Design of State Observers for Interconnected Time-delay Systems via a Coordinate Transformation Approach

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Abstract: This paper considers the design of state observers for interconnected time-delay systems using a coordinate transformation method. Through such a transformation, the system that has interconnection and state delays is metamorphosed into a new system that injects time-delay information into its input and output terms, before reintroducing them back into the latter system, effectively coupling the delay terms into the IO injection terms and eliminating the delay values from the state variables. Next, full-order and reduced-order observers are designed based on the transformed system. Finally, the observed states of the transformed system that correspond to the original system is used to deduce the estimates of the original system. A numerical example is provided of an interconnected time-delay system.

1 INTRODUCTION

High volume information exchange of modern interconnected systems are driving the need for seamless communication capabilities, while at the same time pushing for an optimum usage of already saturated communication channels. In a world where the ideal case of instantaneous interactions between subsystems is limited by the communication capabilities, the occurrence of time delays is therefore inevitable. In the context of state estimation for these subsystems, the notion of time delays inflicted on the state variables of these subsystems, put constraints on the employment of prominent observer structures and its existence conditions.

Understanding the main constraints that come from the observability properties is an important state estimation issue to be addressed. For time-delay free systems that are observable, the design of observers is straightforward as many existing observer design techniques can be applied (Luenberger, 1971; Darouach et al., 1994; Darouach, 2000; Hou and Müller, 1992, 1994; Trinh, 1999; Trinh and Fernando, 2012) and others. However, the opposite is true for time-delay systems (Bhat and Koivo, 1976; Fairman and Kumar, 1986; Pearson and Fiagbedzi, 1989; Darouach et al., 1999; Trinh, 1999; Darouach, 2001; Germani et al., 2002; Hou et al., 2002; Subbarao and Muralidhar, 2008; Nam et al., 2014; Leong et al.,

2015) that put forward challenges and complications into devising effective observer structures. In this paper, it will be shown that the design of asymptotic observers for time-delay systems can be approached in such a way that a coordinate transformation (Hou et al., 2002) is performed to convert the system to another domain or coordinate that guarantees the observability of its new system matrices. Subsequently, an observer can be designed to estimate the state vector of the transformed system that is algebraically linked to the state vector of the original system. For example, for $z(t) = Mx(t)$, M is the transformation matrix that formularizes the transitional relationship between the state vector $z(t)$ and $x(t)$ of the transformed and original system, respectively. In short, the state variables of the original system is related to that of the transformed system through a coordinate transformation relationship (Hou et al., 2002). Eventually, the prediction of the states of the former system $\hat{x}(t)$ can be deduced from the estimates of the latter system $\hat{z}(t)$ on the basis of the same coordinate transformation relationship.

The implementation of such a concept requires the establishment of a coordinate transformation (Hou et al., 2002) that governs the relationship between the original system and the transformed system in such a way that the transformation is bi-directional - the transformed system can be reversed-transformed to its original system. In other words, the state vector of an

x-domain system can be conveniently convertible to that of the state vector of a *z*-domain system and vice-versa.

To the best of our knowledge, application of co-ordinate transformation in enabling the design of observers for interconnected time-delay systems has not been well considered. In view of the potential that such a transformation would act as a stepping stone towards the construction of observers for such a class of systems, it is the aim of this paper to tap into the incentive such a coordinate transformation approach has to offer.

The rest of the paper is organized as follows:

Session 2 provides the structure of a general time-delay system represented in the delay operator form. It further explores the underlying motivation and possibility of a coordinate transformation inspired by the work of Hou et al. (2002), upon the satisfaction of the given Theorem 1. Next, a typical two-area interconnected time-delay system is given and an explanation provided on how best to take advantage of the benefits inherent in the transformed system to overcome the distinct limitations of an interconnected time-delay system in terms of state observation. In section 4, the construction of a coordinate transformation based on the idea of Hou et al. (2002) is carried out. Subsequently, full-order and reduced-order observers are designed for the transformed system using well-known and straightforward techniques. Finally, conclusion is given in section 5.

2 SYSTEM DESCRIPTION

To begin, assume that a time-delay system can be represented in two different coordinate systems, *x*-coordinate and *z*-coordinate. In a conventional *x*-coordinate system, the state vector is known as $x(t)$ and as $z(t)$ in the new *z* domain.

If we define τ as a time-delay operator, a time-delay system can be represented as the following general form such that

$$\dot{x}(t) = A(\tau)x(t) + B(\tau)u(t), \tag{1}$$

$$x(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}, \forall t \in [-\tau_{max}, 0], \tag{2}$$

$$y(t) = C(\tau)x(t), \tag{3}$$

where $\tau = \{\tau_i\}$ for systems with multiple time-delay constants for $i = 1, 2, ..., K$ and $\tau_{max} = max\{\tau_i\}$. Vectors $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, control input and output measurement, respectively. Matrices $A(\tau) \in \mathbb{R}^{n \times n}$, $B(\tau) \in \mathbb{R}^{n \times m}$ and $C(\tau) \in \mathbb{R}^{p \times n}$ are known system polynomial matrices as a function of τ operator.

The purpose of the embedment of time-delay operator into the system matrices is twofold. The first reason is to simplify the representation of the structure of time-delay systems to accommodate for potentially a larger class of interconnected systems with time-delay appearing in diverse forms and values; and secondly, to standardize the time-delay system to the notational form that is conducive for the subsequent coordinate transformation to take place. To demonstrate the usage of delay operator in a particular polynomial matrix, the multiplication of the time-delay operator of an element of a polynomial matrix with a state variable, for instance

$$A(\tau)x(t) = \begin{bmatrix} 3 + 2\tau_1 - \tau_2^2 & \tau_3 \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

produces

$$\begin{bmatrix} 3x_1(t) + 2x_1(t - \tau_1) - x_1(t - 2\tau_2) + x_2(t - \tau_3) \\ \cdots \end{bmatrix}.$$

According to Hou et al. (2002), a coordinate transformation can be performed to transform the system into an equivalent *z*-coordinate system which offers distinct advantages. With the computation of a polynomial matrix $T(\tau)$ having a coordinate transformation relationship as

$$z(t) = T(\tau)x(t), \tag{4}$$

the system described in (1)-(3) can be transformed into the following

$$\dot{z}(t) = \bar{A}z(t) + \bar{E}(\tau)y(t) + \bar{B}(\tau)u(t), \tag{5}$$

$$y(t) = \bar{C}z(t), \tag{6}$$

where $\bar{E}(\tau)y(t)$ and $\bar{B}(\tau)u(t)$ are the output and input injection terms respectively. $\bar{A}$ and $\bar{C}$ are constant matrices of appropriate dimensions that carry the forms of

$$\bar{A} = \begin{bmatrix} 0 & I_p & & \\ & \ddots & \ddots & \\ & & \ddots & I_p \\ & & & 0 \end{bmatrix}, \tag{7}$$

$$\bar{C} = \begin{bmatrix} I_p & 0 & \cdots & 0 \end{bmatrix}, \tag{8}$$

in which *p* is the dimension of the system output with reference to (3).

Apparently, one would now realize the distinguishable benefit that the coordinate-transformed system (5)-(6) is unquestionably observable. Such a characteristic is pivotal because the conformity to the

observability criteria signifies the affirmation of the existence of an observer for such a system. Additionally, one would notice that the delay values associated with the state vector $x(t)$ brought about by the system polynomial matrix $A(\tau)$ in (1) is no longer existent in the new system of (5)-(6) as the value of $\bar{A}$ follows (7). This is an indication that a shift of time-delay association has happened, that the time-delay constants are now tied to the output and input injection terms, through $\bar{E}(\tau)$ and $\bar{B}(\tau)$ polynomial matrices, rather than having a direct association with the state vector. In principal, the use of coordinate transformation has redefined the time-delay problem in the state vector into a whole new problem of having time-delay terms in the input and output which is much easier to be dealt with. In comparison, the same time-delay system in the pre-transformed form of (1)-(3) suffers from a limited viability of observer construction.

In order to obtain a transformed system that possesses system matrices that are observable, one has to construct a transform or coordinate change matrix which in this case denoted as $T(\tau)$. Now, an important question arises as to under what situation the coordinate transformation is available? The possibility of constructing such a matrix depends largely on a set of conditions. The existence condition of the transformation matrix is given in Hou et al. (2002) and for convenience, it will be quoted as follows.

Theorem 1. *(Hou et al., 2002) There exists a coordinate transformation that transforms an original system of* (1)-(3) *into an observable form of* (5)-(6) *if the observability matrix $Q_k(\tau)$ defined in the following is column unimodular.*

The observability matrix for (1)-(3) *is given by*

$$Q_k(\tau) = \begin{bmatrix} C(\tau) \\ C(\tau)A(\tau) \\ \vdots \\ C(\tau)A^{k-1}(\tau) \end{bmatrix}, \qquad (9)$$

where $k \leq n$ is the smallest integer such that $rank(Q_k(\tau)) = p$ for all τ.

Proof. The reader may refer to the proof detailed in Hou et al. (2002). □

In the sequel, the succeeding section explores the construction of observers for an interconnected time-delay system via coordinate transformation by capitalizing on the benefits of inherent observability in the matrix pair $(\bar{C}, \bar{A})$.

3 COORDINATE TRANSFORMATION OF AN INTERCONNECTED TIME-DELAY SYSTEM

Similar to that of (1)-(3) that is represented in τ time-delay operator form, a class of interconnected time-delay system having the form of

$$\dot{x}_i(t) = A_{ii}x_i(t) + \sum_{j=1, j\neq i}^{N} A_{ij}x_j(t - \tau_{ji})$$
$$+ A_{d_{ii}}x_i(t - \tau_{ii}) + B_iu_i(t) \qquad (10)$$
$$y_i(t) = C_ix_i(t); \qquad i = 1, 2, ..., N, \qquad (11)$$

can be conveniently expressed in the similar fashion. It is important to note that the common notation would be slightly different when the system is expressed in the general time-delay operator form. In order to illustrate the process of coordinate transformation in terms of the mapping of notations, an example of the following two-area interconnected system will be utilized.

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t - \tau_{21}) + A_{d_{11}}x_1(t - \tau_{11})$$
$$+ B_1u_1(t), \qquad (12)$$
$$y_1(t) = C_1x_1(t), \qquad (13)$$
$$\dot{x}_2(t) = A_{22}x_2(t) + A_{21}x_1(t - \tau_{12}) + A_{d_{22}}x_2(t - \tau_{22})$$
$$+ B_2u_2(t), \qquad (14)$$
$$y_2(t) = C_2x_2(t), \qquad (15)$$

where $i = 1, 2$, $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$ and $y_i(t) \in \mathbb{R}^{p_i}$ are the state, input and the measured output for the $i-th$ subsystem, respectively. Matrices $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $A_{d_{ii}} \in \mathbb{R}^{n_i \times n_i}$, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_i \in \mathbb{R}^{n_i \times m_i}$ and $C_i \in \mathbb{R}^{p_i \times n_i}$ are real known system matrices.

The system matrices for the system described in (12)-(15) are chosen as

$$A_{11} = \begin{bmatrix} -0.2 & -0.2 \\ 0 & -0.1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -0.6 & 0 \\ -0.1 & 0.3 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -0.6 & -0.3 \\ 0.4 & -0.2 \end{bmatrix},$$
$$A_{d11} = \begin{bmatrix} -0.6 & 0 \\ 0 & -0.1 \end{bmatrix}, A_{d22} = \begin{bmatrix} -0.6 & 0 \\ 0.3 & -0.3 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

For the convenience of representation, the delay constants in the system equations are mapped to τ_i notation as follows:

$\tau_1 = \tau_{11}$, $\tau_2 = \tau_{21}$, $\tau_3 = \tau_{22}$ and $\tau_4 = \tau_{12}$.

Obviously, the system is entangled with four different delay terms in its state variables. For this reason, the application of standard design techiques that
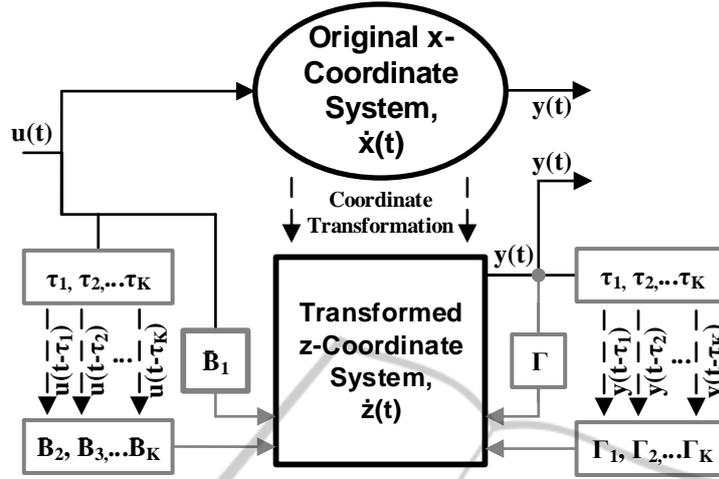
Figure 1: Coordinate transformation of the system.

requires that the observability criteria to be true becomes out of the question when it comes to the design of observer for such a system. Traditionally, this results in a state observation problem that requires a complicated solution or observer structure to account for the time-delay terms implicit in the system state variables.

It is therefore of crucial importance to have a means to overcome this observability problem and limitation of the time-delay terms in the state variables by working on a less restrictive alternative system that is backward convertible to the original system. Before a coordinate transformation is taking place, it is convenient to represent the system in the time-delay operator form as previously shown. In this case, the system polynomial matrices in the form of (1)-(3) would be easily computed as

$$A(\tau) = \begin{bmatrix} -0.6\tau_1 - 0.2 & -0.2 \\ 0 & -0.1\tau_1 - 0.1 \\ 0 & 0 \\ 0 & -0.1\tau_4 \end{bmatrix}$$

$$\begin{matrix} -0.6\tau_2 & 0 \\ -0.1\tau_2 & 0.3\tau_2 \\ -0.6\tau_3 - 0.6 & -0.3 \\ 0.3\tau_3 + 0.4 & -0.3\tau_3 - 0.2 \end{matrix} \quad (16)$$

$$B(\tau) = \begin{bmatrix} -0.1 & 0 \\ 0.1 & 0 \\ 0 & 0.4 \\ 0 & 0.1 \end{bmatrix}, \quad (17)$$

$$C(\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad (18)$$

Now, a coordinate transformation relationship $T(\tau)$ can be obtained and is given as

$$T(\tau) = \begin{bmatrix} C(\tau) \\ C(\tau)A(\tau) - \bar{E}_1(\tau)C(\tau) \end{bmatrix}, \quad (19)$$

such that

$$z(t) = T(\tau)x(t), \quad (20)$$

where $\begin{bmatrix} \bar{E}_2(\tau) & \bar{E}_1(\tau) \end{bmatrix} = C(\tau)A(\tau)^2 Q_k^+(\tau)$, and that $Q_k^+(\tau)$ is the pseudoinverse of $Q_k(\tau)$ obtained in (9).

The system can now be transformed into

$$\dot{z}(t) = \bar{A}z(t) + \bar{E}(\tau)y(t) + \bar{B}(\tau)u(t), \quad (21)$$

$$y(t) = \bar{C}z(t), \quad (22)$$

where $\bar{E}(\tau) = \begin{bmatrix} \bar{E}_1(\tau) \\ \bar{E}_2(\tau) \end{bmatrix} \in \mathbb{R}^{n_z \times p}$, $\bar{B}(\tau) = T(\tau)B(\tau)$.

Or equivalently, when (16)-(18) are used, the specific transformed system obviously becomes that of

$$\begin{aligned} \dot{z}(t) = {} & \bar{A}z(t) + \Gamma y(t) + \Gamma_1 y(t - \tau_1) + \Gamma_2 y(t - \tau_2) \\ & + \Gamma_3 y(t - \tau_3) + \Gamma_4 y(t - \tau_4) + \Gamma_5 y(t - 2\tau_1) \\ & + \Gamma_6 y(t - 2\tau_3) + \Gamma_7 y(t - \tau_1 - \tau_2) \\ & + \Gamma_8 y(t - \tau_1 - \tau_4) + \Gamma_9 y(t - \tau_2 - \tau_3) \\ & + \Gamma_{10} y(t - \tau_2 - \tau_4) + \bar{B}_1 u(t) + \bar{B}_2 u(t - \tau_1) \\ & + \bar{B}_3 u(t - \tau_2) + \bar{B}_4 u(t - \tau_3) \\ & + \bar{B}_5 u(t - \tau_4), \quad t \geq 0, \quad (23) \end{aligned}$$

$$y(t) = \bar{C}z(t), \quad (24)$$

where pair $(\bar{C}, \bar{A})$ is observable, $\sum_{i=1}^{10} \Gamma_i$ and $\sum_{j=2}^{5} \bar{B}_j$ are the delayed output and input injection terms to be

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computed during the coordinate transformation process and will be covered in the next section. $z(t) \in \mathbb{R}^{n_z}$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are respectively the state, input and the measured output of the transformed system. Matrices $\bar{A} \in \mathbb{R}^{n_z \times n_z}$, $\bar{C} \in \mathbb{R}^{p \times n_z}$, $\Gamma_i \in \mathbb{R}^{n_z \times p}$, $1 \le i \le 10$ and $\bar{B}_j \in \mathbb{R}^{n_z \times m}$, $2 \le j \le 5$, are known constant matrices.

Note that the state vector $z(t)$ has no association of delay terms in comparison to the original system. The coordinate transformation is illustrated in Figure 1 in which the transformed system in z-coordinate, receives the same control input information $u(t)$ as the original x-coordinate system, and eventually produces the same output measurements $y(t)$ as the original system. In other words, the functionality of the transformed and original system is virtually the same.

3.1 Design of a Full-order Observer

The structure of a typical Luenberger observer for the transformed system is shown below. It is worthnoting that part of this observer structure follows the standard Luenberger structure except for the additional delayed input and output injection terms. The use of these injection terms renders the relaxation of the observer existence condition which in this case the observability criteria.

$$
\begin{aligned}
\dot{\hat{z}}(t) = {} & (\bar{A} - L\bar{C})\hat{z}(t) + \{L + \Gamma\}y(t) + \Gamma_1 y(t - \tau_1) \\
& + \Gamma_2 y(t - \tau_2) + \Gamma_3 y(t - \tau_3) + \Gamma_4 y(t - \tau_4) \\
& + \Gamma_5 y(t - 2\tau_1) + \Gamma_6 y(t - 2\tau_3) \\
& + \Gamma_7 y(t - \tau_1 - \tau_2) + \Gamma_8 y(t - \tau_1 - \tau_4) \\
& + \Gamma_9 y(t - \tau_2 - \tau_3) + \Gamma_{10} y(t - \tau_2 - \tau_4) \\
& + \bar{B}_1 u(t) + \bar{B}_2 u(t - \tau_1) + \bar{B}_3 u(t - \tau_2) \\
& + \bar{B}_4 u(t - \tau_3) + \bar{B}_5 u(t - \tau_4), \qquad t \ge 0, \quad (25)
\end{aligned}
$$

where L is a matrix of appropriate dimension.

3.2 Design of a Reduced-order Observer

The objective of a reduced order observer is to estimate a function $f(t) \in \mathbb{R}^{(n_z - p)}$ defined as:

$$
f(t) = Lz(t), \qquad (26)
$$

where $L = \begin{bmatrix} 0_{(n_z - p) \times p} & I_{n_z - p} \end{bmatrix} \in \mathbb{R}^{(n_z - p) \times n_z}$ is a full-row rank matrix such that $\begin{bmatrix} \bar{C} \\ L \end{bmatrix}$ is non-singular.

The reduced-order observer structure that we pro-

pose for the transformed z-domain system is

$$
\hat{f}(t) = w(t) + Ey(t), \qquad (27)
$$

$$
\begin{aligned}
\dot{w}(t) = {} & Nw(t) + Jy(t) + M_1 y(t - \tau_1) + M_2 y(t - \tau_2) \\
& + M_3 y(t - \tau_3) + M_4 y(t - \tau_4) + M_5 y(t - 2\tau_1) \\
& + M_6 y(t - 2\tau_3) + M_7 y(t - \tau_1 - \tau_2) \\
& + M_8 y(t - \tau_1 - \tau_4) + M_9 y(t - \tau_2 - \tau_3) \\
& + M_{10} y(t - \tau_2 - \tau_4) + Hu(t) + K_1 u(t - \tau_1) \\
& + K_2 u(t - \tau_2) + K_3 u(t - \tau_3) \\
& + K_4 u(t - \tau_4), \qquad t \ge 0, \qquad (28)
\end{aligned}
$$

where $\hat{f}(t) \in \mathbb{R}^{n_z}$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the reduced-order state estimates, input and output respectively. Matrices E, N, J, M_i, $1 \le i \le 10$, H, K_j, $1 \le j \le 4$ are matrices of appropriate dimensions. The construction of a reduce-order observer requires that these unknown matrices to be obtained.

As soon as the state estimates of the z-coordinate system are obtained, the state of the x-coordinate system can be approximated by recovering them from the z-coordinate state estimates, for instance, $\hat{x}(t)$ and $\hat{z}(t)$ are related through the linear function of

$$
\hat{x}(t) = T^+(\tau)\hat{z}(t), \qquad (29)
$$

where $T^+(\tau)$ of appropriate dimension is the pseudoinverse of $T(\tau)$ obtained from equation (19).

4 NUMERICAL EXAMPLE

Consider the previous example of an interconnected time-delay system where the system polynomial matrices have been worked out to be (16)-(18). The time-delay values are chosen to be $\tau_1 = 5$, $\tau_2 = 5$, $\tau_3 = 5$ and $\tau_4 = 5$ for simulation purposes.

According to Theorem 1, an observability matrix $Q_k(\tau)$ has to be computed, and provided that it is column unimodular, the system can be transformed into an observable form of (5)-(6). It is found that $\mathrm{rank}(Q_2(\tau)) = p$, which signifies that the following $Q_2(\tau)$ is column unimodular.

$$
Q_2(\tau) =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-0.6\tau_1 - 0.2 & -0.2 & -0.6\tau_2 & 0 \\
0 & 0 & -0.6\tau_3 - 0.6 & -0.3
\end{bmatrix},
$$

$$(30)$$

It then follows that a left-inverse $Q_2^+(\tau)$ exists to

be

$$Q_2^+(\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3\tau_1 - 1 & -3\tau_2 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2\tau_3 - 2 & 0 & -\frac{10}{3} \end{bmatrix}.$$

By definition of (19), the coordinate transformation $T(\tau)$ is derived as

$$T(\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.1\tau_1 + 0.1 & -0.2 & -0.2\tau_2 & 0 \\ 0.15\tau_4 & 0 & 0.3\tau_3 + 0.2 & -0.3 \end{bmatrix}, \tag{31}$$

which eventually produces

$$\bar{E}(\tau) = \begin{bmatrix} -0.7\tau_1 - 0.3 \\ -0.15\tau_4 \\ -0.08\tau_1 - 0.06\tau_1{}^2 - 0.02 \\ -0.03\tau_4 - 0.09\tau_1\tau_4 \\ -0.4\tau_2 \\ -0.9\tau_3 - 0.8 \\ 0.08\tau_2 - 0.06\tau_1\tau_2 + 0.12\tau_2\tau_3 \\ -0.09\tau_2\tau_4 - 0.39\tau_3 - 0.18\tau_3{}^2 - 0.24 \end{bmatrix} \tag{32}$$

and

$$\bar{B}(\tau) = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.4 \\ -0.03 - 0.01\tau_1 & -0.08\tau_2 \\ -0.015\tau_4 & 0.12\tau_3 + 0.05 \end{bmatrix}. \tag{33}$$

This z-coordinate system can be easily converted into the form of (23)-(24) and therefore the detailed conversion process will be omitted here.

4.1 Full-order Observer

A full-order observer is designed for such a system utilizing the structure given in (25). Selecting the poles to be $\begin{bmatrix} -3; & -4; & -5; & -6; \end{bmatrix}$, the observer gain matrix L can be obtained through the well-known pole-assignment technique and the calculated observer parameters are given as follows.

$$L = \begin{bmatrix} 11 & 0 \\ 0 & 7 \\ 30 & 0 \\ 0 & 12 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -0.3 & 0 \\ 0 & -0.8 \\ -0.02 & 0 \\ 0 & -0.24 \end{bmatrix},$$

$$\Gamma_1 = \begin{bmatrix} -0.7 & 0 \\ 0 & 0 \\ -0.08 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} 0 & -0.4 \\ 0 & 0 \\ 0 & 0.08 \\ 0 & 0 \end{bmatrix},$$

$$\Gamma_3 = \begin{bmatrix} 0 & 0 \\ 0 & -0.9 \\ 0 & 0 \\ 0 & 0.39 \end{bmatrix}, \quad \Gamma_4 = \begin{bmatrix} 0 & 0 \\ -0.15 & 0 \\ 0 & 0 \\ -0.03 & 0 \end{bmatrix},$$

$$\Gamma_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.06 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Gamma_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -0.18 \end{bmatrix},$$

$$\Gamma_7 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -0.06 \\ 0 & 0 \end{bmatrix}, \quad \Gamma_8 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -0.09 & 0 \end{bmatrix},$$

$$\Gamma_9 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0.12 \\ 0 & 0 \end{bmatrix}, \quad \Gamma_{10} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -0.09 \end{bmatrix},$$

$$\bar{B}_1 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.4 \\ -0.03 & 0 \\ 0 & 0.05 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.01 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{B}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -0.08 \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.12 \end{bmatrix}, \quad \text{and}$$

$$\bar{B}_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -0.015 & 0 \end{bmatrix}.$$

Subsequently, with the attainment of $\hat{z}(t)$ from the observer, state estimates of x-coordinate system $\hat{x}(t)$ can be deduced directly from the state estimates of z-coordinate system $\hat{z}(t)$ through equation (29) where $T^+(\tau)$ is computed as:

$$T^+(\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2}\tau_1 + \frac{1}{2} & -\tau_2 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2}\tau_4 & \tau_3 + \frac{2}{3} & 0 & -\frac{10}{3} \end{bmatrix}.$$

Figure 2: State vector, $x_1(t)$ from the original system vs. $\hat{x}_1(t)$ derived from the observer of the transformed system.

Simulation of the z-domain full-order observer was carried out and comparisons between the x state variables and its estimates are plotted as below.
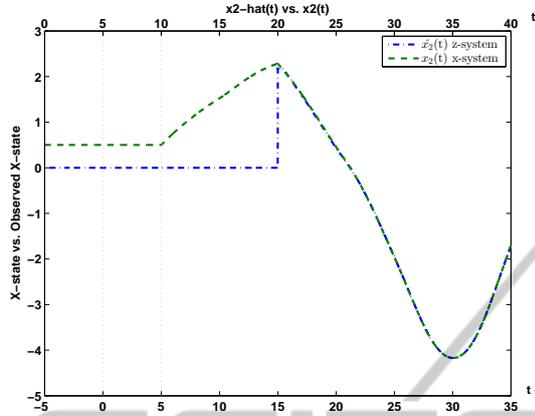

Figure 3: State vector, $x_2(t)$ from the original system vs. $\hat{x}_2(t)$ derived from the observer of the transformed system.
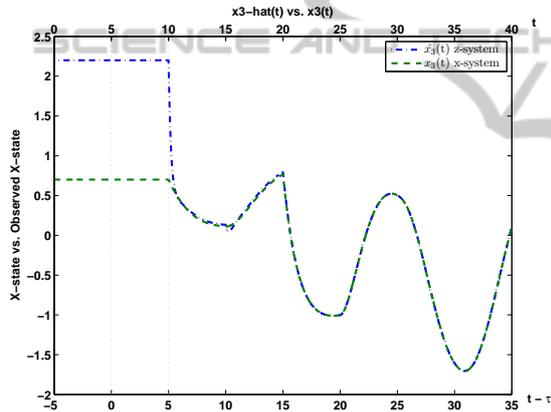

Figure 4: State vector, $x_3(t)$ from original system vs. $\hat{x}_3(t)$ derived from the observer of the transformed system.
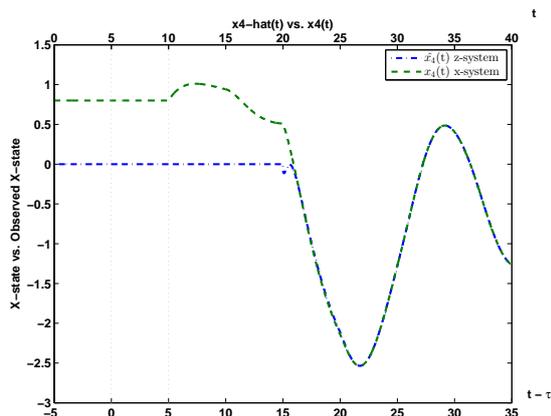

Figure 5: $x_4(t)$ from original system vs. $\widehat{x}_4(t)$ derived from the observer of the transformed system ($x_4(t) \neq z_4(t)$)

4.2 Reduced-order Observer

A reduced-order observer of the structure mentioned in (27)-(28) is constructed. The computed parameters are as follows.

$$E = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \qquad N = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix},$$

$$J = \begin{bmatrix} -8.12 & 0 \\ 0 & -13.04 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 2.02 & 0 \\ 0 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & 1.28 \\ 0 & 0 \end{bmatrix}, \qquad M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 3.21 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0 & 0 \\ 0.57 & 0 \end{bmatrix}, \qquad M_5 = \begin{bmatrix} -0.06 & 0 \\ 0 & 0 \end{bmatrix},$$

$$M_6 = \begin{bmatrix} 0 & 0 \\ 0 & -0.18 \end{bmatrix}, \quad M_7 = \begin{bmatrix} 0 & -0.06 \\ 0 & 0 \end{bmatrix},$$

$$M_8 = \begin{bmatrix} 0 & 0 \\ -0.09 & 0 \end{bmatrix}, \qquad M_9 = \begin{bmatrix} 0 & 0.12 \\ 0 & 0 \end{bmatrix},$$

$$M_{10} = \begin{bmatrix} 0 & 0 \\ 0 & -0.09 \end{bmatrix}, \quad H = \begin{bmatrix} 0.27 & 0 \\ 0 & -1.55 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -0.01 & 0 \\ 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & -0.08 \\ 0 & 0 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0.12 \end{bmatrix}, \text{ and } K_4 = \begin{bmatrix} 0 & 0 \\ -0.015 & 0 \end{bmatrix}.$$

From (26), since essentially, $f(t) = \begin{bmatrix} z_3(t) \\ z_4(t) \end{bmatrix}$, therefore, $\hat{f}(t) = \begin{bmatrix} \hat{z}_3(t) \\ \hat{z}_4(t) \end{bmatrix}$. Again, $\hat{x}_2$ and $\hat{x}_4$ can be deduced according to equation (29) as soon as $\hat{f}(t)$ and the measurable output $y(t)$ are available.

Simulation of the z-domain reduced-order observer was carried out and the comparison between the x state vector and its deduced estimates are shown below.
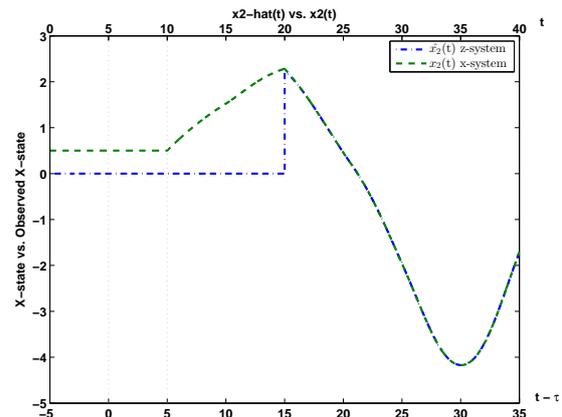

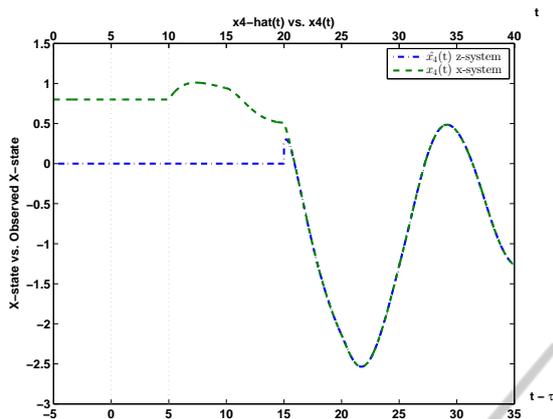Figure 6: $x_2(t)$ from the original system vs. $\hat{x}_2(t)$ derived from the observer of the transformed system.

Figure 7: $x_4(t)$ from the original system vs. $\hat{x}_4(t)$ derived from the observer of the transformed system.

5 CONCLUSIONS

This paper has applied a state transformation method of Hou et al. (2002) into the design of state observers for interconnected time-delay systems. Through the use of a coordinate transformation, an equivalent system of different coordinates has been established, effectively redefining the restrictive time-delay problem in the state vector into a less complex problem of having time-delay terms in the input and output. This in turn opens up the opportunity of accommodating well-established standard observer design techniques for delay-free linear systems which have otherwise lacked viability in the original interconnected time-delay system. Numerical results show that, for the coordinate-transformed time-delay system, observers of desirable asymptotic convergence properties may be designed using estimation theory available for delay-free systems. Further work is needed in order to meet constraints imposed on the flow of information in an interconnected system. Hence some forms of distributed or decentralized observer schemes will be a possible topic for future research.

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