Adaptive Gauss Hermite Filter for Parameter and State Estimation of Nonlinear Systems

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Abstract: This paper presents an adaptive Gauss Hermite filter for nonlinear signal models in the situation when the measurement noise statistics is unknown. The proposed nonlinear filter, based on the Gauss Hermite quadrature rule, can ensure satisfactory estimation performance despite the problem of unknown measurement noise statistics by online adaptation. Results of Monte Carlo Simulation demonstrate the efficacy of the proposed filter for joint estimation of parameters and states using an object tracking problem.

1 INTRODUCTION

Optimal filtering and estimation require knowledge about the covariances of process and measurement noise (Simon, 2006). Estimation performance is known to deteriorate when such noise covariances are unknown. One solution to overcome the above problem is to use adaptive estimators. In this paper, an yet unreported adaptive sigma point filter has been proposed for nonlinear systems where the measurement noise covariance is unknown.

The estimator proposed here is based on the Gauss Hermite quadrature rule (Ito, 2000; Arasaratnam, 2007) and belongs to the family of sigma point filters. Sigma point filters (Lefebvre, 2004) are derivative free filters and had been widely reported in literature on nonlinear estimation as these filters overcome the well known shortcomings of the Extended Kalman Filter (EKF). Despite the extensive computation effort Gauss Hermite filters (GHF) stand out in certain situations in comparison to Unscented Kalman filters (UKF), Central Difference filter (CDF) (Ito, 2000) and simulation based filters like Particle filters (Arasaratnam, 2007).

Even this sophisticated filtering algorithm fails to provide accurate estimation results in the face of unknown noise covariances as discussed before. This paper attempts to overcome this limitation proposing an adaptive Gauss Hermite quadrature filter which has been developed by incorporating adaptation steps in the framework of Gauss Hermite quadrature filter. The adaptation steps in the proposed filter employ "covariance matching method" as inspired from the adaptive linear filters (Mehra, 1972; Maybeck, 1982; Myer, 1976). Like the cited previous work the proposed method also makes use of the statistics of 'innovation' (defined as the difference between the a priori estimate of measurement and the actual measurement) sequence for adaptation. Unlike (Myer, 1976), in the work of (Mehra, 1972; Maybeck, 1982) the algorithm has been made computationally more efficient by eliminating the need to use previous history of a priori error covariance.

Adaptive nonlinear filters like adaptive EKF (Busse, 2003), adaptive UKF (Das, 2013) or other adaptive derivative free sigma point filters like adaptive Divided Difference filter (adaptive DDF) (Karlgaard, 2011) are also reported in the literature.

In general, adaptive filters are categorized into two classes depending on the adaptation of the process noise covariance (Q-adaptive filter), or the measurement noise covariance (R-adaptive filter). As the present work is based on measurement noise adaptation (R-adaptation) the literature review focuses on R adaptive nonlinear filters. The R adaptive UKF based on innovation sequence by (Das, 2013) reports direct adaptation of R while R adaptive UKF by (Hajiyev, 2014) prefers scaling factor based adaptation, an equally accepted method of adaptation. A Robust adaptive DDF presented in (Karlgaard, 2011) emphasizes on robustness in presence of outliers and also adapts the unknown

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noise covariances using the innovation based Q and R adaptation.

The proposed filter uses Gauss Hermite quadrature rule for evaluation of the integrals encountered in nonlinear Bayesian filtering problem (Ito, 2000) and also incorporates the steps for R adaptation. Only the R adaptive version of adaptive GHF based on innovation sequence is reported here. The Q adaptive version (addresses the complementary problem of R adaptation) was proposed by the present authors in (Dey, 2014) for joint estimation of states and parameters.

The adaptive GHF, which has not yet been reported in the recent literature to the best knowledge of the authors, has the following advantages:
(i) Like other derivative free filters the proposed filter replaces computation of Jacobian and Hessian matrices by some functional evaluations, (ii) It has been demonstrated in (Ito, 2000) that Gauss Hermite quadrature filters provide better estimation performance compared to UKF and CDF for certain nonlinear systems and such advantages are expected to be inherited by its adaptive version. It has also been reported in (Arasaratnam, 2007) that in certain situations Gauss Hermite quadrature filters can ensure estimation accuracy comparable to that of much more computationally intensive simulation based filters like Particle filters, (iii) Being a direct quadrature formula, the proposed filter does not need the discerning choice of tuning parameters like the UKF.

The proposed filter is evaluated with the help of two case studies. The case studies which use a benchmark nonlinear estimation problem and a well known ballistic object tracking problem demonstrate that the proposed filter is capable of joint estimation of parameters and states.

2 ADAPTIVE GAUSS HERMITE FILTER

2.1 Problem Statement

We consider nonlinear dynamic equations of a system as given below

$$x_k = f(x_{k-1}) + \theta_k \tag{1}$$

$$y_k = g(x_k) + \upsilon_k \tag{2}$$

where $x_k \in R^n$ is a state vector, $y_k \in R^m$ is output vector. The zero mean process and measurement noises (assumed Gaussian) are denoted as $\theta_k \in R^n \sim (0,Q), \upsilon_k \in R^m \sim (0,R_k)$. The process noise covariance is a known constant matrix. However, the measurement noise covariance being unknown it is to be adapted at every time instants.

2.2 Filter Algorithm

For the above described estimation problem, the algorithm of Adaptive Gauss Hermite filter is presented below.

$\bar{x}_k$ is a priori estimate, P_k^- is a priori error covariance, $\hat{x}_k$ is a posteriori estimate, P_k^+ is a posteriori error covariance.

Step (i) Initialization:

$\hat{x}_0, P_0^+, Q, \hat{R}_0$

Step (ii) Computation of Quadrature Points and corresponding weights:

- Compute J, a symmetric tri-diagonal, defined as $J_{i,i} = 0$ and $J_{i,i+1} = \sqrt{\frac{i}{2}}$ for $1 \le i \le N-1$ with N-quadrature points.
- The quadrature points are chosen as $q_i = \sqrt{2}x_i$ where x_i are the eigen values of J matrix.
- The corresponding weight (w_i) of q_i is computed as $|(v_i)_1|^2$ where $(v_i)_1$ is the first element of the ith normalized eigenvector of J.

Step (iii) Gauss Hermite Quadrature Rule:

Following Gauss Hermite Quadrature Rule, $I_N = \int_{R^n} \tilde{F}(s) \frac{1}{(2\pi)^{n/2}} e^{-(1/2)|s|^2} ds$ can be equivalently expressed as

$$I_N = \sum_{i_1=1}^{N} \dots \sum_{i_n=1}^{N} \tilde{F}(q_{i_1}, q_{i_2}, \dots, q_{i_n}) w_{i_1} w_{i_2} \dots w_{i_n} \tag{3}$$

In order to evaluate I_N for n^{th} order system, N^n number of quadrature points and weights are required.

Step (iv) Time update step:

Compute the Cholesky Factor

$$S_x^+(k-1) = CholeskyFactor(P_{k-1}^+) \tag{4}$$

Select quadrature points as

$$\chi_i^+ = S_x^+(k-1) q_i + \hat{x}_{k-1} \tag{5}$$

$$\bar{x}_k = \sum_{i=1}^{N} f(\chi_i^+) w_i \tag{6}$$

$$P_k^- = Q + \sum_{i=1}^{N} \left(f(\chi_i^+) - \bar{x}_k\right)\left(f(\chi_i^+) - \bar{x}_k\right)^T w_i \tag{7}$$

Step (v) Measurement update step:
Compute the Cholesky Factor

$$S_x^-(k) = Cholesky\ Factor\left(P_k^-\right) \tag{8}$$

Select sigma points as $\chi_i^- = S_x^-(k) q_i + \bar{x}_k$ (9)

A priori estimate of measurement becomes

$$z_k = \sum_{i=1}^{N} h(\chi_i^-) w_i \tag{10}$$

The following covariance can be computed as -

$$P_k^{xz} = \sum_{i=1}^{N} \left(\chi_i^- - \bar{x}_k\right)\left(g(\chi_i^-) - z_k\right)^T w_i \tag{11}$$

$$P_k^{zz} = \sum_{i=1}^{N} \left(g(\chi_i^-) - z_k\right)\left(g(\chi_i^-) - z_k\right)^T w_i \tag{12}$$

Step (vi) R – Adaptation:
Compute the innovation sequence as

$$\vartheta_k = y_k - z_k \tag{13}$$

The estimated innovation covariance can be computed from a sliding window of epoch length L. using (14)

$$\hat{C}_k^{\vartheta} = \frac{1}{L} \sum_{j=k-L+1}^{k} \vartheta(j) \vartheta^T(j) \tag{14}$$

The adapted R is computed using (15)

$$\hat{R}_k = \hat{C}_k^{\vartheta} - P_k^{zz} \tag{15}$$

Step (vii) step for computation of filter gain and a posteriori estimates:

$$K_k = P_k^{xy}\left(P_k^{zz} + \hat{R}_k\right)^{-1} \tag{16}$$

$$\hat{x}_k = \bar{x}_k + K_k \vartheta_k \tag{17}$$

$$P_k^+ = P_k^- - K_k\left(P_k^{zz} + \hat{R}_k\right)K_k^T \tag{18}$$

Step (viii) Recursion:
Starting from *k=1* the steps from (i) to (vii) are repeated for subsequent time instants.

2.3 Notes on the Algorithm

- Though the proposed algorithm considers the additive noise, the extension to the more general cases is straight forward.
- The adaptation step is executed before computation of filter gain so that adapted $\hat{R}_k$ of current instant can be incorporated for computation of filter gain, a posteriori state estimate and a posteriori error covariance.
- It is to be noted that the innovation sequence from a sliding window has been employed for computation of estimated innovation covariance, which subsequently computes the adapted *R*.
- The window length or epoch length is a parameter which needs experimentation. A large choice of window size smoothens the estimate of *R* at the cost of computational burden and low tractability. A small choice of window length is appropriate to track the short term variation in *R* but makes the filter prone to divergence.
- It is to be also noted that until the step index *k* is less than epoch length *L*, the adapted *R* is computed based on available size of innovation sequence (length *k*). Afterwards *R* is adapted from sliding window as given in (14).

3 CASE STUDY-1

State estimation of a single dimensional system with a considerably strong nonlinearity has been chosen in this section. The nonlinear system possesses two stable equilibrium points at 1, –1 and an unstable equilibrium point at 0. The measurement equation, having a weak bi-modal tendency, fails to distinguish between the stable equilibrium points decisively. The problem is well known for its ability to detect the limitation of estimators if any. Improper tuning of the filter because of unknown noise statistics may consequently enforce the estimates to settle at the wrong equilibrium point. In context of this problem, the comparison of the adaptive and the non adaptive Gauss Hermite is justified when the measurement noise covariance is unknown.

3.1 System Dynamics

The system dynamics and the measurement equations are presented in this section. The system dynamics, taken from (Ito, 2000), is given below.

$$x_k = \phi(x_{k-1}) + \theta_k \tag{19}$$

$$\phi(x) = x + 5\tau x(1 - x^2) \tag{20}$$

θ_k is an additive Gaussian noise, $\theta_k \sim N(0, b^2\tau)$.The measurement equation presented in (Sadhu, 2007) has been considered as it has weaker bimodal tendency compared to (Ito, 2000).

$$y_k = \gamma(x_k) + \upsilon_k \tag{21}$$

$$\gamma(x) = \tau x(1 - 0.5x) \tag{22}$$

υ_k is an additive measurement noise(Gaussian), $\upsilon_k \sim N(0, d^2\tau)$. The parameters used to generate the true state trajectory have the values as given below. $\tau = 0.01$ sec, $x_0 = -0.2$, $b = 0.5$, $d = 0.1$. For the filter, the initial values are chosen as $\hat{x}_0 = 0.8$, $P_0^+ = 2$, $Q = 0.25\tau$. However, measurement noise covariance is unknown to the filter. We initialize both the filters assigning an arbitrary choice of measurement noise which is thousand times greater than the truth value to induce sufficient uncertainty. The window length is considered to be 100.

3.2 Simulation Results

- The tracking performance of both adaptive and non adaptive Gauss Hermite filter (both of them have 5 quadrature points) for a representative run is presented by Figure 1. It is observed that, although initialized with an arbitrary initial choice of measurement noise covariance with large error, the proposed AGHF can track the true trajectory. The non adaptive GHF, however, loses the track and get settled at the wrong equilibrium point.
- The error settling performance of both the filter is compared from the results of Monte Carlo study with 10,000 runs. The RMS errors of both the filters are represented by Figure 2. The results indicate that the RMS error of AGHF is much lower than that of the non adaptive GHF. This also signifies numerous occurrence of track loss in case of non adaptive GHF.
- Figure 3 illustrates the adapted R obtained from the adaptive GHF for a representative run. It is observed that the adapted R converges and successfully tracks the truth value the truth value for subsequent time instants.

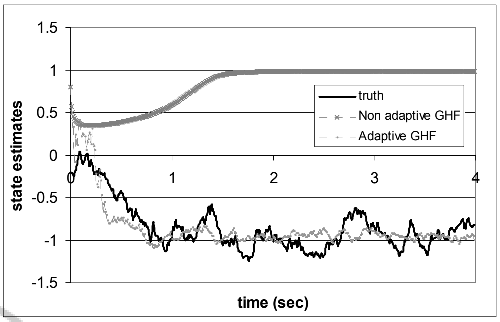

Figure 1: Performance comparison of Adaptive and Non adaptive GHF for a representative run.

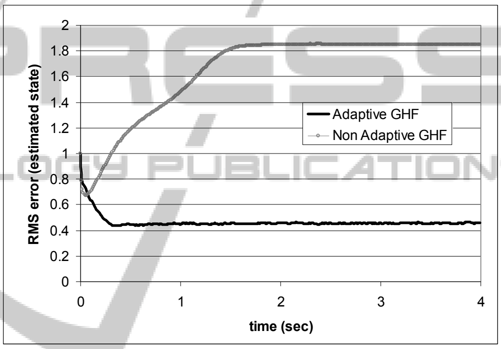

Figure 2: RMS error plot of Adaptive and Non adaptive GHF for 10,000 MC runs.

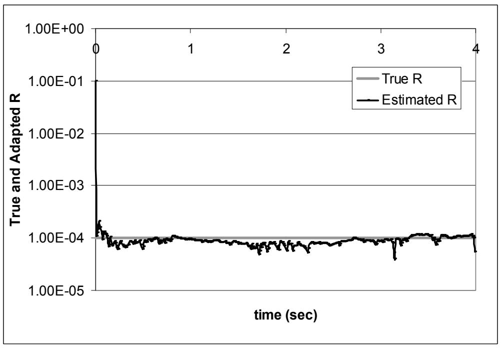

Figure 3: Plot of estimated measurement noise covariance (R) for a representative run.

4 CASE STUDY-2

Suitability of the AGHF for joint estimation of parameters and states is demonstrated with the help of a well known problem of ballistic object tracking during re-entry. The object is considered to be tracked by a radar with range only measurement.

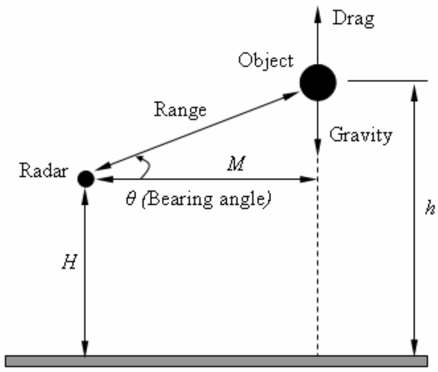

Figure 4: Radar Tracking of a ballistic object during re-entry: A schematic diagram.

4.1 System Dynamics

This section presents the dynamic model of the object during re-entry. As the drag force becomes pronounced during endo-atmospheric phase the dynamics becomes highly nonlinear. The effect of gravity is assumed to be negligible compared to drag force as reported in (Athans, 1968).

The dynamic model is given by

$$\dot{h} = -V \tag{23}$$

$$\dot{V} = -\frac{C_D A \rho(h) V^2}{2m} \tag{24}$$

Below are given the details of the symbols often encountered in this particular section:

h :altitude of the object (ft), V : object velocity (ft/sec), C_D : drag coefficient (dimensionless), A : reference area for drag evaluation (sq. ft), ρ :air density(slug/ft^3), m : mass of ballistic object(slug)

Air density varies exponentially with altitude following a model $\rho(h) = \rho_0 e^{-\gamma h}$ with $\gamma = 5\times10^{-5}$ ft^{-1}. On contrary of the ballistic coefficient, a ballistic parameter $\xi \overset{\Delta}{=} \frac{C_D A \rho_0}{2m}$, reportedly defined in (Athans, 1968), is considered as a parameter to estimate. However, the ballistic coefficient is usually defined as $\beta = \frac{mg}{C_D A}$ and related with the ballistic parameter as $\xi = \frac{\rho_0 g}{2\beta}$.

For estimation of ballistic parameter, it is augmented with state vector and modelled as a constant. The differential equation of object dynamics is modified as given below:

$$\dot{h} = -V \tag{25}$$

$$\dot{V} = -e^{-\gamma h} V^2 \xi \tag{26}$$

$$\dot{\xi} = 0 \tag{27}$$

Using Euler's approximation with a sampling time τ the corresponding discrete state space model of object is obtained (Ristic, 2003).The kinematic states of the ballistic object and the ballistic parameter are perturbed with additive process noise w_k (Gaussian).

The discrete time model is given by:

$$x_k = f(x_{k-1}) + w_k \tag{28}$$

$f(x_{k-1})$ indicates the discrete nonlinear model for system dynamics.

$$f(x_{k-1}) = \varphi x_{k-1} - G[D(x_{k-1})] \tag{29}$$

Here, $\varphi = \begin{bmatrix} 1 & \tau & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $x_{k-1} = \begin{bmatrix} h_{k-1} & V_{k-1} & \xi_{k-1} \end{bmatrix}^T$ and $G = \begin{bmatrix} 0 & \tau & 0 \end{bmatrix}^T$. The drag experienced by the objectis defined by

$$D(x_{k-1}) = \exp(-\gamma x_{k-1}^T e_1)(x_{k-1}^T e_2)^2 \left(x_{k-1}^T e_3\right) \tag{30}$$

where e_i denotes the i^{th} unit vector.

The process noise covariance of w_k is considered as $Q = \begin{bmatrix} q_1\tau^3/3 & q_1\tau^2/2 & 0 \\ q_1\tau^2/2 & q_1\tau & 0 \\ 0 & 0 & q_2\tau \end{bmatrix}$ where q_1 and q_2 are parameters for describing the process noise as given in (Ristic, 2003). w_k is independent of measurement noise v_k.

The range measured by the radar has a nonlinear measurement equation. The interval of measurement is same as sampling interval, i.e., τ sec. For this problem τ is considered to be equal to 0.1sec.

$$y_k = \sqrt{M^2 + (x_k^T e_1 - H)^2} + v_k \tag{31}$$

Here, $e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, represents the unit vector. H is the altitude of radar and M is the shortest horizontal distance from the path of the ballistic object during re-entry as shown in the Figure 4. v_k indicates zero mean Gaussian noise with an unknown noise covariance R_k.

To generate the true state trajectories of object, the truth value of initial kinematic states and truth value of the ballistic parameter are chosen following (Norgaard, 2000) as specified in Table 1. As for the filter necessary parameters are also provided in the same table. The truth value of R_k being unknown to the filter, it is deliberately assigned with an arbitrary value which has wide range of uncertainty ($R_{filter} = R_{true} \times 100$).

4.2 Simulation Results

A comparative study between the adaptive and the non adaptive GHF is carried out using Monte Carlo simulation in the situation with unknown measurement noise statistics. Both the filters are initialized with an arbitrary choice of R which is hundred times higher than the value of true R. However, knowledge of process noise covariance, Q, is considered to be known to both the filters. The performance is evaluated analysing the RMS errors obtained from both adaptive and non adaptive GHF. From Figure 5, Figure 6 and Figure 7 it has been observed that, for all the states and the parameter, RMS errors of the adaptive GHF converged quickly to a lower value compared to the non adaptive GHF.

Table 1: Numerical values and description of the parameters used in simulation.

Symbols	Value	Description
x_0	$[300000\ 20000\ 10^{-3}]^T$	Initial value for true trajectories
q_1	$5\ ft^2s^{-3}$	A parameter of true Q
q_2	$10^{-12} ft^{-2}s^{-1}$	A parameter of true Q
M	100000 ft	Horizontal distance of object from radar
H	100000 ft	Height of the radar
R_{true}	$100^2\ ft^2$	Measurement noise covariance
P_0^+	$diag\ (10^6, 4\times10^6, 10^{-4})$	Initial a posteriori error covariance
$\hat{x}_0$	$N(x_0, P_0^+)$	Initialization of filter estimates.
L	100	Actual window length

This indicates that the AGHF can adapt the unknown measurement noise covariance and produces more reliable estimation than non adaptive GHF in case of the joint estimation of parameters and states.

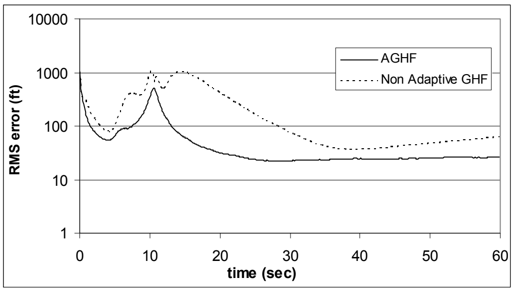

Figure 5: Comparison of RMS error (altitude estimation) of AGHF & GHF for 1000 MC runs.

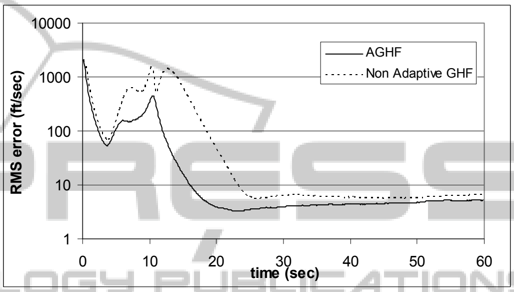

Figure 6: Comparison of RMS error (velocity estimation) of AGHF & GHF for 1000 MC runs.

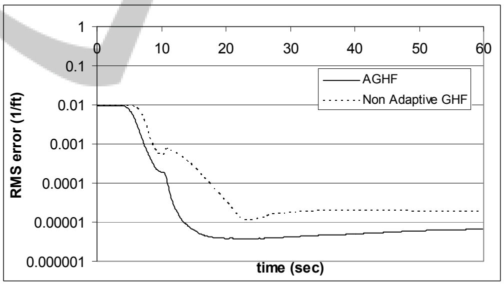

Figure 7: Comparison of percentage of RMS error (ballistic parameter estimation) of AGHF & GHF for 1000 MC runs.

5 CONCLUDING DISCUSSIONS

An adaptive Gauss Hermite filter has been proposed and evaluated with different bench mark nonlinear estimation problems. It can be inferred from the results of the Monte Carlo simulation that the adaptive GHF can successfully adapt the unknown measurement noise covariance and presents substantially improved estimation performance over the non adaptive filter for a wide range of initial choice of measurement noise covariance. The suitability of the proposed filter for joint estimation of parameters and states of nonlinear systems is also demonstrated using a well known object tracking problem.

As the non adaptive GHF reportedly excels other non adaptive sigma point filters like UKF and CDF, the performance of the proposed adaptive GHF has been compared with its non adaptive version.

In the absence of analytical proof of convergence, each adaptive nonlinear filter, including the proposed one, are to be thoroughly evaluated with the help of extensive simulation studies or real time experiments in several fields of application before such filtering techniques may be widely applied in practice with confidence.

However, the proposed filter may be recommended for state and parameter estimation of nonlinear systems because of its improved estimation performance, good convergence, simple adaptation rule, capacity to accommodate wide uncertainty in the initial choice of measurement noise covariance.

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