A Multi-Agent Min-Cost Flow problem with Controllable Capacities
Complexity of Finding a Maximum-flow Nash Equilibrium

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Abstract: A Multi-Agent Minimum-Cost Flow problem is addressed in this paper. It can be seen as a basic multi-agent transportation problem where every agent can control the capacities of a set of elementary routes (modeled as arcs inside a network), each agent incurring a cost proportional to the chosen capacity. We assume that a customer is interesting in transshipping a product flow from a source to a sink node through the transportation network. It offers a reward that is proportional to the flow that the agents manage to provide. The reward is shared among the agents according to a pre-established policy. This problem can be seen as a non-cooperative game where every agent aims at maximizing its individual profit. We take interest in finding stable strategies (i.e., Nash Equilibrium) such that no agent has any incentive to modify its behavior. We show how such equilibrium can be characterized by means of augmenting or decreasing path in a reduced network. We also focus on the problem of finding a Nash equilibrium that maximizes the flow value and prove its NP-hardness.

1 INTRODUCTION

Multi-agent network games have become a promising interdisciplinary research area with important links to many application fields such as transportation, project scheduling, computer network, etc. In these applicative areas, decision processes often involve several actors, each one having its own autonomy, its own objectives and its own constraints. These actors, often referred to as agents, aim at maximizing their own profits, provided a global objective should be fulfilled. This kind of problem, called multi-agent optimization, can be met in many real-life problems such as transportation networks, supply chain management, web services, production, etc. The nature and complexity of network optimization problems change significantly when the multi-agent context is considered. Besides optimizing a global objective, a solution should also satisfy additional criteria related to multi-agent games. In fact, on the one hand, a solution should optimize the agent's objective and, on the other hand, should also be stable in the sense of Nash (i.e.; no agent is able to improve its profit, to the detriment of the others). Those additional features are connected with multi-objective optimization and Game Theory, respectively. This work sits at the crossroads between two disciplines, namely multi-agent systems and social networks. The former ties in with distributed resolution of multi-agent problems, while the latter is connected to game theory, which formalizes the multi-agent optimization problem with strategic game between different agents.

Recently, some researchers have paid attention to a particular multi-agent network problem: the Multi-Agent Project Scheduling (MAPS) problem. In the seminal work of (Evaristo and Fenema, 1999), the authors proposed a special framework for distributed projects, with costs and rewards shared among agents. In an earliest work (De et al., 1997), the authors consider a MAPS problem where each agent can control the duration of its activities at a given cost. The project activities and precedence constraints are classically modeled by an activity-on-arc graph. A reward is offered to agents when they manage to finish the project earlier than expected, as proposed in (Fernandez, 2012). It was demonstrated in (Agnetis et al., 2013) and (Briand et al., 2012a) that finding a Nash equilibrium minimizing the project makespan is NP-hard in the strong sense . Moreover, based on the concepts of increasing and decreasing cuts, as defined in (Kamburowski, 1994), and, on duality between maximum flow and minimum cut problems,

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Briand *et al.* (2012b) proposed an efficient integer linear program formulation for this problem (Briand et al., 2012b).
Another important application to the network optimization is the well-known Network Flow theory (Ford and Fulkerson, 1958). Several algorithms have been developed in order to find a maximum flow in a network. Ford and Fulkerson (1956) were the first to develop a clever algorithm based on the duality between minimum cut and maximum flow (Ford and Fulkerson, 1956). Later on, efficiency improvements were proposed see eg; (Edmunds and Karp, 1972), (Goldberg and Tarjan, 1986), etc. The minimum cost maximum flow problem, which is equivalent to the minimum cost circulation problem, is solvable in polynomial time (Tardos, 1985).
As regards to social networks, the prediction of agents' behavior is of interest. Several papers focus on games associated with various forms of networks, see (Tardos and Wexler, 2007) for an overview. In a recent work, Apt and Markakis (2011) studied the complexity of finding a Nash Equilibrium for the multi-agent social networks with multiple products, in which the agents, influenced by their neighbors, can choose one out of several alternatives (Apt and Markakis, 2011).

Specifically, this work considers a transportation network that involve a set of agents, each one being in charge of a part of the network. It is assumed that each agent is able to control the transportation capacities of its arcs. A lot of features of this work are inspired by the multi-agent project scheduling problems, as presented in (Briand et al., 2012a), especially concerning the reward sharing policy. In fact, the outcome of an agent depends on its own strategy and on the satisfaction of a customer, which depends on the network flow. As proposed by Fernandez (2012) (Fernandez, 2012), we assume that the customer gives a reward proportional to the maximum-flow that can circulate inside the network. This reward is shared among agents according to some ratios predefined in the network design phase (Cachon and Lariviere, 2005).
To the best of our knowledge, the research presented here is an original way of presenting a transportation problem using multi-agent network flow with controllable arcs capacities. One important application to the problem proposed in this paper is the distributed control of transportation networks, like traffic, water, where the road of the network are distributed among several agents which can control the amount of product or water to circulate on the network.
This paper mainly discusses the complexity of finding a Nash Equilibrium that maximizes the flow in the network.

The paper is organized as follows: Section 2 defines formally the Multi-Agent Minimum-Cost Flow problem and introduces some important notations. Thereafter, Section 3 introduces the duality between efficiency and stability of a strategy and presents some important definitions and properties. In Section 4 and 5, we illustrate some basic notions for the single agent and the multi-agent cases, respectively. In Section 6, an example is provided to illustrate the notions introduced in previous sections. Section 7 deals with the complexity of the problem of finding a Nash equilibrium with bounded flow. Finally, conclusions and future directions are drawn in Section 8.

2 PROBLEM STATEMENT AND NOTATIONS

We focus on a Minimum-Cost Flow problem under a Multi-Agent context. This problem will be further referred to as MA-MCF. Considering a transportation network with limited arc capacities, this problem consists in sending a maximum amount of products from a source node to a sink node, at minimum cost. In this work, a major assumption is that arc capacities are controlled by agents, each arc being assigned to a specific agent.

2.1 Problem Definition

The MA-MCF problem can be described as a tuple $< G, \mathcal{A}, \underline{Q}, \overline{Q}, C, \pi, W >$, where:

- $G = (V, E)$ is a flow network. V is the set of nodes, $s, t \in V$ being the source and the sink nodes of the flow network G, respectively. E is the set of arcs, each one having its capacity and receiving a flow. An arc e from node i to node j is denoted by $e = (i, j)$.
- $\mathcal{A}$ is a set of m agents: $\mathcal{A} = \{A_1, \ldots, A_u, \ldots, A_m\}$. Arcs are distributed among agents. An agent A_u owns a set of m_u arcs, denoted $\mathcal{E}_u$. Each arc (i, j) belongs to exactly one agent (i.e., $\mathcal{E}_u \cap \mathcal{E}_v = \emptyset$ for each agent's pair $(A_u, A_v) \in \mathcal{A}^2$ such that $u \neq v$).
- $q_{i,j}$ is the capacity of arc (i, j) which takes value in an interval $[\underline{q}_{i,j}, \overline{q}_{i,j}]$. $\underline{q}_{i,j}$ (resp. $\overline{q}_{i,j}$) is the normal (resp. maximum) arc capacity. $\underline{Q} = (\underline{q}_{i,j})_{(i,j)\in E}$ and $\overline{Q} = (\overline{q}_{i,j})_{(i,j)\in E}$ referred to as the vectors of normal and maximum arc capacities, respectively.

For any circulating flow $f_{i,j}$, it classically holds $f_{i,j} \leq q_{i,j}$ with $q_{i,j} \in [\underline{q}_{i,j}, \overline{q}_{i,j}]$.

- $C = \{c_{i,j}\}$ is the vector of costs where $c_{i,j}$ is the unitary cost incurred by agent A_u, such that $(i,j) \in \mathcal{E}_u$, for increasing $q_{i,j}$ by one unit.
  The vector C_u denotes the cost vector incurred by augmenting the capacity of arcs by the agent A_u.
- π referred to as a reward given by a final client. This reward is proportional to the maximum flow that can circulate from s to t.
- $W = \{w_u\}$ defines the sharing policy of rewards among the agents. The A_u reward for a gain of one unit of maximum flow equals $w_u\pi$.

We denote by Q_u, $u = 1, \ldots, m$ the vector of capacities chosen by the agent A_u for the arcs belonging to him. $\underline{Q} \leq Q_u \leq \overline{Q}$ represents the individual strategy of the agent A_u. We further refer to $S = (Q_1, \ldots, Q_m)$ as the vector of individual strategies of all agents. A strategy S_{-u} denote the strategies of the $(m-1)$ players, but agent A_u, such that $S_{-u} = (Q_1, Q_2, .., Q_{u-1}, Q_{u+1}, .., Q_m)$.
Given a strategy S, $F(S)$ denotes the maximum flow that can circulate on the network flow given the current values of capacities. It is equal to the sum of flow circulating in the forward arcs of source node (i.e., $F = \sum_{(s,j)\in E} f_{s,j}$). $\underline{F}$ corresponds to the maximum flow when capacities $q_{i,j}$ are set to $\underline{q}_{i,j}$, in other words, the largest possible flow at zero cost. $\overline{F}$ is the maximum flow that can circulate when capacities are set to their maximum values ($q_{i,j} = \overline{q}_{i,j}$). Therefore, for any strategy S, it holds that $\underline{F} \leq F(S) \leq \overline{F}$.
It is assumed, in this paper, that the share of reward among agents w_u did not depend on the arcs used by agents. It is nevertheless possible to extend this work to the case in which the reward depends on how much the resource owned by each agent is used at optimum.

The cost incurred by agent A_u for a strategy Q_u equals:

$$C_u(Q_u - \underline{Q}_u) = \sum_{(i,j)\in\mathcal{E}_u} c_{i,j}(q_{i,j} - \underline{q}_{i,j})$$

With respect to the above payment scheme, the total reward given for a circulating flow $F(S)$ under strategy S is $\pi\ (F(S) - \underline{F})$.
The profit $Z_u(S)$ of agent A_u under strategy S is equal to:

$$Z_u(S) = w_u\ \pi\ (F(S) - \underline{F}) - \sum_{(i,j)\in\mathcal{E}_u} c_{i,j}(q_{i,j} - \underline{q}_{i,j}) \tag{1}$$

We denote by $Z(S) = (Z_1(S), \ldots, Z_m(S))$ the overall profit vector.

Example of a MA-MCF Network
The network flow $G(V,E)$ displayed in Fig. 1 is composed of five arcs $E = \{a,b,c,d,e\}$ distributed between two agents A_1 and A_2 such that $\mathcal{E}_1 = \{b,c,d\}$ and $\mathcal{E}_2 = \{a,e\}$ (their assigned arcs are represented with plain and dotted arcs, respectively). The set of vertex is $V = \{A,B,C,D\}$ where the source node is A and sink node is D. Each arc in the graph 1 is denoted by the interval of normal and maximum capacities, and by the cost of increasing arc capacities $([\underline{q}_{i,j}, \overline{q}_{i,j}], c_{i,j})$. Costs and capacities are such that $C_{AB} = C_{BD} = 50$, $C_{AC} = C_{CD} = 30$, $C_{BC} = 10$ and $q_{AB}, q_{BD}, q_{AC}, q_{CD}, q_{BC} \in [0,1]$. When increasing arc capacities which leads to one additional unit of flow circulating, a final client gives reward $\pi = 120$ which will be shared between agents following the sharing policy $w_1 = w_2 = \frac{1}{2}$.

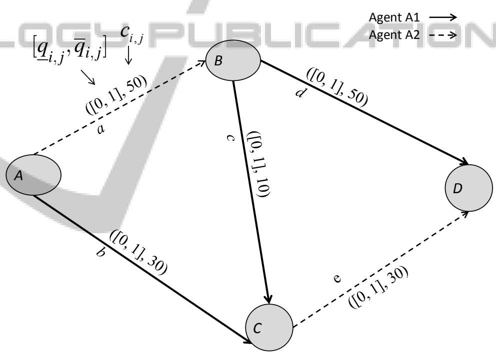


Figure 1: Problem description of example 1.

2.2 Mathematical Formulation

Each agent should choose the capacities of its arcs, having the objective of maximizing its own profit. The problem can be formalized as the following multi-objective mathematical program:

$$\begin{array}{ll}
Max & (Z_1(S), Z_2(S), \ldots, Z_m(S)) \\
s.c. & \\
(i) & f_{i,j} \leq q_{i,j}, \forall (i,j) \in E \\
(ii) & \sum_{(i,j)\in E} f_{i,j} - \sum_{(j,i)\in E} f_{j,i} = \begin{cases} 0 \ \forall i \neq s,t \\ F\ ,\ i = s \\ -F\ ,\ i = t \end{cases} \\
(iii) & \underline{q}_{i,j} \leq q_{i,j} \leq \overline{q}_{i,j}, \forall (i,j) \in E \\
 & f_{i,j} \geq 0, \forall (i,j) \in E
\end{array}$$

Where $Z_u(S)$, $u = 1, \ldots, m$ is the profit of agent A_u given by the equation (1) for each strategy S.
Constraints (i) represent the capacity constraints.

Constraints (ii) impose the conservation of the flow. The aim of this problem is to find an overall strategy S that maximizes agents' profit. Each agent A_u has to decide the arc capacity $q_{i,j}$, $\forall(i,j) \in \mathcal{E}_u$ in order to maximize its profit.

3 EFFICIENCY VS. STABILITY

A strategy is said efficient if it corresponds to a Pareto-optimal solution with respect to the above multi-objective program. The notion of Pareto optimality is concerned with social efficiency (Ehrgott, 2005). A Pareto strategy is preferred to any other strategy dominated by it.

Definition 1. *Pareto Optimality: A strategy S is Pareto-optimal if it is not dominated by any other strategy S'. In other words, it does not exist any strategy S' such that $Z_u(S') \geq Z_u(S)$ for all A_u, with at least one inequality being strict.*

The set of Pareto optimal strategies is denoted by S^P. On the other hand, a strategy is stable if there is no incentive for any agent to modify its decision in order to improve its profit. The stability of a strategy ensures that agents can trust each other. It is connected to the notion of Nash equilibrium in non-cooperative game.

Definition 2. *Nash Equilibrium: given a sharing reward policy w_u, a strategy $S = (Q_1, \ldots, Q_m)$ is a Nash Equilibrium if for any agent A_u with strategy Q'_u, the following equation holds:*

$$Z_u(Q_u, S_{-u}) \geq Z_u(Q'_u, S_{-u}), \quad \forall Q'_u \neq Q_u \qquad (2)$$

We refer to S^N as the set of Nash equilibria.

Let us also define the concept of a poor strategy. This concept will be useful for characterizing properly Nash equilibria.

Definition 3. *Poor Strategy: A strategy $S = (Q_1, \ldots, Q_m)$ with flow $F(S)$ is a poor strategy if and only if it exists an agent A_u and an alternative strategy Q'_u such that $Z_u(S) < Z_u(S')$ and $F(S') = F(S)$, where $S' = (S_{-u}, Q'_u)$.*

In other words, S is a poor strategy if and only if one agent is able to increase its profit by changing unilaterally its strategy (modifying the capacity of some of its arcs), without modifying the overall flow in the network, nor the profits of other agents. It is obvious that for any poor strategy S, $S \notin S^N \bigcup S^P$.
The set of non-poor strategies will be denoted by $\hat{S}$.

Ideally, agents should choose a strategy which satisfy both Pareto optimality and Nash stability (i.e., $S \in S^N \bigcap S^P$). Nevertheless, since $S^N \bigcap S^P$ can be empty, such a strategy is not always attainable. In this case, we are looking for a Nash equilibrium that is as efficient as possible with respect to the customer viewpoint. A Nash equilibrium that maximizes the flow circulating is indeed suitable both for maximizing the total reward and the customer satisfaction.
The aim of this study is to find an optimal strategy profile S^* such that the solution is a Nash Equilibrium that maximizes the flow circulating, the share of reward among agents w_u being fixed.
Assumptions: For sake of simplicity, it is assumed throughout this paper, that $\underline{q}_{i,j} = 0$. Therefore, the initial minimum circulating flow at zero cost is equal to $\underline{F} = 0$. This assumption does not modify the fundamental results of this work.

4 THE SINGLE-AGENT CASE

This section presents some basic properties related to classical network flow theory. In the single agent case (all the arcs belong to a single agent), a non-poor strategy S for a given flow $F(s)$ is a strategy that minimizes the overall cost. Such minimization problem is well-identified in the literature as the minimum-cost flow problem (Busacker and Gowen, 1961).
Let us recall in the following section how the total flow can be either increased or decreased, at minimum cost, using increasing or decreasing paths. These notions will be used in section 5.

4.1 Increasing the Max-Flow

Given a flow $F(S)$ for strategy S, we are interested in increasing the flow value at minimum cost. For this purpose, we recall the well-known notion of an *augmenting path* based on the concept of residual graph $G_f(S)$, which is defined below.

Definition 4. *Residual Graph: Given a network $G = (V, E)$ and a flow $F(S)$, the corresponding residual graph $G_f(S) = (V, E_r)$ is defined as follows: each arc $(i,j) \in E$, having a maximum capacity $\overline{q}(i,j)$ and a flow $f_{i,j}$ in G, is replaced by two arcs (i,j) and (j,i). The arc (i,j) has cost $c_{i,j}$ and residual capacity $r_{i,j} = \overline{q}_{i,j} - f_{i,j}$ and the arc (j,i) has cost $c_{j,i} = -c_{i,j}$ and residual capacity $r_{j,i} = f_{i,j}$.*

Definition 5. *Augmenting Path: An augmenting path is a path P in $G_f(S)$ from s to t, where $e_1 = s$ and $e_k = t$.*

We refer to $\mathcal{P}$ as the set of augmenting paths.
The greatest flow augmentation that can be achieved using P is $r_p = min\{r_{ij} : (i,j) \in P\}$.

An augmenting path in $G_f(S)$ is made of forward arcs (having the same direction in G) and backward arcs (having the opposite direction than the ones in G). The set of forward and backward arcs are denoted P^+ and P^-, respectively.
The cost of augmenting the flow by one unit using the augmenting path $P \in \mathcal{P}$ is denoted $cost(P)$. It is expressed as follows:

$$cost(P) = \sum_{(i,j)\in P^+} c_{i,j} - \sum_{(i,j)\in P^-} c_{i,j} \tag{3}$$

4.2 Decreasing the Max-Flow

When considering the problem of decreasing the flow at minimum cost in the network, we introduce the new concept of *decreasing path*.

Definition 6. *Decreasing Path: a decreasing path $\overline{P}$ is a path in $G_f(S)$ from node t to node s through which the flow can be decreased.*

We refer to $\overline{\mathcal{P}}$ as the set of decreasing paths.
Similarly, a decreasing path in $G_f(S)$ is made of forward arcs (having the opposite direction than the one in $G_f(S)$) and backward arcs (having the same direction in $G_f(S)$). The set of forward and backward arcs are denoted $\overline{P}^+$ and $\overline{P}^-$, respectively.

$$profit(\overline{P}) = \sum_{(i,j)\in\overline{P}^+} c_{i,j} - \sum_{(i,j)\in\overline{P}^-} c_{i,j} \tag{4}$$

5 THE MULTI-AGENT CASE

In the multi-agent context, one agent can decrease (or increase) unilaterally its arc capacities to improve its profit. In this context, we introduce the concept of profitability of an augmenting or a decreasing path and provide a characterization of Nash equilibrium.

5.1 Increasing the Max-Flow

Let us introduce the notion of profitable augmenting path. In a similar way, in the multi-agent context, an augmenting path is composed by a set of forward and backward arcs $P = \{P^+, P^-\}$ such that if $q_{i,j}$ is increased by one unit $\forall(i,j) \in P^+$ and decreased by one unit $\forall(i,j) \in P^-$, it is possible to increase the overall flow by one unit.
The cost of an augmenting path for agent A_u, $cost_u(P)$, is defined as the net change in cost flow for one unit of flow augmentation throughout this path. It is expressed as follows:

$$cost_u(P) = \sum_{(i,j)\in P^+\cap\mathcal{E}_u} c_{i,j} - \sum_{(i,j)\in P^-\cap\mathcal{E}_u} c_{i,j} \tag{5}$$

Definition 7. *Profitable augmenting path. An augmenting path $P \in \mathcal{P}$ is said profitable path for all agents if, for every agent A_u, $cost_u(P) < w_u \times \pi$.*

This means that through a profitable augmenting path, increasing the flow by one unit, is profitable for all the agents owning the arcs of the path.

5.2 Decreasing the Max-Flow

Now, the notion of profitable decreasing path is introduced. In the multi-agent context, a decreasing path $\overline{P} = \{\overline{P}^+, \overline{P}^-\}$ is composed of forward and backward arcs. If $q_{i,j}$ is decreased by one unit, $\forall(i,j) \in \overline{P}^+$, and increased by one unit, $\forall(i,j) \in \overline{P}^-$, the overall flow is decreased by one unit.
The profit $profit_u(\overline{P})$ generated by decreasing capacity through a decreasing path, for an agent A_u, is defined as follows:

$$profit_u(\overline{P}) = \sum_{(i,j)\in\overline{P}^+\cap\mathcal{E}_u} c_{i,j} - \sum_{(i,j)\in\overline{P}^-\cap\mathcal{E}_u} c_{i,j} \tag{6}$$

Definition 8. *Profitable decreasing path. A decreasing path $\overline{P} \in \overline{\mathcal{P}}$ is profitable if there is one agent A_u such that $profit_u(\overline{P}) > w_u \times \pi$.*

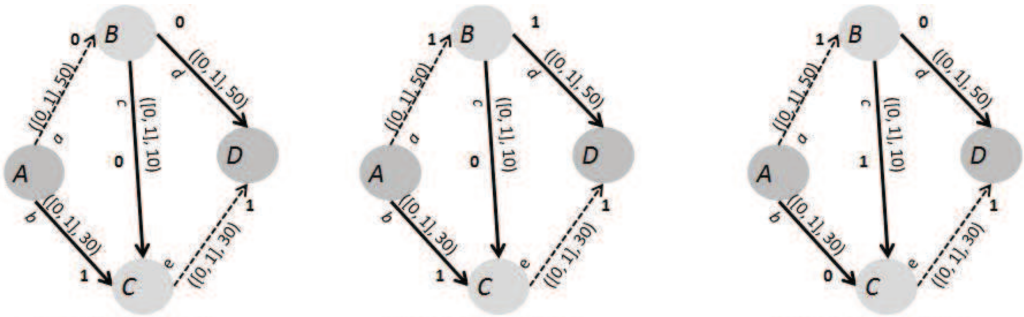
In other words, through a profitable decreasing path, decreasing the flow by one unit is profitable for one agent, to the detriments of the others.

In the multi-agent context, it is important to characterize strategies in which an agent can decrease or increase the overall flow. Therefore, it is important to find profitable augmenting paths in order to increase flow without generating decreasing paths that are profitable for some agent, hence preserving stability.

Proposition 5.1. *Nash Equilibrium.*
For a given non-poor strategy profile S, S is a Nash Equilibrium if and only if:

- $\forall A_u \in \mathcal{A}, \forall P \in \mathcal{P}$ *such that* $(i,j) \in \mathcal{E}_u$
$$cost_u(P) > w_u \times \pi \tag{7}$$
- $\forall A_u \in \mathcal{A}, \forall \overline{P} \in \overline{\mathcal{P}}$
$$profit_u(\overline{P}) < w_u \times \pi \tag{8}$$

Proof. Consider a strategy S and an agent A_u. If S is non poor, A_u can improve its situation only by increasing or decreasing the flow. In the former case, for an additional unit of flow, A_u receives $w_u \times \pi$. Since, such increase is profitable to A_u if and only if there is an augmenting path P such that $cost_u(P) < w_u \times \pi$. In the latter case, viceversa, it is profitable for an agent A_u to decrease the flow by one unit if and only if there is a decreasing path $\overline{P}$ such that $profit_u(\overline{P}) > w_u \times \pi$. Therefore, if and only if for no agent any of those conditions holds, no agent A_u can individually improve its profit, and S is a Nash equilibrium. ☐



(a) Strategy profile S1 (b) Strategy profile S2 (c) Strategy profile S3

Figure 2: A multi-agent network flow with two agents and five arcs.

6 ILLUSTRATIVE EXAMPLE

Let us come back to the previous example (section 2.1) to illustrate the optimality-stability duality of a strategy.

The initial flow on the network is equal to its minimum value $\underline{F} = 0$. It is possible to increase it along the profitable augmenting path (A-C-D), which leads to the strategy $S_1 = (0,1,0,0,1)$ (see Figure 2(a)) with $F(S_1) = 1$ and $Z_1(S_1) = Z_2(S_1) = 30$. From this strategy, it is still possible to increase the flow along the profitable augmenting path (A-B-D), which leads to the strategy profile $S_2 = (1,1,0,1,1)$ (see Figure 2(b)) with $F(S_2) = 2$ and $Z_1(S_2) = Z_2(S_2) = 40$. From this strategy, we observe that there exists a profitable decreasing path $(D-B-C-A)$ from sink node D to source node A which is profitable for agent A_1. In fact, A_1 can improve its own profit, by decreasing back the flow on b and d by one unit and increasing the flow on arc c by one unit. This leads to the strategy $S_3 = (1,0,1,0,1)$ (see Figure 2(c)) with $F(S_3) = 1$ and profits $Z_1(S_3) = 50$ and $Z_2(S_3) = -20$, which is obviously bad for A_2. Therefore, although the strategy S_2 corresponds to a Pareto Optimum, which leads to a maximization of agent's profits, it is not a stable strategy. Strategy S_1 is a Nash Equilibrium but not Pareto Optimum. Therefore, in our example there is no a strategy which is both in S^N and S^P. The motivation of this paper is to search for a Nash-stable solution which is as efficient as possible, i.e., which maximizes $F(S)$.

7 PROBLEM COMPLEXITY

In this section, we discuss the complexity of finding a Nash equilibrium that maximizes the flow in the network. This problem can be described by the following mathematical model.

$$
\begin{array}{ll}
\mathcal{P}_{MA-MCF} & \\
Max & F \\
s.c. & \\
(i) & f_{i,j} \leq q_{i,j},\ \forall (i,j) \in E \\
(ii) & \sum_{j \in P^+(i)} f_{i,j} - \sum_{j \in P^-(i)} f_{j,i} = \begin{cases} 0 \ \forall i \neq s,t \\ F \ ,\ i = s \\ -F \ ,\ i = t \end{cases} \\
(iii) & \underline{q}_{i,j} \leq q_{i,j} \leq \overline{q}_{i,j},\ \forall (i,j) \in E \\
(iv) & profit_u(\overline{P}) < w_u \times \pi,\ \forall \overline{P} \in G_f(S) \\
 & \\
 & f_{i,j} \geq 0,\ \forall (i,j) \in E
\end{array}
$$

Constraints (i), (ii) and (iii) are the same as the one of the multi-objective mathematical formulation presented above 2. They represent the constraints of arcs capacities and flow conservation, respectively. Constraints (iv) impose that no decreasing path $\overline{P}$ exists in solution S with profit $profit_u(\overline{P})$ greater or equal to $w_u \times \pi$. In other words, it represent the constraints for a solution to being Nash stable. Even if the constraint (iv) is linear, we notice that the number of possible paths in the residual graph can grow exponentially. Moreover, a non-linearity can be recognized since the residual graph $G_f(S)$ depends on the strategy chosen by each agent.

Constraints (iv) impose that no decreasing path $\overline{P}$ exists in solution S with $profit_u(\overline{P})$ greater or equal to $w_u \times \pi$ (See Proposition 5.1). Increasing paths do not need to be bounded since since the network flow is maximized.

7.1 Finding a Nash Equilibrium with Bounded Flow

We consider the decision problem to find a strategy which is a Nash equilibrium, with a flow greater than a given value. This problem can be defined as follows.

Nash-Equilibrium Bounded Flow:

Instance: a tuple $< G, \mathcal{A}, \underline{Q}, \overline{Q}, C, \pi, W >$ as defined in section 2 and an integer Φ

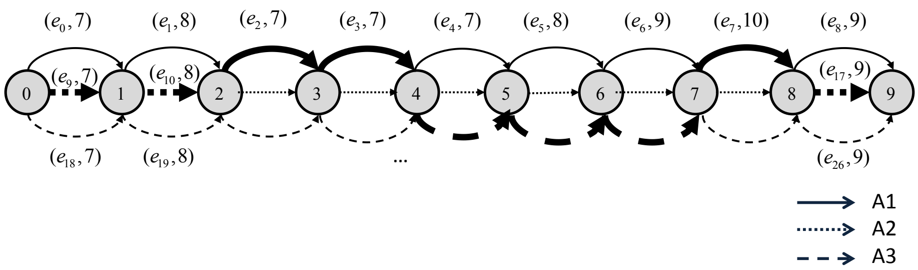


Figure 3: Reduction from 3-PARTITION problem with $k = 3$.

Problem: Is there a Nash Equilibrium strategy profile S such that $F(S) > \Phi$?

Proposition 7.1. *Problem NE-Bounded Flow is strongly NP-complete.*

Proof. The NP-completeness of this problem can be proved using a reduction from the well-known 3-partition problem, which is known to be NP-complete in the strong sense (Garey and Johnson, 1979).

3-Partition:

Instance: a set $\zeta = \{a_0, \ldots, a_{K-1}\}$ of $K = 3k$ positive integers, such that $\sum_{l=0}^{K-1} a_l = k \times B$ and $a_l \in]B/4, B/2]$

Problem: Deciding whether ζ can be partitioned into k subsets so that the sum of integers in each subset is equal to B?

An instance of the MA-MCF problem with controllable capacities can be generated from an arbitrary instance of the 3-partition problem as follows.
From the 3-partition problem instance, we build up a network G with $k \times K$ arcs and $K+1$ nodes where the first one is source node $V_0 = s$ and the last one is the sink node $V_K = t$. An agent $A_u \in \mathcal{A} = \{A_1, \ldots, A_k\}$ owns K arcs.

The tail of an arc e_i is $V_{i \operatorname{div} K}$, its head is $V_{(i \operatorname{div} K)+1}$. Between nodes $V_{i \operatorname{div} K}$ and $V_{i \operatorname{div} K+1}$, there are k parallel arcs, indexed from i to $i+K$ step k, each of them belonging to a specific agent: arc e_i belongs to $A_{i \operatorname{div} K}$. The cost of arc e_i is $c_{e_i} = a_{i \operatorname{mod} K}$. In other words, to every positive integer $a_l \in \zeta$ is associated k parallel arcs with, same head and tail, maximum capacity $\overline{q}_{e_i} = 1$ and cost a_l. The total reward is set to $\pi = (B+\varepsilon)k$, ε being an arbitrary small positive value. The sharing policy is defined by $w_u = 1/k$. Therefore, agent's unit reward is $w_u\pi = B+\varepsilon$, identical for all agents. The objective is to determine whether it exists a Nash strategy such that $F(S) > \Phi$?
For instance, Figure 3 illustrates the resulting flow network obtained from the 3-partition instance defined by $k = 3$, $\zeta = \{7,8,7,7,7,8,9,10,9\}$ and $B = 24$. We have $k = 3$ agents and $K * k = 27$ arcs. Between nodes i and $i+1$, we find $k = 3$ arcs with cost a_{i+1}. The problem is to find, whether it exists, a Nash strategy such that the flow is strictly greater than 0. In that case, using the path with bold arcs allows to obtain a one-unit total flow, which is a Nash equilibrium since every agent does not pay more than its part of reward ($w_u\pi = B+\varepsilon = 24+\varepsilon$). Any equivalent stable path is also a solution to the original 3-Partition problem.

Let us prove this last property in a general way. Consider the strategy $\underline{S}$ where all arcs have normal capacity, $q_{i,j} = 0$. The resulting flow obviously equals to $F(\underline{S}) = 0$. With respect to $\underline{S}$, we observe that an agent can increase the flow by the amount $\delta \in]0,1]$, increasing the capacities of all its arcs by the same amount δ. However, doing so, the agent pays $kB\delta$ and only gains $(B+\varepsilon)\delta$. Hence, the new strategy is not profitable and cannot be a Nash equilibrium. In order to obtain a Nash equilibrium, the total cost incurred by each agent for increasing its arc capacities must not exceed B, otherwise at least one agent will be interested in decreasing back its capacities.
Due to the topology of the network, in order to increase the flow, exactly $K = 3k$ arcs must be involved in an augmenting path. In any Nash equilibrium strategy with flow strictly greater than 0, the augmenting path having to be profitable for every agent, it must be made of exactly three arcs per agent. The total compression cost for every agent equals exactly B. $\square$

8 CONCLUSIONS

This paper presents a new game theory framework for a multi-agent flow network problem with controllable capacities. We consider that a final customer gives a reward, shared among agent, for any additional unit of flow circulating in the network. Each agent has the possibility to modify the capacities of its arcs at a given cost. We particularly point out the notions of efficiency and stability of a strategy. We also prove that finding a Nash Equilibrium with maximum flow is NP-hard in the strong sense. Further works are ongoing to linearize the mathematical model of finding a Nash Equilibrium as a Mixed Integer linear programming. Distributed heuristics able to find a Nash equilibrium are also under study.

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REFERENCES

Agnetis, A., Briand, C., Billaut, J.-C., and Sucha, P. (2013). Nash equilibria for the multi-agent project scheduling problem with controllable processing times. Technical report, LAAS-CNRS.

Apt, R. K. and Markakis, E. (2011). Diffusion in social networks with competing products. In *Proceedings of the 4th international conference on Algorithmic game theory*, SAGT'11, pages 212–223. Springer-Verlag.

Briand, C., Agnetis, A., and Billaut, J. (2012a). The multiagent project scheduling problem : complexity of finding an optimal nash equilibrium. In *Proceedings of the 13th International Conference on Project Management and Scheduling, Leuven, Belgium*, pages 106–109.

Briand, C., Sucha, P., and Ngueveu, S. (2012b). Minimisation de la durée totale d'un projet multi-agent : recherche du meilleur équilibre de nash. In *Proceedings of the 13th edition of ROADEF*, pages 111–112.

Busacker, R. and Gowen, P. (1961). *A Procedure for Determining a Family of Minimal-Cost Network Flow Patterns*. Defense Technical Information Center, Johns Hopkins University, 15 edition.

Cachon, G. and Lariviere, M. (2005). Supply chain coordination with revenue-sharing contracts: Strengths and limitations. *International journal of Management Science*, 51(1):30 – 44.

De, P., Dunne, E., Ghosh, J., and Wells, C. (1997). Complexity of the discrete time-cost tradeoff problem for project networks. *European journal of Operations Research*, 45(2):302–306.

Edmunds, J. and Karp, R. (1972). Theoretical improvements in algorithmic efficiency for netowrk flow problems. *Journal of the Association for Computing Machinery*, 19:248–264.

Ehrgott, M. (2005). *Multicriteria Optimization*. Springer, Berlin, second edition edition.

Evaristo, R. and Fenema, P. V. (1999). A typology of project management: emergence and evolution of new forms. *International Journal of Project Management*, 17(5):275–281.

Fernandez, M. E. (2012). A game theoretical approach to sharing penalties and rewards in projects. *European Journal of Operational Research*, 216(3):647–657.

Ford, L. and Fulkerson, D. (1956). Maximum flow through a network. *Canadian Journal of Mathematics*, 8:399–404.

Ford, L. and Fulkerson, D. (1958). *Flows in networks*. Princeton University Press, New Jersey.

Garey, M. and Johnson, D. (1979). *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman & Co, New York, USA.

Goldberg, A. and Tarjan, R. (1986). A new approach to the maximum flow problem. In *Proceedings of the eighteenth annual ACM symposium on Theory of Computing (STOC'86)*, pages 136–146. New york.

Kamburowski, J. (1994). On the minimum cost project schedule. *International Journal of Management Science (Omega)*, 22(4):401– 407.

Tardos, E. (1985). A strongly polynomial minimum cost circulation algorithm. *International journal Combinatorica*, 5(3):247–255.

Tardos, E. and Wexler, T. (2007). *Network formation games and the potential function method*. Algorithmic Game Theory, Cambridge University Press.