,0.5>\}.$$

Since the reduced HFE of H is

$$h_H=\{0.2,0.3,0.4,0.5\},$$

And according to Definition 4, we can get

$$A_{env}(h_H)=<0.2,0.5>.$$

It's clear that $A_{env}(h_H)$ is an IFN in $A_{env}(H)$, if $H=\{(0.2,0.3,0.5)\}$, then

$$A_{env}(H)=\{<0.2,0.5>\}=A_{env}(h_H).$$

Thus, $A_{env}(h)$ is a particular case of $A_{env}(H)$.

Proposition 10. For any three GHFEs H, H_1 and H_2, $\lambda>0$, we have

1) $A_{env}(H^\lambda)=(A_{env}(H))^\lambda$;
2) $A_{env}(\lambda H)=\lambda(A_{env}(H))$;
3) $A_{env}(H_1\oplus H_2)=A_{env}(H_1)\oplus A_{env}(H_2)$;
4) $A_{env}(H_1\otimes H_2)=A_{env}(H_1)\otimes A_{env}(H_2)$.

Proof. For any three GHFEs H, H_1 and H_2, $\lambda>0$, we have

1) $$\begin{aligned}A_{env}(H^\lambda)&=A_{env}(\cup_{u\in H}\{u^\lambda\})\\&=\cup_{u^-,u^+\in u}\{((u^-)^\lambda,1-(u^+)^\lambda)\mid u\in H\}\\&=\cup_{u^-,u^+\in u}\{((u^-)^\lambda,1-(1-(1-u^+))^\lambda)\mid u\in H\}\\&=\cup_{u^-,u^+\in u}\{((u^-,1-u^+)^\lambda)\mid u\in H\}=(A_{env}(H))^\lambda\end{aligned}$$
;

2) $$\begin{aligned}A_{env}(\lambda H)&=A_{env}(\cup_{u\in H}\{\lambda u\})\\&=\cup_{u^-,u^+\in u}\left\{\begin{matrix}(1-(1-u^-)^\lambda,1-(1-(1-u^+)^\lambda))\\ \mid u\in H\end{matrix}\right\}\\&=\cup_{u^-,u^+\in u}\left\{\begin{matrix}(1-(1-u^-)^\lambda,(1-u^+)^\lambda))\\ \mid u\in H\end{matrix}\right\}\\&=\lambda(\cup_{u^-,u^+\in u}\{(u^-,1-u^+)\mid u\in H\})\\&=\lambda(A_{env}(H));\end{aligned}$$

3) $$\begin{aligned}A_{env}(H_1\oplus H_2)&=A_{env}(\cup_{u_1\in H_1,u_2\in H_2}\{u_1\oplus u_2\})\\&=\cup_{u_1^-,u_1^+\in u_1,u_2^-,u_2^+\in u_2}\left\{\begin{matrix}(u_1^-+u_2^--u_1^-u_2^-,\\1-(u_1^++u_2^+-u_1^+u_2^+))\\ \mid u_1\in H_1,u_2\in H_2\end{matrix}\right\}\\&=\cup_{u_1^-,u_1^+\in u_1,u_2^-,u_2^+\in u_2}\left\{\begin{matrix}(u_1^-+u_2^--u_1^-u_2^-,(1-u_1^+)(1-u_2^+))\\ \mid u_1\in H_1,u_2\in H_2\end{matrix}\right\}\\&=(\cup_{u_1^-,u_1^+\in u_1}\{(u_1^-,1-u_1^+)\mid u_1\in H_1\})\oplus\\&\quad(\cup_{u_2^-,u_2^+\in u_2}\{(u_2^-,1-u_2^+)\mid u_2\in H_2\})\\&=A_{env}(H_1)\oplus A_{env}(H_2);\end{aligned}$$

4) $$\begin{aligned}A_{env}(H_1\otimes H_2)&=A_{env}(H_1\otimes H_2)\\&=A_{env}(\cup_{u_1\in H_1,u_2\in H_2}\{u_1\otimes u_2\mid u_1\in H_1,u_2\in H_2\})\\&=\cup_{u_1^-,u_1^+\in u_1,u_2^-,u_2^+\in u_2}\left\{\begin{matrix}(u_1^-u_2^-,1-u_1^+u_2^+)\\ \mid u_1\in H_1,u_2\in H_2\end{matrix}\right\}\\&=\cup_{u_1^-,u_1^+\in u_1,u_2^-,u_2^+\in u_2}\left\{\begin{matrix}((u_1^-u_2^-,(1-u_1^+)+(1-u_2^+)\\-(1-u_1^+)(1-u_2^+)))\\ \mid u_1\in H_1,u_2\in H_2\end{matrix}\right\}\\&=\cup_{u_1^-,u_1^+\in u_1}\{(u_1^-,1-u_1^+)\mid u_1\in H_1\}\oplus\\&\quad\cup_{u_2^-,u_2^+\in u_2}\{(u_2^-,1-u_2^+)\mid u_2\in H_2\}\\&=A_{env}(H_1)\otimes A_{env}(H_2).\end{aligned}$$

For a given GHFS H on X, we have $H(x)$ for all x in X. Then, we can define the GHFS as the fuzzy multiset (FMS):

$$FMS_H=\oplus_{x\in X}\oplus_{\gamma\in u}\{(x,\gamma)\mid u\in H(x)\}\tag{12}$$

Thus, we can give the relationship between the GHFS and the FMS below.

Proposition 11. GHFSs can be represented as FMSs.

Similar to HFSs, the operations for FMSs do not apply correctly to the GHFSs.

Given a GHFS H on X, for all x in X, we can also define the GHFS as the following type-2 fuzzy set (T2FS):

$$\mu_{H(x)}(\gamma)=\begin{cases}1,\gamma\in u,u\in H(x)\\0,\gamma\notin u,u\in H(x)\end{cases}\quad(x\in X)\tag{13}$$

Similarly, we can derive the following result:

Proposition 12. GHFSs can be represented as T2FSs.

4 CONCLUSIONS

We have developed the generalized hesitant fuzzy set (GHFS) to resolve the information loss problem of hesitant fuzzy sets (HFSs) in this paper. We have shown that HFSs and the intuitionistic fuzzy sets (IFSs) are two particular cases of GHFSs. Given several hesitant fuzzy elements (HFEs), we can construct a generalized hesitant fuzzy element (GHFE) by their Cartesian product. Given a GHFE, we also can get its reduced HFE. We have also built the relationship between intuitionistic fuzzy numbers (IFNs) and the GHFE via the envelope of GHFEs. As an extension of HFSs, GHFSs can save all the information associated with different decision makers (DMs) in group decision making, consider the difference of DMs, and widen the applications of HFSs in practice.

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