Existence of Fractional Solutions in NTU DEA Game

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Abstract: This paper deals with the problem of fairly allocating a certain amount of benefit among individuals or organizations with multiple criteria for their performance evaluation. It is an extension work of our paper on game theoretic approaches to weight assignments in data envelopment analysis (DEA) problems (Sekine et al., 2014). One of the main conclusions in our previous work is that the core of the TU (transferable utility) DEA game is non-empty if and only if the game is inessential, that is, the evaluation indices are identical for all the criteria for each player. This condition is equivalent to a trivial single-criterion setting, which motivates us to turn to the NTU (non-transferable utility) situation and check the existence of the fractional solutions. In this study, we contribute on showing the existence of α-core, and giving two sufficient conditions such that β-core exists and is identical to α-core in NTU DEA game. Our discussion is also interesting in light of a direction to improve the robustness of the β-core existence condition by relaxing the inessential condition in TU DEA game.

1 INTRODUCTION

The problem dealt in this paper is the consensus-making among individuals or organizations in sharing a fixed amount of benefit with multiple criteria for evaluating their performance. DEA problem originates from the bankruptcy problem by (O'Neil, 1982), which has been subsequently analyzed in a variety of contexts. Among them the multi-issue allocation game by (Calleja et al., 2005) has the most similar context to our DEA game. The example given in their paper is

"The central government has to decide how to allocate the taxpayers' money to various public services. The system of government is such that it does not allocate this money directly to these services, but indirectly through various government departments. Each department (player) has a number of claims on the amount of money available, arising from those public services (issues) for which it has responsibility. Some of these services are provided by just a single department (i.e., tax collection vs. the Department of Finance), while more departments may be responsible for other services (i.e., foreign trade vs. the Department of Economic Affairs, Foreign Affairs and Defense). If we were to add up all the claims of a department into one single claim, an ordinary bankruptcy problem would arise."

As argued by (Calleja et al., 2005), the underlying issues should play a role in determining an outcome. If the departmental claims are combined, however, this crucial information is lost. DEA game differs from the multi-issue allocation game in two aspects: first, what a player directly claims is a strategy based on his performance on different criteria, instead of an amount of the benefit available. Here a criterion is the standard to evaluate the performance of a player, in other words, it is an attribute of a player's performance. Whilst in the multi-issue allocation game, an issue is an act a player involved in and gives rise to a claim. Second, with the concept of DEA, the allocation of the benefit is determined endogenously based on all the players' strategy combination. Let us explain it this way. Each player is empowered to vote for the importance of different criteria (strategy), and our mechanism determines the allocation by referring to this voting result (strategy combination). It is more "democratic" and can potentially increase the perceived fairness of the players on the allocation result. On the contrary, the basic assumption in the multi-issue allocation game is that once the central organization is paying out money according to one particular issue, this issue must first be fully dealt with before moving on to the next issue. Hence they have

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Table 1: Pollution Emission Level Comparison between IGCC and Conventional PCT.

	SO_X	NO_X	Particulate
IGCC [1]	1.0 ppm	3.4 ppm	$< 0.1\ mg/m^3N$
PCT [2]	$< 17.7\ ppm$	$< 49.4\ ppm$	$< 20\ mg/m^3N$

[1] Joban Joint Power: http://www.joban-power.co.jp/igccdata/en/
[2] Shenhua Guohua Beijing Cogeneration: http://www.shghrd.com/HistoryData.aspx

to define the allocation rule either by a proportional game or a queue game exogenously.

To illustrate our model, consider the following case. According to a statement released after the State Council's executive meeting presided by Premier Li Keqiang, Chinese central government has laid out a number of detailed measures that identify large cities and regions with the most frequent smog and haze as key areas in the battle against air pollution. A special fund of 10 billion yuan ($1.65 billion) is set up in 2014 to reward efforts to curb air pollution in the key areas. Suppose the central government is providing a fixed amount of clean air fund (benefit) and calls for proposals from both domestic and overseas organizations. Successful applicants (players) have advantages in different criteria, namely, the sustainability indicators. For example, applicant A (i.e., Joban Joint Power, Japan) owns IGCC (integrated gasification combined cycle) technology, which is a type of gasification technology that turns coal into gas, and can potentially improve the efficiency of coal-fired power compared to conventional pulverized coal technology (PCT) as well as the environmental performance (Table 1). Applicant B (i.e., Shenhua Guohua Beijing Cogeneration, China) has advantage in good source of clean coal with low sulfur content and low installation cost of new coal-fired power plant. The central government has to decide a reasonable allocation of the fund to these successful applicants, which should reach the consensus among them. In DEA game, as assumed by (Nakabayashi and Tone, 2006), the successful applicants are egoistic and each of them sticks to his superiority regarding the sustainability indicators, i.e., efficiency, cleaner resource, pollution emission, installation cost, maintenance cost and etc.

The process above is similar to a two-stage tendering with technical and financial proposals submitted separately. In the first stage, organizations A, B, C ... submit their technical proposals without fund claim, in accordance with the specifications by the central government and their respective performance on different criteria. The technical proposals are evaluated and successful applicants are announced. The second stage is to invite the successful applicants to vote for the importance of all the criteria. The fund allocation is endogenously determined by DEA game with reference to this voting result. Our paper focuses on the second stage.

In the literature, DEA is generally introduced as a mathematical programming approach for measuring relative efficiencies of decision-making units (DMUs, in DEA game we use "player" instead), where multiple inputs and multiple outputs are present. The interest of this paper can be best justified by the latest technical note by (Cook et al., 2014), they emphasized that

"Ultimately DEA is a method for performance evaluation and benchmarking against best-practice. The inputs are usually the 'less-the-better' type of performance measures and the outputs are usually the 'more-the-better' type of performance measures. This case is particularly relevant to the situations where DEA is employed as a MCDM (multiple criteria decision making) tool. DEA then can be viewed as a multiple-criteria evaluation methodology where DMUs are alternatives, and DEA inputs and outputs are two sets of performance criteria where one set (inputs) is to be minimized and the other (outputs) is to be maximized."

In the case above, the central government has taken the trouble to apply multiple criteria to determine the allocation of the fund, as a single criterion is hardly to become an acceptable basis of a fair judgment. Different interest group, either a single applicant, or a coalition, has different opinion on the importance of each criterion for estimating a reasonable allocation. Hence DEA is employed as a MCDM tool, where the DMUs are the successful applicants, and output-to-input ratio is the relative contribution of one DMU compared to the total contribution of all DMUs. Here "contribution" is equivalent to "performance", which is quantitatively measured by the score given to each DMU by different criteria. This representation, presented in the next section, is not a typical efficiency measure within the context of maximizing the ratio of weighted outputs to weighted inputs subject to constraints reflecting the performance of the other DMUs; whilst it is the ratio of the fund each DMU may receive by integrating the performance of other DMUs in the objective function. (Nakabayashi et al., 2009) justified this representation by addressing the fundamental concept of DEA–variable weights. The weights are derived directly from the existing data and they are chosen in a manner that assigns a best set of weights to each DMU. The term "best" means that the resulting output-to-input ratio for each DMU is maximized relative to all other DMUs when these weights are assigned to these outputs and inputs for every DMU.

(Nakabayashi and Tone, 2006) first proposed DEA game to solve this kind of multi-criteria benefit allocation problem. Their DEA max game was, however, subadditive. Namely, players gain less in total, or lose their power when they cooperate. To make the game superadditive, they took the dual of the game, called DEA min game, in which each player or each coalition picks up the weight that minimizes their evaluation. No reasonable justification was given in their paper for picking up the minimizing weight under the assumption that players are egoistic and want to maximize their own evaluation. (Sekine et al., 2014) have improved the DEA game above by reassigning the total weights for the coalition members, and studied different solution concepts of TU DEA game and the equilibria of the strategic form DEA game. In particular, we have proved that the core of the TU DEA game is non-empty if and only if the game is inessential, which is equivalent to a trivial single-criterion situation as the inessentiality requires the evaluation indices for all the criteria to be identical for each player. Hence the core of the TU DEA game is generally empty, and this extension paper turns the focus to the NTU coalitional game associated with the strategic form DEA game (NTU DEA game for short).

An NTU coalitional game is a specification of payoff vectors attainable by members of each coalition through some joint course of action. A classical example is an exchange economy, where coalitions can reach certain payoff distributions that constitute the feasible set for that coalition by pooling and redistributing their initial endowments. The players confront the problem of choosing a payoff or solution that is feasible for the group as a whole. This paper explores two fractional solutions for the NTU DEA game, namely, the α-core and β-core.

The rest of the paper is organized as follows. In section 2, we review the basic DEA game scheme including the DEA model in (Nakabayashi and Tone, 2006), and the strategic form DEA game in (Sekine et al., 2014). Section 3 defines the NTU DEA game in both α and β fashion. Section 4 gives our main analysis results regarding the existence of α-core and β-core, and section 5 discusses a direction to improve the robustness of β-core existence condition. Finally, section 6 closes this paper with some concluding remarks.

2 A REVIEW ON THE DEA GAME SCHEME

This section reviews the DEA game scheme before we proceed to the formulation of NTU DEA game.

2.1 The DEA Model

Let $E(>0)$ denote the fixed amount of benefit to be allocated to players $1,\ldots,n$. Players' contributions are evaluated by multiple criteria and summarized as the score matrix $C=(c_{ij})_{i=1,\ldots,m,\ j=1,\ldots,n}$, where c_{ij} is player j's contribution measured by criterion i, called the evaluation index. The problem is to find a weight vector on the criteria (strategy) determined endogenously by players themselves, and reasonable allocations of E based on the weight vector combinations (strategy combinations). Following the DEA analysis, each player k chooses an nonnegative weight vector $w^k=(w_1^k,\ldots,w_m^k)$ such that $\sum_{i=1}^m w_i^k=1,\ w_i^k\geq 0\ \forall i=1,\ldots,m$, where w_i^k is the weight given to criterion i by player k. Then the contribution of player k relative to the total contribution of all players measured by the weight vector w^k is given by

$$\frac{\sum_{i=1}^m w_i^k c_{ik}}{\sum_{i=1}^m w_i^k(\sum_{j=1}^n c_{ij})}$$

Player k chooses the weight vector that maximizes this ratio. The weight vector is found by solving the following fractional program

$$\max_{w^k} \frac{\sum_{i=1}^m w_i^k c_{ik}}{\sum_{i=1}^m w_i^k(\sum_{j=1}^n c_{ij})}$$
$$s.t.\ \sum_{i=1}^m w_i^k=1,\ w_i^k\geq 0\ \forall i=1,\ldots,m$$

Each of the other players similarly maximizes the ratio produced by his own weight vector. This representation deviates from the output-to-input ratio in the traditional DEA approach, as the weighted sum of player k's contribution is hardly to say as "output", and the weighted sum of all players' total contributions is not a typical "input" as well. However, it is classical in nature as each egoistic player is trying to present to the authority that he contributes more or performs better than other players under his own valuation system, in order to win the consensus of the authority and others that his ability and effort worth more benefit. Hence player k's contribution is the "more-the-better", whereas the total contribution of all players is the "less-the-better", consistent with the argument by (Cook et al., 2014).

This maximization problem can be reformulated as the following much simpler form. For each row $(c_{i1},\ldots,c_{in})$, $i=1,\ldots,m$, divide each element by the row-sum $\sum_{j=1}^n c_{ij}$. Let

$$c'_{ij}=c_{ij}/\sum_{j=1}^n c_{ij}\quad i=1,\ldots,m$$

The matrix $C' = (c'_{ij})_{i=1,\dots,m,\ j=1,\dots,n}$ is called the normalized score matrix and $\sum_{j=1}^{n} c'_{ij} = 1$ is satisfied. By Charnes-Cooper transformation (Charnes et al., 1978), the maximization problem is not affected by this operation. Then the fractional maximization program above can be expressed as the following linear maximization program.

$$\max_{w^k} \sum_{i=1}^{m} w_i^k c'_{ik}$$

$$s.t. \sum_{i=1}^{m} w_i^k = 1,\ w_i^k \geq 0\ \forall i = 1,\dots,m$$

Let $c(k)$ be the maximal value of the program. Apparently the maximal value is attained by letting $w_{i(k)}^k = 1$ for the criterion $i(k)$ such that $c'_{i(k)k} = \max_{i=1,\dots,m}\ c'_{ik}$ and letting $w_i^k = 0$ for all other criteria $i \neq i(k)$. Thus $c(k)$ is the highest relative contribution of player k, namely

$$c(k) = \max_{i=1,\dots,m}\ c'_{ik}$$

For simplicity, we assume that the score matrix is given in the normalized form. That is, $C = (c_{ij})_{i=1,\dots,m,\ j=1,\dots,n}$, where $\sum_{j=1}^{n} c_{ij} = 1\ \forall i = 1,\dots,m$; $c_{ij} \geq 0\ \forall i = 1,\dots,m,\ \forall j = 1,\dots,n$.

(Nakabayashi and Tone, 2006) showed that if each player k claims the portion $c(k)$ of E, the sum of the claims generally exceeds the total benefit E. Then the problem arises: how to allocate E reasonably to players? To find a fair allocation of E, they proposed to apply cooperative game theory. As is mentioned above, we have improved their DEA game in (Sekine et al., 2014). Let us first review the definition of the strategic form DEA game, and then proceed to introduce the NTU DEA game.

2.2 Strategic Form DEA Game

Let $N = \{1,\dots,n\}$ be the set of players and $M = \{1,\dots,m\}$ be the set of criteria. The basic DEA model is as follows. Each player $j \in N$ chooses a weight vector $w^j = (w_1^j,\dots,w_m^j)$ with $w_1^j + \dots + w_m^j = 1, w_i^j \geq 0\ \forall i \in M$ on the criteria so as to maximize his relative contribution, $\sum_{i=1}^{m} w_i^j c_{ij}$. The fixed amount of benefit E is shared by the players in accordance with their relative contribution.

Therefore the strategic form game reflecting the DEA model can be defined as

$$G^{DEA} := (N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$$

where $N = \{1,\dots,n\}$ is the set of players, $W^j = \{w^j = (w_1^j,\dots,w_m^j) | w_1^j + \dots + w_m^j = 1, w_i^j \geq 0\ \forall i \in M\}$ is the strategy set of player $j \in N$, and $f^j : W = W^1 \times \dots \times W^n \to \Re$ is the payoff function of player $j \in N$, which is given by

$$f^j(w^1,\dots,w^n) = (\sum_{i=1}^{m}(\frac{1}{n}\sum_{j=1}^{n} w_i^j)c_{ij})E$$

Namely, the reward E is shared by players in proportion to the weighted sum of their evaluation indices where the weights are the average weights over all players. Suppose that there are 4 out of 5 players receiving the highest evaluation on criterion $i(k)$. Obviously egoistic them will all put their whole weight on $i(k)$, and $i(k)$ will receive an average weight of at least 0.8 out of 1. This payoff function well reflects the overall ranking on the importance of different criteria by all the players, and can potentially increase the perceived fairness on the allocation result. Hereafter we call this game the strategic form DEA game.

3 NTU DEA GAME

NTU coalitional game is somewhat more complicated than the TU one by the fact that there is no standard definition, and many results are sensitive to details in the definition of the game. First let us present the definition we stick to in this research.

Definition 3.1 (NTU Coalitional Game). *The pair (N,V) is called NTU coalitional game if and only if V is a correspondence from any coalition $S \subseteq N$ into a set of real vectors $V(S) \subseteq \Re^N$ satisfying the following conditions:*

(1) If $S \neq \emptyset, V(S)$ is an non-empty closed subset of $\Re^N$; and $V(\emptyset) = \emptyset$.

(2) $\forall x,y \in \Re^N$, if $x \in V(S)$, and $x^j \geq y^j\ \forall j \in S$, then $y \in V(S)$.

(3) $\forall j \in N, \exists V^j \in \Re$ such that $\forall x \in \Re^N : x \in V(\{j\})$ if and only if $x^j \leq V^j$.

(4) $\{x \in V(N) : x^j \geq V^j\}$ is a compact set.

The interpretation of the NTU coalitional game (N,V) is that $V(S)$ is the set of feasible payoff vectors for the coalition S if that coalition forms. Only the coordinates for players $j \in S$ in elements of $V(S)$ matter. A consequence of property (2) is that, if x is feasible for S, then any $y \leq x$ is feasible for S as well; this property is often called *comprehensiveness*, and it can be interpreted as free disposability of utility. Property (3) says that for any player $j \in N$, its feasible payoff is always upper bounded. Property (4) ensures that the individually rational part of $V(N)$ is bounded.

Next, based on the strategic form DEA game, the NTU DEA game can be defined in both α and β fashion depending on a coalition's reactions against the deviations by its counter-coalition.

Definition 3.2 (α-coalitional NTU DEA Game)**.** *Suppose a strategic form DEA game $G^{DEA} := (N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$ is given. The α-coalitional NTU game (N, V_α) associated with G^{DEA} is a correspondence from any coalition $S \subseteq N$ into a set of real vectors $V_\alpha(S) \subseteq \Re^N$, which satisfies*

(1) For all non-empty $S \subsetneq N, x \in V_\alpha(S)$ if and only if there exists $w^S \in W^S$ such that, for all $u^{N\setminus S} \in W^{N\setminus S}$ and for all $j \in S$, $x^j \le f^j(w^S, u^{N\setminus S})$.

(2) $x \in V_\alpha(N)$ if and only if there exists $w^N \in W^N$ such that, for all $j \in N, x^j \le f^j(w^N)$.

where $W^S = \prod_{j\in S} W^j$. Coalition $S \subseteq N$ is said to α-improve upon $x \in V_\alpha(S)$ if there exists $w^S \in W^S$ such that for all $u^{N\setminus S} \in W^{N\setminus S}$, $x^j < f^j(w^S, u^{N\setminus S})$ $\forall j \in S$ is satisfied. The **α-core** *is the set of payoff vectors $x \in V_\alpha(N)$ upon which no coalition α-improves.*

Definition 3.3 (β-coalitional NTU DEA Game)**.** *Suppose a strategic form DEA game $G^{DEA} := (N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$ is given. The β-coalitional NTU game (N, V_β) associated with G^{DEA} is a correspondence from any coalition $S \subseteq N$ into a set of real vectors $V_\beta(S) \subseteq \Re^N$, which satisfies*

(1) For all non-empty $S \subsetneq N, x \in V_\beta(S)$ if and only if for all $u^{N\setminus S} \in W^{N\setminus S}$, there exists $w^S \in W^S$ such that for all $j \in S, x^j \le f^j(w^S, u^{N\setminus S})$.

(2) $x \in V_\beta(N)$ if and only if there exists $w^N \in W^N$ such that, for all $j \in N, x^j \le f^j(w^N)$.

where $W^S = \prod_{j\in S} W^j$. Coalition $S \subseteq N$ is said to β-improve upon $x \in V_\beta(S)$ if for all $u^{N\setminus S} \in W^{N\setminus S}$, there exists $w^S \in W^S$ such that $x^j < f^j(w^S, u^{N\setminus S})$ $\forall j \in S$ is satisfied. The **β-core** *is the set of payoff vectors $x \in V_\beta(N)$ upon which no coalition β-improves.*

Corollary 3.1. *From the definition of the α- and β-coalitional NTU DEA game, it directly follows that*

$$\beta - core \subseteq \alpha - core$$

4 MAIN RESULTS

For the convenience of the reader, we first introduce the beautiful theorem by (Scarf, 1971).

Theorem 4.1 (Scarf)**.** *Assume that for each $j \in N$, W^j is a compact convex set, and f^j is quasi-concave in $w \in W$, then the α-core is non-empty.*

Applying Scarf's theorem, we can easily show the non-emptiness of the α-core in the NTU DEA game.

Theorem 4.2. *Suppose a strategic form DEA game $G^{DEA} := (N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$ is given. The α-core of the NTU coalitional game (N, V_α) associated with G^{DEA} is non-empty.*

Proof. For all $j \in N$, for all $w^j \in W^j$, we have $w_i^j \in [0,1]$ $\forall i \in M$ and $\sum_{i=1}^m w_i^j = 1$, thus W^j is a compact convex set.

From the definition of the strategic form DEA game, we know that

$$f^j(w^1,\dots,w^n) = (\sum_{i=1}^m (\frac{1}{n}\sum_{j=1}^n w_i^j) c_{ij})E$$

Let $w, v \in W$, and $\lambda \in (0,1)$, then

$$\begin{aligned} & f^j(\lambda w^1 + (1-\lambda)v^1, \dots, \lambda w^n + (1-\lambda)v^n) \\ & = (\sum_{i=1}^m (\frac{1}{n}\sum_{j=1}^n (\lambda w_i^j + (1-\lambda)v_i^j)) c_{ij})E \\ & = \lambda f^j(w^1,\dots,w^n) + (1-\lambda) f^j(v^1,\dots,v^n) \\ & \ge min\{f^j(w^1,\dots,w^n), f^j(v^1,\dots,v^n)\} \end{aligned}$$

Thus f^j is quasi-concave in $w \in W$. The α-core of the NTU DEA game is non-empty.

Definition 3.3 says that β-core $\ni x$ of the NTU DEA game is non-empty if and only if for any coalition $S \subseteq N$, there exists $u^{N\setminus S} \in W^{N\setminus S}$ such that for all $w^S \in W^S$, there exists $j \in S$ such that

$$x^j \ge f^j(w^S, u^{N\setminus S})$$

is satisfied. This two-folded "exists" requirement makes β-core somehow more intricate than α-core.

(Masuzawa, 2003) presented a specific class of NTU games with non-empty cores different from that of (Scarf, 1971), without quasi-concavity of payoff functions. Although Theorem 4.2 has shown that the payoff function of the strategic form DEA game is quasi-concave, his concept of dominant punishment strategy has high applicability, and we will apply it to find an existence condition for β-core. First let us define the dominant punishment strategy.

Definition 4.1 (Dominant Punishment Strategy)**.** *Let $\emptyset \ne S \subseteq N$, and $w^S, v^S \in W^S$. We say that w^S is weakly punishment-dominant over v^S against $N \setminus S$ if for all $j \in N \setminus S$ and for all $u^{N\setminus S} \in W^{N\setminus S}$*

$$f^j(u^{N\setminus S}, v^S) \ge f^j(u^{N\setminus S}, w^S)$$

is satisfied. w^S is a ***dominant punishment strategy*** *against $N \setminus S$ if w^S is weakly punishment-dominant over all $v^S \in W^S$ against $N \setminus S$.*

If this concept is applied to the fixed benefit allocation problem such as the DEA game, a dominant punishment strategy w^S actually secures coalition S the largest portion of the benefit it can receive whatever strategies the complementary coalition may choose. Let us give a lemma regarding the existence of dominant punishment strategy in the strategic form DEA game.

Lemma 4.1. *Suppose a strategic form DEA game* $G^{DEA} := (N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$ *is given. Let* $\underline{C}^j = \{i(j) \in M \mid c_{i(j)j} \le c_{ij}\ \forall i \in M\} \quad \forall j \in N$. *Each player* $k \in N$ *has a dominant punishment strategy against others if and only if for all* $k \in N$, $\underline{C}^{N\setminus\{k\}} = \underline{C}^1 \cap \ldots \cap \underline{C}^{k-1} \cap \underline{C}^{k+1} \cap \ldots \cap \underline{C}^n \ne \emptyset$.

Proof. Sufficiency: from the definition, we know that $\underline{C}^j$ is the set of criteria giving player j the lowest evaluation. Let $k \in N$, $\underline{C}^{N\setminus\{k\}} \ne \emptyset$ means that there is at least one common criterion $i(j) \in M$ giving all the player $j \in N\setminus\{k\}$ the lowest evaluation. Assume for all $k \in N$, w^k corresponds to a strategy such that player k puts his whole weight on criterion $i(j) \in \underline{C}^{N\setminus\{k\}}$. Then for all $j \in N\setminus\{k\}$, for all $v^k \in W^k$

$$\sum_{i=1}^{m}(v_i^k - w_i^k)c_{ij} \ge 0$$

is satisfied. Hence for all $j \in N\setminus\{k\}$, for all $u^{N\setminus\{k\}} \in W^{N\setminus\{k\}}$ and for all $v^k \in W^k$

$$\begin{aligned} & f^j(u^{N\setminus\{k\}}, v^k) - f^j(u^{N\setminus\{k\}}, w^k) \\ &= (\sum_{i=1}^{m}(\frac{1}{n}(v_i^k - w_i^k))c_{ij})E \\ &\ge 0 \end{aligned}$$

w^k weakly punishment dominates over all $v^k \in W^k$ and is the dominant punishment strategy of player k.

Necessity: suppose that there exists player $k \in N$ such that $\underline{C}^{N\setminus\{k\}} = \emptyset$. Let $w^k \in W^k$ be a strategy such that for certain player $j \in N\setminus\{k\}$, for all $u^{N\setminus\{k\}} \in W^{N\setminus\{k\}}$ and for all $v^k \in W^k$

$$\begin{aligned} & f^j(u^{N\setminus\{k\}}, v^k) - f^j(u^{N\setminus\{k\}}, w^k) \\ &= (\sum_{i=1}^{m}(\frac{1}{n}(v_i^k - w_i^k))c_{ij})E \\ &\ge 0 \end{aligned}$$

is satisfied. Then w^k must correspond to a strategy such that player k puts his whole weight on the criterion (criteria) $i(j) \in \underline{C}^j$. However, as $\underline{C}^{N\setminus\{k\}} = \emptyset$, there must exist player $j' \in N\setminus\{k\}$ and $j' \ne j$ such that criterion (criteria) $i(j) \notin \underline{C}^{j'}$. Let $w^k(j')$ be a strategy such that player k puts his whole weight on the criterion (criteria) $i(j') \in \underline{C}^{j'}$, then for player j', for all $u^{N\setminus\{k\}} \in W^{N\setminus\{k\}}$

$$\begin{aligned} & f^{j'}(u^{N\setminus\{k\}}, w^k(j')) - f^{j'}(u^{N\setminus\{k\}}, w^k) \\ &= (\sum_{i=1}^{m}(\frac{1}{n}(w_i^k(j') - w_i^k))c_{ij'})E \\ &< 0 \end{aligned}$$

in which $w^k(j')$ punishes player j' more severely than w^k. Hence player k has no dominant punishment strategy.

Each player $k \in N$ has a dominant punishment strategy against others if and only if for all $k \in N$, $\underline{C}^{N\setminus\{k\}} \ne \emptyset$.

Now we will give a proposition showing the existence of β-core with the concept of dominant punishment strategy.

Proposition 4.1. *Suppose a strategic form DEA game* $G^{DEA} := (N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$ *is given. If each player* $j \in N$ *has a dominant punishment strategy against others, the* β*-core of the NTU coalitional game* (N, V_β) *associated with* G^{DEA} *exists and is identical to the* α*-core.*

Proof. With Theorem 4.2 and Corollary 3.1, it suffices to show that if each player $j \in N$ has a dominant punishment strategy against others, $\alpha - core \subseteq \beta - core$.

To see this, assume a payoff vector $x \in V_\beta(S)$ can be β-improved. Then, for coalition $S \subseteq N$, for all $u^{N\setminus S} \in W^{N\setminus S}$, there exists $v^S \in W^S$ such that $x^j < f^j(v^S, u^{N\setminus S})\ \ \forall j \in S$ is satisfied.

Because each player $j \in N$ has a dominant punishment strategy against others, with Lemma 4.1, it can be easily verified that each non-empty coalition $S \subseteq N$ has a dominant punishment strategy against $N\setminus S$. Then for all $j \in S$, for all $v^S \in W^S$, for all $u^{N\setminus S} \in W^{N\setminus S}$, there exists $d^{N\setminus S} \in W^{N\setminus S}$ such that

$$f^j(v^S, u^{N\setminus S}) \ge f^j(v^S, d^{N\setminus S})$$

It follows that for $d^{N\setminus S}$, there exists $w^S \in W^S$ such that $x^j < f^j(w^S, d^{N\setminus S})$. Then, there exists $w^S \in W^S$ such that for all $u^{N\setminus S} \in W^{N\setminus S}$

$$x^j < f^j(w^S, d^{N\setminus S}) \le f^j(w^S, u^{N\setminus S})\ \ \forall j \in S$$

is satisfied. Hence x can be α-improved as well, and $\beta - core = \alpha - core$.

Even though it is a bit strong, next we will present a sufficient and necessary condition under which each non-empty coalition $S \subseteq N$ has a dominant strategy. Similar to the dominant punishment strategy, let us first give the definition of the dominant strategy.

Definition 4.2 (Dominant Strategy)**.** *Let* $\emptyset \ne S \subseteq N$, *and* $w^{*S} \in W^S$. *We say that* w^{*S} *is a* ***dominant strategy*** *of* S *if for all* $u^{N\setminus S} \in W^{N\setminus S}$, *for all* $w^S \in W^S$, *and for all* $j \in S$

$$f^j(w^{*S}, u^{N\setminus S}) \ge f^j(w^S, u^{N\setminus S})$$

is satisfied.

Lemma 4.2. *Suppose a strategic form DEA game* $G^{DEA} := (N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$ *is given. Let* $\bar{C}^j = \{i(j) \in M \mid c_{i(j)j} \ge c_{ij}\ \forall i \in M\} \quad \forall j \in N$. *Each non-empty coalition* $S \subseteq N$ *has a dominant strategy if and only if* $\bar{C}^N = \bar{C}^1 \cap \bar{C}^2 \cap \ldots \cap \bar{C}^n \ne \emptyset$.

Proof. Sufficiency: from the definition, we know that $\bar{C}^j$ is the set of criteria giving player j the highest evaluation. $\bar{C}^N \neq \emptyset$ means that there is at least one common criterion $i(j) \in M$ giving all the player $j \in N$ the highest evaluation. Assume for each non-empty coalition $S \subseteq N$, $w^{*S} = \prod_{j \in S} w^{*j}$ corresponds to a strategy combination such that each player $j \in S$ puts his whole weight on the criterion (criteria) $i(j) \in \bar{C}^N$. Then for all $u^{N \setminus S} \in W^{N \setminus S}$, for all $w^S \in W^S$ and for all $k \in S$

$$\begin{aligned} & f^k(w^{*S}, u^{N\setminus S}) - f^k(w^S, u^{N\setminus S}) \\ &= (\sum_{i=1}^{m} (\frac{1}{n}(\sum_{j\in S} w_i^{*j} - \sum_{j \in S} w_i^j)) c_{ik}) E \\ &\geq 0 \end{aligned}$$

Hence w^{*S} is the dominant strategy of coalition S.

Necessity: suppose $\bar{C}^N = \emptyset$. Take the grand coalition N, and let $w^{*N} = \prod_{j \in N} w^{*j}$ be a strategy combination such that for certain player $k \in N$, for all $w^N \in W^N$

$$\begin{aligned} & f^k(w^{*N}) - f^k(w^N) \\ &= (\sum_{i=1}^{m} (\frac{1}{n}(\sum_{j\in N} w_i^{*j} - \sum_{j \in N} w_i^j)) c_{ik}) E \\ &\geq 0 \end{aligned}$$

is satisfied. Then w^{*N} must be a strategy combination such that each player $j \in N$ puts his whole weight on the criterion (criteria) $i(k) \in \bar{C}^k$. However, as $\bar{C}^N = \emptyset$, there must exist player $k' \in N$ and $k' \neq k$ such that $i(k) \notin \bar{C}^{k'}$. With the strategy combination $w^N(k') = \prod_{j \in N} w^j(k')$ such that each player $j \in N$ puts his whole weight on the criterion (criteria) $i(k') \in \bar{C}^{k'}$, we have

$$\begin{aligned} & f^{k'}(w^{*N}) - f^{k'}(w^N(k')) \\ &= (\sum_{i=1}^{m} (\frac{1}{n}(\sum_{j\in N} w_i^{*j} - \sum_{j \in N} w_i^j(k'))) c_{ik'}) E \\ &< 0 \end{aligned}$$

The grand coalition N has no dominant strategy.

Each non-empty coalition $S \subseteq N$ has a dominant strategy if and only if $\bar{C}^N \neq \emptyset$.

The following proposition gives another condition for an non-empty β-core with the concept of dominant strategy.

Proposition 4.2. *Suppose a strategic form DEA game $G^{DEA} := (N, \{W^j\}_{j\in N}, \{f^j\}_{j \in N})$ is given. If each non-empty coalition $S \subseteq N$ has a dominant strategy, the β-core of the NTU coalitional game (N, V_β) associated with G^{DEA} exists and is identical to the α-core.*

Proof. Again with Theorem 4.2 and Corollary 3.1, it suffices to show that if each non-empty coalition $S \subseteq N$ has a dominant strategy, $\alpha - core \subseteq \beta - core$.

Assume a payoff vector $x \in V_\beta(S)$ can be β-improved. Then, for coalition $S \subseteq N$, for all $u^{N\setminus S} \in W^{N \setminus S}$, there exists $w^S \in W^S$ such that $x^j < f^j(w^S, u^{N\setminus S})$ $\forall j \in S$ is satisfied.

As S has a dominant strategy, then there exists $w^{*S} \in W^S$ such that for all $u^{N\setminus S} \in W^{N\setminus S}$, for all $w^S \in W^S$, for all $j \in S$

$$f^j(w^{*S}, u^{N\setminus S}) \geq f^j(w^S, u^{N\setminus S})$$

is satisfied. It follows that there exists $w^{*S} \in W^S$ such that for all $u^{N\setminus S} \in W^{N\setminus S}$, for all $j \in S$

$$x^j < f^j(w^S, u^{N\setminus S}) \leq f^j(w^{*S}, u^{N\setminus S})$$

Hence x can be α-improved as well, and $\beta - core = \alpha - core$.

Propositions 4.1 and 4.2 establish two important conditions under which the β-core of NTU DEA game is non-empty. Formally, we give the following theorem.

Theorem 4.3. *Suppose a strategic form DEA game $G^{DEA} := (N, \{W^j\}_{j\in N}, \{f^j\}_{j \in N})$ is given. For all $j \in N$, let $\underline{C}^j = \{i(j) \in M \mid c_{i(j)j} \leq c_{ij}\ \forall i \in M\}$, and $\bar{C}^j = \{i(j) \in M \mid c_{i(j)j} \geq c_{ij}\ \forall i \in M\}$. If either condition A or B below is satisfied, the β-core of the NTU coalitional game (N, V_β) associated with G^{DEA} exists and is identical to the α-core.*

A. $\underline{C}^{N\setminus\{j\}} = \underline{C}^1 \cap \ldots \cap \underline{C}^{j-1} \cap \underline{C}^{j+1} \cap \ldots \cap \underline{C}^n \neq \emptyset\ \forall j \in N$

B. $\bar{C}^N = \bar{C}^1 \cap \bar{C}^2 \cap \ldots \cap \bar{C}^n \neq \emptyset$

This theorem states that in NTU DEA game, if each player has a dominant punishment strategy against others, or each non-empty coalition has a dominant strategy, β-core is non-empty and identical to α-core. However, both *A* and *B* are sufficient conditions and criticized to be a bit strong. In the next section, we will discuss a direction to improve the robustness of the β-core existence condition.

5 DISCUSSION

Although we haven't yet proved a robust existence condition for the β-core, we believe that the algorithm in the proposition below opens up new lines of consideration in finding the β-core with relaxation of the inessential condition in the TU DEA game. For the convenience of the reader, we re-state the inessential condition.

Theorem 5.1 (Sekine et al.). *A TU DEA game is inessential if and only if, for all $j \in N$, $c_{ij} = c_{i'j}$ holds for all $i, i' = 1, \ldots, m$.*

(Sekine et al., 2014) has shown that the core of TU DEA game is non-empty if and only if the evaluation indices for all the criteria are identical for each player, which lost the crucial information provided by different criteria. Our proposition below concerns the existence of β-core of a special case.

Proposition 5.1. *Suppose a strategic form DEA game $G^{DEA} := (N, \{W^j\}_{j\in N}, \{f^j\}_{j\in N})$ is given. Let $\bar{C}^j = \{i(j) \in M \mid c_{i(j)j} \geq c_{ij}\ \forall i \in M\}\ \ \forall j \in N$; and for proof simplicity, assume $m = n$, the number of criteria is equal to the number of players. If $\bar{C}^N = \bar{C}^1 \cap \bar{C}^2 \cap \ldots \cap \bar{C}^n = \emptyset$, and $\bar{C}^{N\setminus\{j\}} = \bar{C}^1 \cap \ldots \cap \bar{C}^{j-1} \cap \bar{C}^{j+1} \cap \ldots \cap \bar{C}^n \neq \emptyset\ \forall j \in N$, then the β-core of the NTU coalitional game (N, V_β) associated with G^{DEA} is non-empty and $x = (1/n)_n E \in \beta - core$.*

Proof. It follows directly from the assumption that for each player $k \in N$, $|\bar{C}^k| = m - 1$ is satisfied; and for all $i(k), i'(k) \in \bar{C}^k$, $c_{i(k)k} = c_{i'(k)k}$ is satisfied. Here $|\bar{C}^k| = m - 1$ is a relaxation of the inessential condition which requires $|\bar{C}^k| = m$.

Let us use $\underline{i}(k)$ to denote the criterion $\underline{i}(k) \in M \setminus \bar{C}^k$, and try to find the β-core following the definition. Suppose $x \in V_\beta(N)$ is a feasible allocation satisfying group rationality and its corresponding strategy profile is w^N. Then for all $k \in N$

$$
\begin{aligned}
x^k &= f^k(w^N) \\
&= (\sum_{i=1}^{m} (\frac{1}{n} \sum_{j\in N} w_i^j) c_{ik}) E \\
&= (c_{i(k)k} - \frac{1}{n}(c_{i(k)k} - c_{\underline{i}(k)k}) \sum_{j\in N} w_{\underline{i}(k)}^j) E
\end{aligned}
$$

Our basic algorithm to find the β-core is to shrink the number of players in S from n to 1 with n steps; at each step, let the corresponding strategy profile w^N meet the criterion by Definition 3.3; and finally find a feasible allocation satisfying all the criteria.

Criterion 1: we begin from the grand coalition N. As is known that $\sum_{k\in N} x^k = E$, there does not exist $v^N \in W^N$ such that $x^k < f^k(v^N)\ \forall k \in N$. Because the increase of one player's payoff will definitely reduce some other players' payoff in this fixed benefit allocation problem. Hence for the moment, any payoff vector $x \in V_\beta(N)$ is in the β-core.

Criterion 2: next let us examine $N \setminus \{p\}\ \ \forall p \in N$. No β-improvement on $x \in V_\beta(N)$ by $N \setminus \{p\}$ requires that there exists $u^p \in W^p$ such that for all $v^{N\setminus\{p\}} \in W^{N\setminus\{p\}}$, there should exist at least one $k \in N \setminus \{p\}$ with $f^k(w^N) \geq f^k(v^{N\setminus\{p\}}, u^p)$ satisfied, which means

$$
\begin{aligned}
& f^k(w^N) - f^k(v^{N\setminus\{p\}}, u^p) \\
&= \frac{1}{n}(c_{i(k)k} - c_{\underline{i}(k)k})(\sum_{j\in N\setminus\{p\}} v_{\underline{i}(k)}^j + u_{\underline{i}(k)}^p - \sum_{j\in N} w_{\underline{i}(k)}^j) E \\
&\geq 0
\end{aligned}
$$

The inequality constraint above is equivalent to

$$
\sum_{j\in N\setminus\{p\}} v_{\underline{i}(k)}^j + u_{\underline{i}(k)}^p - \sum_{j\in N} w_{\underline{i}(k)}^j \geq 0
$$

Suppose $u_{\underline{i}(k)}^p = 1$ and consider the worst situation that $\sum_{j\in N\setminus\{p\}} v_{\underline{i}(k)}^j = 0$, at this point $x \in V_\beta(N)$ should be corresponding to a strategy profile w^N such that there exists $k \in N \setminus \{p\}$ with $\sum_{j\in N} w_{\underline{i}(k)}^j \leq 1$ satisfied.

...

Criterion n: following similar procedures, we continue shrinking the number of players in S one by one until the single player coalition $\{p\}\ \ \forall p \in N$. No β-improvement on $x \in V_\beta(N)$ by $\{p\}$ requires that there exists $u^{N\setminus\{p\}} \in W^{N\setminus\{p\}}$ such that for all $v^p \in W^p$, $f^p(w^N) \geq f^p(v^p, u^{N\setminus\{p\}})$ is satisfied. w^N should be corresponding to a strategy profile such that $\sum_{j\in N} w_{\underline{i}(p)}^j \leq n - 1$.

With all the criteria above, it can be easily proved that the β-core of the NTU DEA game is non-empty and $x = (1/n)_n E \in \beta - core$.

Example 5.1. *For a clearer understanding of Proposition 5.1, here is an numerical example with 3×3 score matrix, and let us find its β-core following the algorithm above.*

$$
\begin{bmatrix}
0.45 & 0.40 & 0.30 \\
0.45 & 0.20 & 0.35 \\
0.10 & 0.40 & 0.35
\end{bmatrix}
$$

Solution. Criterion 1: the weight matrix should be

$$
\begin{bmatrix}
w_1^1 & w_1^2 & w_1^3 \\
w_2^1 & w_2^2 & w_2^3 \\
w_3^1 & w_3^2 & w_3^3
\end{bmatrix}
$$

in which $\sum_{i=1}^{3} w_i^j = 1\ \forall j \in \{1, 2, 3\}$.

Criterion 2: consider the two-player coalition $\{1, 2\}$. Following criterion 2, either $w_3^1 + w_3^2 + w_3^3 \leq 1$ or $w_2^1 + w_2^2 + w_2^3 \leq 1$ should be satisfied. Similar for coalitions $\{1, 3\}$ and $\{2, 3\}$, thus at least two out of the following three conditions have to be satisfied.

(1) $w_1^1 + w_1^2 + w_1^3 \leq 1$

(2) $w_2^1 + w_2^2 + w_2^3 \leq 1$

(3) $w_3^1 + w_3^2 + w_3^3 \leq 1$

Criterion 3: consider the single player coalition, the following three conditions should all be satisfied.

(1) $w_1^1 + w_1^2 + w_1^3 \leq 2$

(2) $w_2^1 + w_2^2 + w_2^3 \leq 2$

(3) $w_3^1 + w_3^2 + w_3^3 \leq 2$

The multivariate linear inequalities above can be solved by MATLAB. One of the feasible strategy combinations

$$w = w^1 \times w^2 \times w^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

satisfies all three criteria above, and hence $x = (1/3, 1/3, 1/3)E$ is in the β-core. ■

We can see that even if the dominant punishment strategy and the dominant strategy for each nonempty coalition do not exist, β-core may still exist under certain assumptions. With relaxation of the inessential condition in TU DEA game, proposition 5.1 shows a new direction in light of improving the robustness of the β-core existence condition.

6 SUMMARY

In this extension paper, we contribute on studying two of the fractional solutions in NTU DEA game. We have shown the existence of the α-core with Scarf's theorem, and given two sufficient conditions under which the β-core is non-empty and identical to the α-core (Theorem 4.3). This paper also indicates a direction to find a more robust existence condition for β-core by relaxing the inessential condition in TU DEA game, and we will follow this line to improve our current work.

Even though our study in DEA game is predominantly theoretical and derivation-based, we are now doing benefit allocation comparison study under different DEA game schemes with the data from Joban Joint Power and Shenhua Guohua Beijing Cogeneration, which will be included in the future work as well.

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