High-Order Analytical Solution of Relative Motion Equation for Satellite Formation Flying in Elliptical Orbit

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Abstract: The paper studied relative motion equation for satellite formation flying with large separations. The configuration is traditionally designed by the periodic solutions of the C-W equation in circle reference orbit or Lawden equation in elliptic reference orbit. Hence, the linear solutions are more suitable for the configuration with small scale formation than large scale formation. However, in some specific situations, it is necessary to use satellites with large separations. Then the paper studied relative motion based on the nonlinear equations in an elliptic reference orbit. The solution is expanded as series form with respect to the eccentricity of the reference orbit, in-plane amplitude and out-of-plane amplitude. Taking the Lawden periodic solution as starting point, the high-order analytical solution is constructed by Lindstedt-Poincare method. Particularly, as the eccentricity is zero, the analytical solution degenerated to express the relative motion in circle reference orbit. Finally, the practical convergence of the analytical solution is discussed in order to examine its validity and applicability.

1 INTRODUCTION

Multiple satellites form a certain configuration to complete a common scientific exploration or task, the mode of operation is called satellite formation flying. Due to its potential technical advantages and widely applications, the study of satellite formation flying is more and more active in astronautics field. In fact, the research on configuration is the essential and important fundamental theory among formation flying technics, such as the dynamics, the capture, maintenance and reconfiguration issues of satellite formation flying.

A successful satellite formation flying relies on the technic of configuration. For example, to sustain the formation, the economic fuel consumption is related to the accuracy of the configuration. Generally, researchers used the linear Hill equation or C-W equation, which is linearized the relative motion equation in a circular orbit. The analytical solution can be directly derived. It is easy to be used to design the configuration of satellites formation. However, the analytical solutions of C-W equation are not capable to solve the problem as satellite's formation has a large separations. Therefore, the paper will generate high-order analytic solution in an elliptic reference orbit based on the non-linear relative motion equation. The aim is to provide the precise mathematic theory for the configuration,capture, reconfiguration, and maintenance of satellite formation flying with a large baseline in an elliptical orbit.

Relative motion problems can be considered as a degenerate case of restricted three-body system, that is, it corresponds to the specific situation when the mass parameter of the three body system μ is 0. Thus, the methods and results under the restricted three-body system can be naturally applied to solve the solution of the relative motion equations. In fact, restricted three-body problem has abundant dynamical properties, such as three collinear libration points which are collinear with the main celestial have instability dynamics quality. However, in most cases, the other two triangular libration points which constitutes an equilateral triangle with main celestial are stable. Numerous researchers studied analytical solutions of relative equation for satellite formation flying both in an elliptical reference orbit and in a circle reference orbit. Richardson applied Lindstedt-Poincar'e (L-P) method to derive the third-order analytic solution of Halo orbit near collinear libration points under the circular restricted three-body system (Richardson.(1980)). Taking the analytical solution as the

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initial solution, more accurate numerical solution of Halo orbit can be acquired by the differential correction method. Jorba and Masdemont expand Lissajous orbit and Halo orbit near collinear libration points as series form of in-plane amplitude and out-of-plane amplitude. And they calculate coefficients corresponding to every order solution (Jorba.(1999)).Since the collinear libration points have unstable dynamic properties, Masdemont expanded invariant manifolds near libration points into series form of four amplitude parameters. Two of amplitudes corresponds to hyperbolic manifolds, the other two amplitudes are corresponding to the center manifold (Masdemont.(2005)). Taking into account the stable dynamics properties of triangular libration points, the motion in the vicinity of the triangular libration points can be expanded to series form with respect to the long period amplitude, short period amplitude and vertical cycle amplitude, under the circle restricted three-body system. Lei and Xu derive arbitrary high-order analytical solution which can provide a precise mathematical tool for designing the missions near triangular libration points (Lei.(2013)). For the relative motion equation in the circle reference orbit, Richardson and Mitchell use L-P method to acquire a third-order analytical solution (Richardson.(2003)). Thereafter, G'omez and Marcote expand it to series solution form of in-plane amplitude and out-of-plane amplitude. It also used L-P method to bring arbitrary high-order analytic solution (Gomez.(2006)). Ren et al derived the third-order analytic solutions in the elliptical reference orbit, and they explained how to generate arbitrary high-order analytical solution (Ren.(2012)). In this paper, the arbitrary high-order analytical solution is generated by L-P method for relative motion of satellite formation flying with large baselines. And the follower satellite's relative motion can be expressed by series form related to orbital eccentricity e, in-plane amplitude α and out-of-plane amplitude β.

From the elliptic restricted three-body problem, the paper deduced the elliptical relative motion equation in the orbital coordinate. In the satellite formation flying problem, the location of the chief corresponds to a libration point under degradation restricted three-body system. Hence, the substantive issue of configuration problem is to study dynamics near libration points in the restricted three-body problem. With our existed work under the restricted three-body system, in the orbital coordinate, the follower satellite's motion related to the chief satellite can be expanded as series form of the orbital eccentricity e, the in-plane amplitude α and out-of-plane amplitude β. The arbitrary-order analytical solution can be generated by the L-P method. Finally, we calculate the convergence domain of high-order analytical solution. In one hand, it verifies the validity of the analysis solution. On the other hand, it gives the scope of the analytical solution according to the specific accuracy requirements.

The rest of paper is organized as follows. The dynamic model of the relative motion is introduced in section 2. Based on the model and used L-P method, the detailed procedures solving analytical solution is described in the section 3. Section 4 analyzed the results of the solutions. Finally, some brief conclusions were given in the last section.

2 THE ELLIPTICAL RELATIVE MOTION EQUATION

Let us derive the elliptical relative motion equation corresponding to an elliptical orbit from the perspective of the elliptic restricted three-body system. The leader satellite P_1 is orbiting the Earth as an eccentricity e elliptical motion. The follower satellite P_0 is flying in the gravitational field produced by the Earth and the leader satellite. Its motion is independent of both the Earth and the leader satellite's motion, such a system is called an elliptic restricted three-body system. The mass of the Earth, the chief satellite and the follower satellite are respectively noted m_0, m_1, m_2. In the study, the dimensionless units are usually adopted. The unit of mass is taken as the mass of the Earth and the chief satellite, noted by $[M] = m_0 + m_1$; and Units of length are the instantaneous distance between the leader satellite and the earth, which is described as: $[L] = a(1 - e^2/(1 + e\cos f))$, in which a is the orbit semi-major axis, f is the chief satellite's true anomaly; to make the gravitational constant equal to 1, $G = 1$, the time dimension is taken as:$[T] = \sqrt{[L]^3/(G[M])}$. In the dimensionless system, the mass of the chief satellite is expressed as $\mu = m_1/(m_0 + m_1)$. Then the mass of the Earth is expressed as $1 - \mu$.

For convenience, the coordinate origin located at the centroid O of the Earth and the chief star. The unit vector x is directed from the Earth to the chief satellite. Z is positive in the direction of the instantaneous angular momentum vector. Axis-Y is determined by the right-handed coordinate system. The coordinate system is called the barycenter synodic system O-XYZ (see Fig. 1). In the barycenter synodic system, the earth located at $(\mu, 0, 0)$ and the follower satellite located at $(1 - \mu, 0, 0)$. The state vector is described as $(x, y, z, x^{'}, y^{'}, z^{'})$, where $*^{'}$ is $'*'$ partial derivatives of true anomaly f, noted $*^{'} = \frac{d*}{df}$. The dimensionless relative motion equation of the follower satellite is as

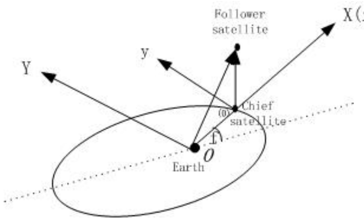

Figure 1: Schematic diagram of barycenter synodic system O-XY and orbital coordinate system o-xy.

follow:(Szebehely.(1967))

$$\begin{cases} \ddot{X} - 2\dot{Y} = \dfrac{1}{1+e\cos f}\dfrac{\partial\Omega}{\partial X} \\ \ddot{Y} + 2\dot{X} = \dfrac{1}{1+e\cos f}\dfrac{\partial\Omega}{\partial Y} \\ \ddot{Z} + Z = \dfrac{1}{1+e\cos f}\dfrac{\partial\Omega}{\partial Z} \end{cases} \tag{1}$$

In Eq. 1, Ω is the potential function,

$$\Omega = \frac{1}{2}(X^2+Y^2+Z^2) + \frac{1-\mu}{R_1} + \frac{\mu}{R_2}. \tag{2}$$

R_1 and R_2 are distances from P_2 to P_0 and from P_2 to P_1 respectively,

$$\begin{cases} R_1 = \sqrt{(X+\mu)^2+Y^2+Z^2} \\ R_2 = \sqrt{(X-1+\mu)^2+Y^2+Z^2} \end{cases} \tag{3}$$

Eq.1 has five libration points, three of them are collinear libration points located in the axis-X, denoted by L_i, i = 1, 2, 3. The other two libration points constitute equilateral triangle with the Earth and the chief satellite. They are called triangle libration points, denoted by L_4 and L_5.

In particular, for satellite formation flying problem, the mass of the chief satellite and the follower satellite is approximately the same. However, their masses are far less than the mass of the Earth. Therefore, $\mu \to 0$. At the same time, in elliptic restricted three-body problem, the collinear libration points L_1 and L_2 degenerate to a point, coinciding with the position of the chief satellite. In this case, the follower satellite 's relative motion converts to study the motion near collinear libration points corresponding to $\mu = 0$, in the elliptical restricted three-body problem. It usually uses orbital coordinate system o-xyz (see Fig. 1). The direction of axis is consistent with the barycenter synodic system. The state vector is denoted by (x, y, z, x', y', z'). In the barycenter synodic system, the chief satellite is located at $(1,0,0)$, then the coordinate transformation from orbital coordinate system to the barycenter synodic system is expressed as:

$$x = X-1, y = Y, z = Z. \tag{4}$$

In orbital coordinate system, the follower satellite's relative motion is written as:

$$\begin{cases} \ddot{x} - 2\dot{y} = \dfrac{1}{1+e\cos f}\dfrac{\partial\Omega}{\partial x} \\ \ddot{y} + 2\dot{x} = \dfrac{1}{1+e\cos f}\dfrac{\partial\Omega}{\partial y} \\ \ddot{z} + z = \dfrac{1}{1+e\cos f}\dfrac{\partial\Omega}{\partial z} \end{cases} \tag{5}$$

where Ω is ,

$$\Omega = \frac{1}{2}(x^2+y^2+z^2) + \frac{1-\mu}{R_1} + \frac{\mu}{R_2}. \tag{6}$$

r_1 is the distance between the follower satellite and the Earth,

$$r_1^2 = (x+1)^2 + y^2 + z^2. \tag{7}$$

Using the following formula, the relative motion equation can be expressed as a polynomial form of x, y and z.

$$\frac{1}{\sqrt{(x-A)^2+(y-B)^2+(z-C)^2}} = \frac{1}{D}\sum_{n\geq 0}(\frac{\rho}{D})^n P_n(\frac{Ax+By+Cz}{D\rho}). \tag{8}$$

In Eq. 8, $D^2 = A^2+B^2+C^2$, $\rho^2 = x^2+y^2+z^2$, P_n is Legendre polynomial, then $\frac{1}{r_1}$ can be written as:

$$\frac{1}{r_1} = \frac{1}{\sqrt{(x+1)^2+y^2+z^2}} = \sum_{n\geq 0}\rho^n P_n(\frac{-x}{\rho}). \tag{9}$$

Setting $T_n = \rho^n P_n(-\frac{x}{\rho})$, $R_{n-1} = \frac{1}{y}\frac{\partial T_{n+1}}{\partial y}$, the following relations can be described as:

$$\frac{\partial T_{n+1}}{\partial x} = -(n+1)T_n, R_{n-1} = \frac{1}{z}\frac{\partial T_{n+1}}{\partial z}. \tag{10}$$

T_n and R_n has the following recurrence relation,

$$\begin{cases} T_n = \dfrac{1-2n}{n}xT_{n-1} - \dfrac{n-1}{n}\rho^2 T_{n-2} \\ R_n = -\dfrac{2n+3}{n+2}xR_{n-1} - \dfrac{2n+2}{n+2}T_n - \dfrac{n+1}{n+2}\rho^2 R_{n-2} \end{cases} \tag{11}$$

The initial values are $T_0 = 1$, $T_1 = -x$, $R_0 = -1$, $R_1 = 3x$. Substituting $\frac{1}{1+e\cos f} = \sum_{i\geq 1}((-e)^i \cos f^i)$ and Eq. 9 into Eq. 5, the relative motion equation in orbital coordinate system can be rewritten as:

$$\begin{cases} \ddot{x}-2\dot{y}-3x=\sum_{i\geq 1}(3x\cdot(-e)^i\cos f^i) \quad (12) \\ -\sum_{i\geq 0}((-e)^i\cos f^i)[\sum_{n\geq 2}(n+1)T_n] \\ \ddot{y}+2\dot{x}=\sum_{i\geq 0}((-e)^i\cos f^i)[y\sum_{n\geq 2}R_{n+1}] \\ \ddot{z}+z=\sum_{i\geq 0}((-e)^i\cos f^i)[z\sum_{n\geq 2}R_{n+1}] \end{cases} \tag{13}$$

3 HIGH-ORDER ANALYTICAL SOLUTION OF ELLIPTICAL RELATIVE MOTION EQUATION

3.1 High-order Expansion of Relative Motion

Linearizing the relative motion Eq. 13 in orbital coordinate, the following equations can be derived:

$$\begin{cases} \ddot{x}-2\dot{y}-3x=3\sum_{i\geq 1}(x\cdot(-e)^i\cos f^i) \\ \ddot{y}+2\dot{x}=0 \\ \ddot{z}+z=0 \end{cases} \tag{14}$$

The Eq.14 is the Lawden equation. Its periodic solution is the Lawdwn solution, which is expressed as:

$$\begin{cases} x(f)=\alpha\cos\theta_1+\frac{1}{2}\alpha e\cos(f-\theta_1)+\frac{1}{2}\alpha e\cos(f+\theta_1) \\ y(f)=-2\alpha\sin\theta_1-\frac{1}{2}\alpha e\sin(f+\theta_1) \\ z(f)=\beta\cos\theta_2 \end{cases} \tag{15}$$

In Eq.15, α and β are in-plane amplitude and out-of-plane amplitude respectively. θ_1 and θ_2 are phase angles. θ_{10} and θ_{20} are initial phase angles.

$$\theta_1=f+\theta_{10},\theta_2=f+\theta_{20}. \tag{16}$$

Considering the perturbation of the non-linear term, the relative motion can be expanded as series solution form, which is related to eccentricity, in-plane amplitude, and out-of-plane amplitude.

$$\begin{cases} x(f)=\sum_{i,j,k,l,m,n}x_{ijk}^{lmn}\cos(lf+m\theta_1+n\theta_2)e^i\alpha^j\beta^k \\ y(f)=\sum_{i,j,k,l,m,n}y_{ijk}^{lmn}\cos(lf+m\theta_1+n\theta_2)e^i\alpha^j\beta^k \\ z(f)=\sum_{i,j,k,l,m,n}z_{ijk}^{lmn}\cos(lf+m\theta_1+n\theta_2)e^i\alpha^j\beta^k \end{cases} \tag{17}$$

In Eq.17, α and β are in-plane amplitude and out-of-plane amplitude respectively. Phase angles are: $\theta_1=wf+\theta_{10}$ and $\theta_2=wf+\theta_{20}$. $i,j,k\in\mathbb{N},l,m,n\in\mathbb{Z}$, and l, m, n have the same parity as i, j, k. In the process of constructing the analytic solution, we only need to consider coefficients that are satisfied with following conditions: $\mid l\mid\leq i,\mid m\mid\leq j$ and $\mid n\mid\leq k$. As taking into account the symmetry of positive cosine function, supposing $l\geq 0$; if l =0, then $m\geq 0$; if l=m=0, then $n\geq 0$. Similarly, the frequency can be expanded as series solution in the form of the eccentricity and the amplitudes.

$$w=\sum_{i,j,k}w_{ijk}e^i\alpha^j\beta^k \tag{18}$$

Only when i, j, k are all even, the frequency coefficient w_{ijk} is non-zero. Based on the Lawden solution Eq.15 , the arbitrary high-order solution of relative motion can be constructed by Lindstedt-Poincare method.

The series solution Eq.17 is the analytical solution of the elliptical relative motion equation. If $\alpha\neq 0 and \beta=0$, Eq.17 describes the plane periodic configuration. If $\alpha=0 and \beta\neq 0$, it corresponds to the vertical periodic configuration. Then if $\alpha\neq 0 and \beta\neq 0$, it depicts the general periodic configuration. Especially, if e=0, Eq.17 may degenerate to describe the relative motion in circular reference orbit. In the process of constructing the analytical solution, we define two orders, represented by (N_1,N_2). $N_1=i$ is the order corresponding to eccentricity, and $N_2=j+k$ is the order corresponding to the amplitude. The total order of the solution is $N=N_1+N_2$. The order of Lawden periodic solution is noted by (n_1,n_2), in which $n_1=1,n_2=1$, its corresponding known coefficients can be expressed in detail as:

$$\begin{cases} x_{010}^{010}=1.0,x_{110}^{i-10}=0.5,x_{110}^{110}=0.5 \\ y_{010}^{010}=-2.0,y_{110}^{110}=-0.5 \\ w_{010}^{010}=1.0,w_{000}=1.0 \end{cases} \tag{19}$$

On the basis of Eq.19, the coefficients $x_{ijk}^{lmn},y_{ijk}^{lmn},z_{ijk}^{lmn}$ and w_{ijk} corresponding to the high-order solution can be solved by L-P method.

Coordinate system relating to the first and second order partial derivatives of the true anomaly is:(take component x as an example)

$$\begin{cases} \dot{x}=\frac{\partial x}{\partial f}+w\frac{\partial x}{\partial\theta_1}+w\frac{\partial x}{\partial\theta_2} \quad (20) \\ \ddot{x}=\frac{\partial^2 x}{\partial f^2}+w^2\frac{\partial^2 x}{\partial\theta_1^2}+w^2\frac{\partial^2 x}{\partial\theta_2^2}+2w\frac{\partial^2 x}{\partial fw\theta_1}+2w\frac{\partial^2 x}{\partial f\partial\theta_2} \\ +2w\frac{\partial^2 x}{\partial\theta_1\partial\theta_2} \end{cases} \tag{21}$$

To construct the analytical solution, it needs to distinguish known items from unknown items in the Eq.13. For instance, the component x, its known items of (n_1, n_2) order solution involves three parts. The first part is corresponding to the right part of the relative motion equation. The second part is from $\dot{y}$ items of the left part corresponding to the relative motion equation. The last part stems from $\ddot{x}$ items corresponding to the left part of the relative motion equation. Merge all known items into the right part of the relative motion equation, denoted by X_{ijk}^{lmn}. Similarly, the known items of y and z corresponding to the relative motion equation are denoted by Y_{ijk}^{lmn} and Z_{ijk}^{lmn}. Unknown coefficients of (n_1, n_2) order solution includes (n_1, n_2) order coordinate coefficients $x_{ijk}^{lmn}, y_{ijk}^{lmn}, z_{ijk}^{lmn}$, and $(n_1, n_2 - 1)$ order frequency coefficients w_{ijk}.

In the relative motion equation corresponding to x component, unknown items corresponding to (n_1, n_2) order solution origin from $\ddot{x}$, $\dot{y}$ and x. And they are respectively expressed as follows:

$$\begin{cases} \ddot{x} \to -(l + mw_0 + nw_0)^2 x_{ijk}^{lmn} - 2w_0 w_{ij-lk} \delta_{l0} \delta_{m1} \delta_{n0} \\ \dot{y} \to (l + mw_0 + nw_0) y_{ijk}^{lmn} - 2w_{ij-lk} \delta_{l0} \delta_{m1} \delta_{n0} \\ x \to x_{ijk}^{lmn}, i = n_1, j + k = n_2 \end{cases} \tag{22}$$

In the relative motion equation corresponding to y component, unknown items are from $\ddot{y}$, $\dot{x}$, they are respectively written as:

$$\begin{cases} \ddot{y} \to -(l + mw_0 + nw_0)^2 y_{ijk}^{lmn} + 4w_0 w_{ij-lk} \delta_{l0} \delta_{m1} \delta_{n0} \\ \dot{x} \to -(l + mw_0 + nw_0) x_{ijk}^{lmn} - w_{ij-lk} \delta_{l0} \delta_{m1} \delta_{n0} \end{cases} \tag{23}$$

In the relative motion equation corresponding to z component, unknown items are from $\ddot{z}$, z, they are respectively written as:

$$\begin{cases} \ddot{z} \to -(l + mw_0 + nw_0)^2 z_{ijk}^{lmn} - 2w_0 w_{ij-lk} \delta_{l0} \delta_{m1} \delta_{n0} \\ \dot{z} \to z_{ijk}^{lmn}, i = n_1, j + k = n_2 \end{cases} \tag{24}$$

The L-P method requires that the same order item must be equal on both sides of the equation. Therefore, the equation of the unknown coefficients can be established as:

$$\begin{cases} A_1 x_{ijk}^{lmn} + B_1 y_{ijk}^{lmn} + C_1 w_{ij-lk} \delta_{l0} \delta_{m1} \delta_{n0} = X_{ijk}^{lmn} \\ A_2 x_{ijk}^{lmn} + B_2 y_{ijk}^{lmn} + C_2 w_{ij-lk} \delta_{l0} \delta_{m1} \delta_{n0} = Y_{ijk}^{lmn} \\ (-\psi^2 + 1) z_{ijk}^{lmn} - 2w_0 w_{ij-lk} \delta_{l0} \delta_{m1} \delta_{n0} = Z_{ijk}^{lmn} \end{cases} \tag{25}$$

In Eq. 25, δ_{ij} is the Dirac symbol. If i=j, then $\delta_{ij} = 1$, otherwise, $\delta_{ij} = 0$. Denoted $\psi = l + mw_0 + nw_0$, $A_i, B_i and C_i, i = 1, 2$ are:

$$\begin{cases} A_1 = -(\psi^2 + 3), B_1 = -2\psi, C_1 = (4 - 2w_0) \\ A_2 = -2\psi, B_2 = -\psi^2, C_2 = (4w_0 - 2) \end{cases} \tag{26}$$

3.2 High-order Solution for Solving the Corresponding Coefficient

In Eq. 25, four unknown coefficients need to be solved. For (n_1, n_2) order solution, three sub-equations all contain unknown frequency coefficient w_{ijk}, $j + k = n_2 - 1$. Thus, the equation corresponding to z component should be solved firstly. Then, calculate coefficients corresponding to z coordinate and frequency coefficients. At last, substituting the frequency coefficients into the equation corresponding to x-y component in order to solve coordinate coefficients corresponding to x and y.

3.2.1 Solving the Equation Corresponding to z Component

Case 1: If $|\psi| = 1$

Case 1.1: If l=0, m=0, n=1, in this case, $-\psi^2 + 1 = 0$. Setting $z_{ijk}^{lmn} = 0$, it can be solved that $w_{ijk-1} = -\frac{Z_{ijk}^{lmn}}{2}$.

Case 1.2: others, setting z_{ijk}^{lmn}.

Case 2: if $|\psi| \neq 1$, in this case, the unique unknown coefficient is z_{ijk}^{lmn}, then $z_{ijk}^{lmn} = \frac{z_{ijk}^{lmn}}{-\psi^2+1}$.

3.2.2 Solving the Equation Corresponding to x-y Component

Since w_{ijk}^{lmn}, $j + k = n_2 - 1$ is known, the unknown coefficients are only x_{ijk}^{lmn} and y_{ijk}^{lmn} , and they are solved in the following cases:

Case 1: if $|l| = 1$

Case 1.1: If $\psi = 1$, setting $x_{ijk}^{lmn} = 0$, $y_{ijk}^{lmn} = -\frac{X_{ijk}^{lmn}}{2}$.

Case 1.2: If $\psi = 0$, setting $x_{ijk}^{lmn} = 0$, $y_{ijk}^{lmn} = 0$, then $x_{ijk}^{lmn} = -\frac{X_{ijk}^{lmn}}{3}$.

Case 1.3: others, the unknown coefficients x_{ijk}^{lmn} and y_{ijk}^{lmn} are satisfied with the following equation:

$$\begin{cases} -(\psi^2 + 3) x_{ijk}^{lmn} - 2\psi y_{ijk}^{lmn} = X_{ijk}^{lmn} \\ -2\psi x_{ijk}^{lmn} - \psi^2 y_{ijk}^{lmn} = Y_{ijk}^{lmn} \end{cases} \tag{27}$$

Case 2: if l=0

Case 2.1: If m=1, n=0,Setting $x_{ijk}^{lmn} = 0$ in order to solve y_{ijk}^{lmn}. (Or setting $y + ijk^{lmn} = 0$ in order to calculate x_{ijk}^{lmn}). Taking the former case, then

$$y_{ijk}^{lmn} = -\frac{[X_{ijk}^{lmn} - (4-2w_0)w_{ij-lk}]}{2}. \qquad (28)$$

Case 2.2: if $m+n=0$, in this case, y_{ijk}^{lmn} are coefficients of $\sin(0)$. Setting $y_{ijk}^{lmn} = 0$, then $x_{ijk}^{lmn} = -\frac{X_{ijk}^{lmn}}{3}$.

Case 2.3: if $\mid m+n \mid = 1$ and $m \neq 1$, setting x_{ijk}^{lmn}, then $y_{ijk}^{lmn} = -\frac{X_{ijk}^{lmn}}{2(m+n)}$.

Case 2.4: others, the unknown coefficients satisfy the following equation:

$$\begin{cases} -((m+n)^2+3)x_{ijk}^{lmn} - 2(m+n)y_{ijk}^{lmn} = X_{ijk}^{lmn} \\ -2(m+n)x_{ijk}^{lmn} - (m+n)^2 y_{ijk}^{lmn} = Y_{ijk}^{lmn} \end{cases} \qquad (29)$$

In summary, the corresponding coordinates and frequency coefficients of arbitrary high-order analytical solution can be solved.

4 RESULTS

Using the Fortran program, the arbitrary high-order solution is constructed, and coefficients of the $(3,3)$ order solution is listed in Table 1. It showed that all of frequencies coefficients are zero except $\omega_{000} = 1$. That is, frequency of the relative motion are neither related to the reference orbital eccentricity nor the amplitude of the formation. The frequency is always 1.0, indicating that the relative motion is a periodic configuration of 2π. As mentioned, Eq.17 can describe the general periodic configuration in the vicinity of the chief satellite. Taking $(7,10)$ order solution as an example, Fig.2 shows the plane periodic orbits with $(e = 0.1)$, $\beta = 0$, $\alpha = 0.1, 0.2, 0.3 adim$. Fig.3 shows the vertical periodic orbits with $(e = 0.1)$, $\alpha = 0$, $\beta = 0.1, 0.2, 0.3 adim$. Fig.4 shows the periodic orbit with $e = \alpha = \beta = 0.1$. As $e = 0$, the solution 17 can describe the periodic orbit corresponding to circular reference orbit. Fig.5 showed plane periodic orbits with e=0, $\alpha = 0.2, \beta = 0$,computed by series expansions up to order $(0,15)$. Fig. 6 showed vertical periodic orbits with $e = 0, \alpha = 0, \beta = 0.2$,computed by series expansions up to order (0,15). Fig. 7 showed the periodic orbits with e=0, $\alpha = 0.2, \beta = 0.2$,computed by series expansions up to order (0,15). In all figures,

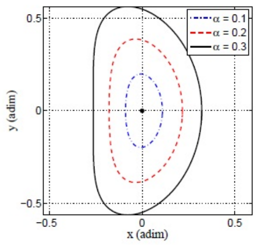

Figure 2: Plane periodic orbits with $e = 0.1$, $\alpha = 0.1, 0.2, 0.3, \beta = 0$,computed by series expansions up to order (7,10).

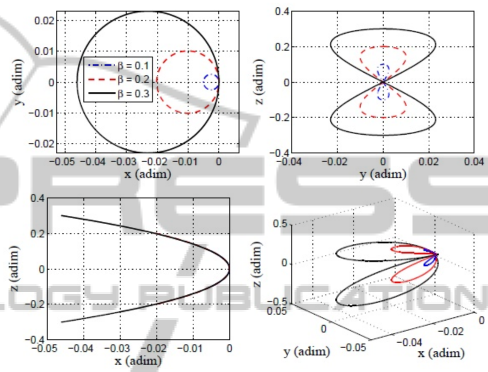

Figure 3: vertical periodic orbits with $(e = 0.1)$, $\alpha = 0$, $\beta = 0.1, 0.2, 0.3 adim$, computed by series expansions up to order (7,10).

the given 'adim' are dimensionless unit of length that is instantaneous distance between the chief satellite and the Earth.

In addition, combining numerical integration methods, the scope and the convergence range of analytic solution is illustrated. In the analytical solution Eq.17 has three parameters , namely the reference orbital eccentricity e, the in-plane amplitude α and the out-of-plane amplitude β. If e is fixed (i.e e = 0.1) , the paper studied the convergence range of in-plane amplitude and out-of-plane amplitude. The results of

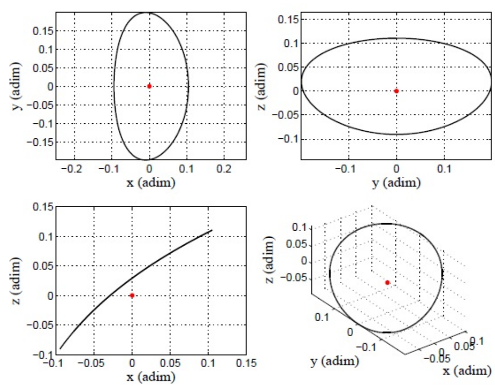

Figure 4: Periodic orbits with $e = \alpha = \beta = 0.1$, computed by series expansions up to order (7,10).

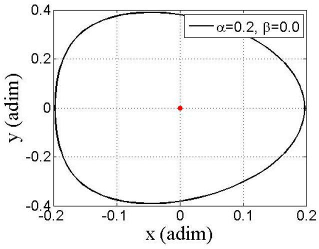

Figure 5: Plane periodic orbits with e=0, $\alpha = 0.2, \beta = 0$,computed by series expansions up to order (0,15).

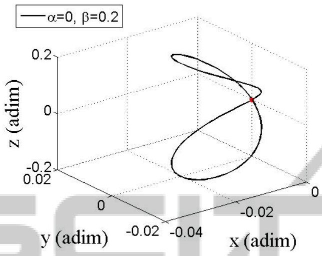

Figure 6: Vertical periodic orbits with e=0, $\alpha = 0, \beta = 0.2$,computed by series expansions up to order (0,15).

(5,5) order and (7,10) order analytical solutions have been displayed. In the computing process, the max in-plane amplitude and the max out-of-plane amplitude are both 0.3 adim. In the region $[0,\alpha_{max}] \times [0,\beta_{max}]$, it is uniformly divided into 100×100 grids. Each grid referred to a set (α,β), represented as (i, j), and the corresponding set of amplitude is denoted by $(\alpha_i,\beta_j), i = 1,2,...100; j = 1,2,...100$. For each (i, j), the state vectors can be obtained by analytical solution Eq.17. Taking the state vectors as the initial value, the relative motion Eq.5 integrate may de integrated through the integrator RKF78 from $f = 0$ to $f = 2\pi$. The position vector corresponding to $f = 2\pi$ is represented as X_N. Meanwhile, the analytical solution 17 can directly derive the position vector, denoted by X_A. Thus , as $f = 2\pi$, the position difference from the numerical solution to analytic solution is denoted by $d_{ij} =| X_A - X_N |$. If the difference is smaller, it means that the analytical solution is more accurate. Fig.8 and Fig. 9 illustrate the convergence range of $(5,5)$ order and $(7,10)$ order respectively. In both figures, the precision coordinates are $\ln d_{ij}$. Obviously, if the order is larger, the convergence range of analytical solution is greater.

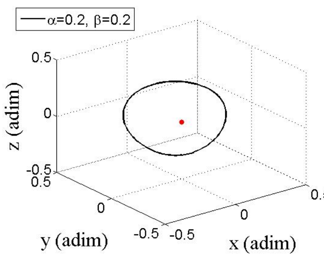

Figure 7: The periodic orbits with e=0, $\alpha = 0.2, \beta = 0.2$,computed by series expansions up to order (0,15).

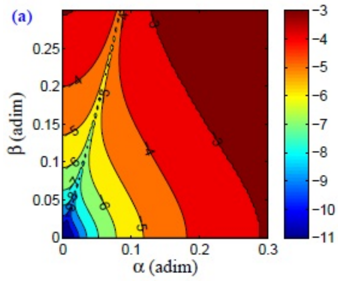

Figure 8: The practical convergence of the high-order analytical solutions of the equations of relative motion with elliptic reference orbit, corresponds to the analytical solutions truncated at order (5,5).

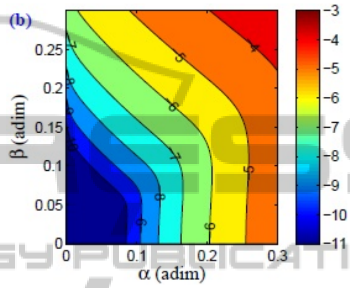

Figure 9: The practical convergence of the high-order analytical solutions of the equations of relative motion with elliptic reference orbit,corresponds to the analytical solutions truncated at order (7,10).

Due to computational efficiency and problem of computer memory, the constructed order of analytical solution is limited. However, taking into account the e, α and β and appropriate adjusting n_1 and n_2 in the order (n_1,n_2), the optimal configuration can be acquired according to specific accuracy requirements. For example, constructing analytical solution of circular orbit, n_1 does not contributed to the accuracy since $e = 0$. Supposing $n_1 = 0$, the higher order analytical solution can be constructed by choosing higher n_2 order.

5 CONCLUSIONS

The paper firstly derived the relative motion equation in an elliptical reference orbit under restricted three-body problem. Then the relative motion equation is linearized in order to obtain the Lawden and Lawden periodic solution. Taking into account the nonlinear term of the relative motion, the motion in the vicinity of the chief satellite is expanded as series form of orbital eccentricity, the in-plane amplitude and out-of-plane amplitude. Adopting Lawden solution as the initial solution, the arbitrary high-order analytical solution is generated by using Lindstedt-Poincar'e method. From results, it can be concluded that the frequency of periodic configuration is always equal to 1.0, that is, the orbital period of the relative motion is always 2π. It can be applied to the control and reconfiguration of the satellite formation flying with large baseline.

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APPENDIX

Table 1: The coefficients of coordinate series of the analytical solution constructed up to order $(3,3)$.

i	j	k	l	m	n	x_{ijk}^{lmn}	y_{ijk}^{lmn}	z_{ijk}^{lmn}
0	0	1	0	0	1	0.0000	0.0000	1.0000
0	1	0	0	1	0	1.0000	-2.0000	0.0000
0	0	2	0	0	2	-0.2500	0.2500	0.0000
0	0	2	0	0	0	-0.2500	0.0000	0.0000
0	1	1	0	1	1	0.0000	0.0000	-0.5000
0	1	1	0	1	-1	0.0000	0.0000	1.5000
0	2	0	0	2	0	0.5000	0.2500	0.0000
0	2	0	0	0	0	-0.5000	0.0000	0.0000
0	1	2	0	1	2	0.1250	-0.1250	0.0000
0	1	2	0	1	-2	0.0000	0.3750	0.0000
0	2	1	0	2	1	0.0000	0.0000	0.3750
0	3	0	0	3	0	-0.3750	-0.2917	0.0000
0	3	0	0	1	0	0.0000	1.1250	0.0000
1	1	0	1	1	0	0.5000	-0.5000	0.0000
1	1	0	1	-1	0	0.5000	0.0000	0.0000
1	2	0	1	2	0	0.1250	0.0000	0.0000
1	2	0	1	0	0	0.0000	0.7500	0.0000
1	1	2	1	1	0	0.2500	-0.2500	0.0000
1	1	2	1	1	-2	0.3125	0.0000	0.0000
1	1	2	1	-1	2	0.2500	-0.2500	0.0000
1	1	2	1	-1	0	0.2500	0.0000	0.0000
1	1	2	1	-1	-2	0.0625	0.0625	0.0000
1	2	1	1	2	1	0.0000	0.0000	0.0625
1	2	1	1	2	-1	0.0000	0.0000	0.3125
1	2	1	1	0	1	0.0000	0.0000	0.5000
1	2	1	1	0	-1	0.0000	0.0000	-1.5000
1	2	1	1	-2	1	0.0000	0.0000	-0.3750
1	2	1	1	-2	-1	0.0000	0.0000	0.1250
1	3	0	1	3	0	-0.1667	-0.1042	0.0000
1	3	0	1	1	0	-0.7500	0.0000	0.0000
1	3	0	1	-1	0	0.6875	0.0000	0.0000
1	3	0	1	-3	0	-0.1458	0.0417	0.0000
2	2	0	2	-2	0	0.1250	0.0000	0.0000
2	2	0	0	2	0	0.0625	-0.0625	0.0000
2	2	0	0	0	0	-0.0625	0.0000	0.0000
2	3	0	2	3	0	-0.0208	-0.0104	0.0000
2	3	0	2	1	0	-0.1250	0.0000	0.0000
2	3	0	2	-1	0	0.0000	-0.1875	0.0000
2	3	0	0	3	0	-0.313	0.0104	0.0000
2	3	0	0	1	0	0.0000	-0.1875	0.0000
3	3	0	3	-3	0	0.0208	0.0000	0.0000
3	3	0	1	1	0	-0.0938	0.0938	0.0000
3	3	0	1	-1	0	-0.1094	0.0000	0.0000
3	3	0	1	-3	0	0.0052	0.0052	0.0000