A Network Dispersion Problem for Non-communicating Agents

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1 THE RESEARCH PROBLEM

Is coordination possible when agents are unable to communicate? This is an abstract problem that can only be made concrete by the specification of a well-defined goal of the coordination processes. For example, the problem of rendezvous of two or more agents on a network is one that has been extensively studied for non-communicating agents (see, e.g., Alpern and Gal, 2003). A less studied and more recent problem is the opposite one of achieving *spatial dispersion*. If there are n agents at k locations, we say that they are *dispersed* if no location has more than one agent (this requires of course that k is at least n). In the case of $n = k$ (the main case studied here), dispersion implies coverage, that is, every location has at least one agent.

Alpern and Reyniers (2002) initiated the study of spatial dispersion of agents in the context of non-communicating agents who could move between any two locations, based only on the knowledge of the populations at all locations.

The research problem addressed in this paper is to see how the problem of Alpern and Reyniers changes if a network (graph) structure is imposed on the set of locations: An agent only knows the population at his current node (location) and he can only stay still or move to an adjacent node. He can see the number of arcs at his node, but if he moves he must choose among them equiprobably. As we are dealing with non-communicating agents with limited amount of global knowledge, we shall limit our discussion to simple and myopic strategies. In particular, our agents adopt common *(Markovian) stunted random walk* strategy, completely specified by the probability p of staying still for another period at the same node. Given such strategies, how long will it take, on average, for the agents to achieve dispersion? The optimization can be simply with respect to p, or with respect to p_i, where i is the population at the agent's current node. For example, in the trivial case of a network with two nodes, where both agents start at the same node, the optimal value of p is ½ and the expected time to dispersion is $T = 2$ periods.

We shall introduce the network dispersion problems by way of the following illustration. Suppose that at 11am the guards at the Louvre have called a wildcat strike for noon. The Louvre by law must be kept open. We know that there are n rooms, but at short notice we don't have a map. We can hire in an hour n guards, but they may speak different languages and will not be able to communicate with each other. We have a single broadcasting line for communication to all these new guards, but we cannot give different guards different instructions as we do not even know their names. Once all the rooms have guards, we can broadcast a command to stay still. To keep things simple, we assume that time is divided into periods and that each guard visits only one location in each period. All we can broadcast is a single number p, the probability that a guard should stay still for that period. We want this to occur before the word gets out that this is a good time to steal a painting, so we want to minimise the expected time to reach full coverage of all the rooms. How can we define a common strategy for the guards that is simple yet still effective? And how fast on average will they arrive at the goal of getting as dispersed as possible?

2 STATE OF THE ART

Our agents are not the smart rational individuals that are normally assumed in game theory models. Instead, this type of agent is normally studied in the domains of Artificial Intelligence and Biology (social animals). Therefore we found that the works that are most relevant to our study comes from those two research communities.

After the paper of Alpern and Reyniers (2002), Grenager et al. (2002) generalized the specific problem to a general class of *dispersion games*. In this type of games, agents win positive payoff only

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when they choose distinct actions. Problems with similar characteristics have been studied in many research areas. Here we list some of them:

1. **Load balancing:** a set of tasks is to be assigned to a set of machines, and to minimise the overall processing time, the load need to be distributed as evenly as possible (see, e.g., Azar et al., 1999).
2. **Niche selection:** this problem has been studied in economics as well as in ecology. In economics, it models a situation in which producers wish to occupy one of a number of different market niches, and they prefer to occupy niches with fewer competitions. A few examples are the Santa Fe problem (Arthur, 1994) and the class of minority games (Challet and Zhang, 1997). In ecology, this problem is also called habitat selection, and an example of this is how animals choose feeding patches with low population density with respect to food supply (see, e.g., Houston and McNamara, 1997). In particular, Fretwell and Lucas (1969) introduced the equilibrium notion of the *ideal free distribution* for habitat distribution in birds. That is a particular analogue of our notion of dispersion, where some nodes have greater capacity (habitat is more supportive of foragers).
3. **Congestion games:** proposed initially by Rosenthal (1973), in this class of games individuals seek facilities or locations of low population density, due to the monotonically increasing cost of using the facilities. One of the subclasses of congestion games that has received significant attention is routing games, in which a set of players in a network choose the route with low congestion level (see, e.g., Qiu et al. 2003).
4. **Multi-agent area coverage:** a team of agents seek to cover an entire area (which may or may not be known a priori), and therefore it is preferred that each agent covers different part of the area. This problem has been studied in several research communities, including robotics/agents, sensor networks, and computational geometry. For a recent survey in the fields of robotics, see Galceran and Carreras (2013).

3 OBJECTIVES

Given a graph G with k nodes, we want to see *which (Markovian) stunted random walk strategies, if simultaneously adopted by all agents, lead to the least expected time T to dispersion*? In all the problems considered, the same optimization criterion is used. We consider both (i) a random initial placement of the n agents onto the k nodes of G, and (ii) an initial placement of all the agents onto a single node j of G. In the latter case, it is of interest to see where the best place to introduce the agents is.

We also will consider two types of stunted random walk strategies, and compare them. First, a simple type, where there is a common parameter p representing the probability of staying still for the next period. Second, a probability vector $\mathbf{p}$ in which p_i (the i-th entry in the vector) gives the probability of staying still (rather than randomly moving) that depends on the number i of agents currently at your node (called the agent population of the node).

A conjecture of Alpern and Reyniers (2002) in a related context says that if you are alone at your node then you should not move and no one should move to your node. While this might work in the context of a complete graph, it clearly fails on the line graph L_n. In the case when there are many agents at node 1 and one agent at node 2, the ones at node 1 could never disperse. However the weaker property of setting $p_1 = 1$ seems to hold for some graphs, which does not have the added assumption that no one goes to a node with current population of 1.

Our objective is to determine the minimum dispersal time and optimal levels of p or of $\mathbf{p}$, for several classes of networks, including line networks L_n, cycle networks C_n, and complete graphs K_n. We also wish to compare on L_n the efficiency of an initial placement of all agents at a node j, as j varies. It would seem likely that it is better to place them as close as possible to the centre.

From now on we assume that the graphs we consider have n nodes and that there are n agents.

4 METHODOLOGY

To evaluate the expected dispersal time T on a graph G as a function of the stationary probability p (or of the vector $\mathbf{p} = (p_1, ..., p_n)$) we use two distinct methodologies. For small graphs (small n), we use the theory of absorbing time for Markov chains. The states of the chain are the population

distributions, allowing for symmetry. For example, the state $[m_1, m_2, m_3]$ says there are m_1 agents at node 1, m_2 at 2 and m_3 at 3. By symmetry of nodes 1 and 3 we may assume m_1 is at least as large as m_3. Thus the states when $G = L_3$ are $[3,0,0]$, $[2,1,0]$, $[2,0,1]$, $[1,2,0]$, $[0,3,0]$, and the absorbing (dispersed) state $[1,1,1]$. In general, the set S_n of all states for the network dispersion problem on L_n can be described as follows, where

$$S_n = \left\{ s = [m_1, m_2, ..., m_n] : m_1 \geq m_n, \sum_{i=1}^{n} m_i = n \right\}$$

When n is larger than 3, the number of states in this type of analysis becomes too large for Markovian analysis, so another methodology is needed – simulation. We use the software application *Mathematica* to simulate the movement of agents adopting a common p-strategy, and minimize the dispersal time numerically. For purposes of comparison, and to check our simulation program, we analyse the line graph L_3 both ways.

5 STAGE OF RESEARCH – RESULTS SO FAR

5.1 Markovian Analysis for L_3

We first analyse the problem on a line graph L_3 of three nodes using Markov chain theory. We define six states in L_3 problem (up to symmetry): $s_1 = [3,0,0]$, $s_2 = [2,1,0]$, $s_3 = [2,0,1]$, $s_4 = [1,2,0]$, $s_5 = [0,3,0]$, and the absorbing state, $s_6 = \hat{s} = [1,1,1]$. We do not put any restrictions on the movement of the agents, other than if an agent chooses to move in the next period, then they can only move to a node adjacent to the current node. This implies that each state in the Markov chain is accessible from any other states (albeit accessing a particular state could take more than one or two steps). A Markov chain with this characteristic is defined as an absorbing Markov chain, for which a set of theoretical formulas has been devised to calculate the expected time to absorption. In this subsection we follow the procedures explained in Grinstead and Snell (1997) to find the time to absorption, or more generally, the expected dispersal time T.

Definition (Absorbing Markov Chain) (Grinstead and Snell, 1997): A Markov chain is absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state (not necessarily in one step).

In an absorbing Markov chain, the probability of absorption is 1. This means that for our problem, the process will eventually end at the absorbing state $\hat{s} = [1,1,1]$. However, we have two special situations in which our Markov chain is no longer absorbing. It is if the value of p, the probability of staying still, is set to either 0 or 1. If p is set to 0, three of the states ($[3,0,0]$, $[2,0,1]$, and $[0,3,0]$) cannot access $\hat{s}$, whereas if p is set to 1, then all states become absorbing states, and thus do not communicate with each other. Therefore, the absorbing Markov chain theory only applies when we restrict the range of p as $0 < p < 1$. For the case of $p = 0$, as well as $p = 1$, we shall describe another approach separately in the following section.

5.1.1 Simple Stunted Random Walk

We begin by considering simple strategies based on a single probability p for staying still. In each period, each agent independently decides to remain at his current node or to move. If he chooses to move, he observes the number of arcs from his node and choses to move along them equiprobably. In the case of a line graph this is elementary: if at an end, he moves to the unique adjacent node; otherwise, he randomly moves left or right. We wish to find the strategy parameter $\overline{p} = \overline{p}_s$ that minimises the expected time $T = T(s, p)$ to reach the unique absorbing state $\hat{s} = [1,1,1]$ from the state s. We denote this minimum time by $\overline{T} = T(s, \overline{p})$. Sometimes we wish to start from a random state, where each state occurs with its probability of occurring from a binomial distribution, that is, if each of the n agents was randomly placed on one of the n nodes. In this case the expected absorption time is called $T = T(r, p)$, where r stands for the random initial state. To calculate T, we first define the transition matrix $\mathbf{Q}$ comprising a set of transition probabilities between all states in the chain.

In the transition matrix Q, q_{ij} gives the transition probability of moving from state s_i to state s_j in one step. For example, for the transition probability from s_3 to s_2, we have $q_{32} = p^2(1-p)$, which means

$$\mathbf{Q}(p)=\begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} & q_{15} & q_{16} \\ q_{21} & q_{22} & q_{23} & q_{24} & q_{25} & q_{26} \\ q_{31} & q_{32} & q_{33} & q_{34} & q_{35} & q_{36} \\ q_{41} & q_{42} & q_{43} & q_{44} & q_{45} & q_{46} \\ q_{51} & q_{52} & q_{53} & q_{54} & q_{55} & q_{56} \\ q_{61} & q_{62} & q_{63} & q_{644} & q_{65} & q_{66} \end{pmatrix}$$

that, if the current state $s_3=[2,0,1]$, then we can reach the state $s_2=[2,1,0]$ in the next period if the two agents at the left stay still (p^2), and the agent at the right moves to the middle $(1-p)$.

We then construct the matrix $\tilde{\mathbf{Q}}$ (a submatrix of $\mathbf{Q}$), which is a 5-by-5 matrix consisting of the transition probabilities of all the transient states (all states except $\hat{s}$), that is, the upper left submatrix.

$$\tilde{\mathbf{Q}}=\begin{pmatrix} q_{11} & q_{12} & q_{13} & q_{14} & q_{15} \\ q_{21} & q_{22} & q_{23} & q_{24} & q_{25} \\ q_{31} & q_{32} & q_{33} & q_{34} & q_{35} \\ q_{41} & q_{42} & q_{43} & q_{44} & q_{45} \\ q_{51} & q_{52} & q_{53} & q_{54} & q_{55} \end{pmatrix}$$

We wish to determine the column vector $\boldsymbol{\tau}=\boldsymbol{\tau}(p)$ whose i-th entry is $\tau_i=T(s_i,p)$, the expected dispersal time from state s_i. Then we can minimise this expression, for each i, with respect to the parameter p. According to the theory of absorbing Markov chains (Grinstead and Snell, 1997), the vector is given by the formula:

$$\boldsymbol{\tau}=\left(\mathbf{I}-\tilde{\mathbf{Q}}\right)^{-1}\cdot\mathbf{c}$$

where $\mathbf{I}$ is the identity matrix, and $\mathbf{c}$ is a column vector, of which all of the entries are 1. In the simple stunted random walk strategy, the value of $\boldsymbol{\tau}$ as a function of p is:

$$\boldsymbol{\tau}(p)=\begin{pmatrix} \dfrac{-8p^6+128p^5-22p^4+58p^3+p^2+10p+1}{3p\left(-8p^6+4p^4+p^3+3p^2-p+1\right)} \\ -\dfrac{16\left(p^2+p+1\right)}{3(p-1)\left(2p^3+3p^2+2p+1\right)} \\ -\dfrac{-8p^5+8p^4+34p^3-8p^2+9p+1}{3(p-1)p\left(8p^4+4p^2-p+1\right)} \\ \dfrac{8p^5+40p^4+26p^3+36p^2-2p+12}{3\left(-8p^6+4p^4+p^3+3p^2-p+1\right)} \\ \dfrac{4p^4+16p^3+15p^2+12p+1}{3p\left(-2p^4-p^3+p^2+p+1\right)} \end{pmatrix}$$

Before we present the results for optimal stunted random walk strategy, we now come back to the two specials situations in which the chain is not absorbing. It is when the value for p is either 0 or 1. For $p=1$, it is quite obvious that if no agent moves, then we will never reach the equidistribution state. We then focus on the case of $p=0$. Let us see the value for the transition matrix in this case:

$$\mathbf{Q}(p=0)=\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

From the transition matrix above, we can see that the equidistribution state $\hat{s}(=s_6)$ is only accessible (not necessarily directly) from two states, s_2 and s_4, meaning that it is only possible to achieve dispersal if the agents start from state $s_2=[2,1,0]$ or $s_4=[1,2,0]$. From s_4, if all agents move, the one at the left end must move to the center. The two at the center equiprobably move to different ends (reaching state s_6) or to the same end (state s_2). We can now simplify the chain into the following diagram:

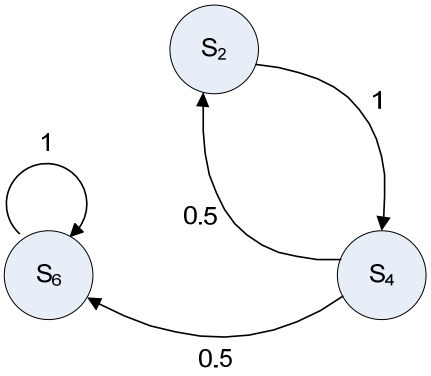

Figure 1: The state transition diagram for L_3 with. $p=0$.

Now we can calculate the expected time to reach s6 from those two states:

$$T(s_2,0)=1+T(s_4,0)$$

$$T(s_4,0)=\frac{1}{2}(2+T(s_4,0))+\frac{1}{2}\cdot 1$$

which gives the values $T(s_2,0)=4$ and $T(s_4,0)=3$. These values happen to be the optimal expected time from s_2 and s_4, as the expected times for other values of p are bigger than 4 and 3, respectively. In other words, for $i=2$ or 4, the minimising value of p is 1. For other values of i, we minimise the expressions for each row of the vector $\boldsymbol{\tau}$ with respect to p in (0,1).

We can now list our results on the optimal expected time using simple stunted random walk strategy as well as the accompanying p values (Table 1).

Table 1: Optimal expected times for simple stunted random walk on L_3.

Starting state	$\bar{T}$	$\bar{p}$
Random, r	4.2274	0.2332
s_1 = [3,0,0]	6.2125	0.2415
s_2 = [2,1,0]	4	0
s_3 = [2,0,1]	5.3333	0.5000
s_4 = [1,2,0]	3	0
s_5 = [0,3,0]	5.2836	0.3219

We can see that on average, the agents can achieve their goal the fastest by starting from state s_4, and setting their probability of staying still p as zero. It is unlikely, though, that setting $p=0$ would provide an optimal solution in larger line graphs. This occurrence is mainly due to the simplicity of the graph. Other solutions that give good values are from random placement $(T=4.2274)$ and when all agents start at the middle node $(T=5.2836)$. These values provide us with a benchmark for checking the validity of our simulation techniques for the L_3.

5.1.2 Compound Stunted Random Walk

In the compound stunted random walk model on L_3, we allow the stationarity probability (of remaining at the current node) to depend on the population of agents at that node. That is, if the total number of agents at your node is i, then stay at that node with probability p_i; otherwise move randomly to a neighbouring node. In this case we get a transition matrix $\mathbf{Q}$ whose elements are functions of the three probabilities p_1, p_2, and p_3. For example the transition probability from $[1,2,0]$ to $[1,1,1]$ is given by $p_1\left(2\,p_2\left(1-p_2\right)(1/2)\right)=p_1p_2\left(1-p_2\right)$. The agent who is alone at the left must stay still, which happens with probability p_1, and one of the two at the middle must move while the other one stays, which happens with probability $2\,p_2\left(1-p_2\right)$. Finally, the one who moves must move to the unoccupied end (probability ½ in a random walk). Then, using the same formula for $\boldsymbol{\tau}$ as in the previous section, we get dispersal times from state i as a function of p_1, p_2, and p_3. Minimising these five expressions (and an average expression for the top line corresponding to the random start r) with respect to varying all three probabilities gives the solution to the minimum problem as seen in Table 2. Note that except for the two states discussed separately above, we have that the optimal value of p_1 is 1. This says that if you are the only agent at your node, stay still. This is similar to a conjecture in Alpern and Reyniers (2002) for a related problem, that singleton agents should stay still, and other agents should not move to their locations.

Table 2: Optimal expected times for compound stunted random walk on L_3.

Starting state	$\bar{T}$	$\bar{p}_1$	$\bar{p}_2$	$\bar{p}_3$
Random, r	3.4006	1.0000	0.4646	0.1331
s_1 = [3,0,0]	5.3672	1.0000	0.4987	0.0831
s_2 = [2,1,0]	4	0	0	any
s_3 = [2,0,1]	2.7405	1.0000	0.5696	0.0537
s_4 = [1,2,0]	3	0	0	any
s_5 = [0,3,0]	4.3076	1.0000	0.4973	0.1425

5.2 Simulation Analysis for L_3

For line graphs L_n with large n $(n>3)$, the Markov solution technique is not feasible, so we will use simulation. To check the validity of the simulation technique, we apply it also to the solved case of L_3. We shall see that we get a close fit to the Markovian solution. For the simulation programme, we modify the state formulation, by numbering the three agents. The state $\langle i,j,k\rangle$ says that agent 1 is at node i, agent 2 is at node j and agent 3 is at node k. For example, state $\langle 1,3,2\rangle$. The first entry, 1, shows that the first agent is at location 1. The second entry, 3, means that the second agent is at location 3, and the final entry means the third agent is at location 2. So this state is actually one of our absorbing states,

in which every agent choose distinct locations. There are in total $3^3 = 27$ states in this formulation.

5.2.1 Simple Stunted Random Walk

We perform 50000 runs of simulation using simple stunted random walk strategies. We found the results are similar to the ones using exact method using Markov chain theory (see Table 3). We also show the comparison between the two results for the case of initial random placement, $T(r, p)$, on the graph in Figure 2. This graphs shows that our simulation provides outcomes which closely approximate the ones calculated using the exact method.

Table 3: Simulation results of optimal expected times, simple stunted random walk on L_3 (50000 runs).

Starting state	$\overline{T}$	$\overline{p}$
Random, r	4.1950	0.2350
$s_1 = [3,0,0]$	6.2000	0.2300
$s_2 = [2,1,0]$	3.9760	0.0600
$s_3 = [2,0,1]$	5.3110	0.4900
$s_4 = [1,2,0]$	2.9810	0.0400
$s_5 = [0,3,0]$	5.2220	0.3100

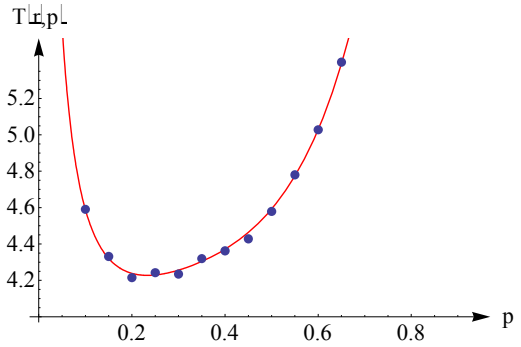

Figure 2: Comparison of the expected times between the two methods (exact method and simulation) for the case of random initial placement T(r,p).

5.2.2 Compound Stunted Random Walk

From Table 4 below, we can see that the simulation also gives results similar to the exact method shown in Table 2. In particular, simulation can handle both absorbing and non-absorbing chain in the same way, unlike in the exact method where we need to differentiate the approach between the two classes of Markov chain.

Table 4: Simulation results of optimal expected times, compound stunted random walk on L_3 (10000 runs).

Starting state	$\overline{T}$	$\overline{p}_1$	$\overline{p}_2$	$\overline{p}_3$
Random, r	3.3115	1.0000	0.4900	0.1200
$s_1 = [3,0,0]$	5.3159	0.9900	0.4900	0.1000
$s_2 = [2,1,0]$	3.9385	0.0000	0.0400	0.6100
$s_3 = [2,0,1]$	2.7392	0.9900	0.5400	0.0600
$s_4 = [1,2,0]$	2.9532	0.0000	0.0000	0.0600
$s_5 = [0,3,0]$	4.2666	0.9900	0.4700	0.1600

6 EXPECTED OUTCOME – FURTHER RESEARCH

There are several directions of further research to be carried out. The most obvious is to continue the investigation to larger line graphs, such as L_4, L_5, and eventually the class of L_n in general. We also plan to study other classes of graphs, in particular the class K_n of complete graphs and the class C_n of cycle graphs. As both are transitive, all initial placements of agents at a single node are equivalent, which will simplify the analysis.

A second direction of research is to analyse how strategies work on unknown graphs (mazes) or random graphs. When sending in robots to search a large house whose interconnecting room structure is unknown, what is a good stationary probability p to program them with?

Another possibility is the manager of the agents observes the state and then broadcasts a 'state-dependent' value of p or of the vector **p**.

Finally, we could give our agents some limited eyesight, so for example they might know the current population of all nodes adjacent to their own. The walk they choose in this case need not be random, as neighboring nodes are distinguished by their populations. Presumably one would go more likely to a less populated node.

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