An Explicit Bound for Stability of Sinc Bases

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Abstract: It is well known that exponential Riesz bases are stable. The celebrated theorem by Kadec shows that $1/4$ is a stability bound for the exponential basis on $L^2(-\pi, \pi)$. In this paper we prove that α/π (where α is the Lamb-Oseen constant) is a stability bound for the sinc basis on $L^2(-\pi, \pi)$. The difference between the two values $\alpha/\pi - 1/4$, is ≈ 0.15, therefore the stability bound for the sinc basis on $L^2(-\pi, \pi)$ is greater than Kadec's stability bound (i.e. $1/4$).

1 INTRODUCTION

It is well known that exponential Riesz bases $\{e^{inx}\}_{n\in\mathbb{Z}}$ are stable, this means that a small perturbation of a Riesz basis produces a Riesz basis (Paley and Wiener, 1934). See also the Young's textbook (Young, 2001). The proof of the Paley-Wiener theorem does not provide an explicit stability bound. The celebrated theorem by Kadec shows that $1/4$ is a stability bound for the exponential basis on the space $L^2(-\pi, \pi)$ of square integrable functions.

In this paper we prove, in the spirit of the Kadec's result, but with a different mathematical approach, a stability result for cardinal sine sequence $\{\text{sinc}(x - n)\}_{n\in\mathbb{Z}}$ showing that α/π, where α is the Lamb-Oseen constant (Oseen, 1912), is a stability bound for the sinc basis on $L^2(-\pi, \pi)$. The difference between the two values $\alpha/\pi - 1/4$, is ≈ 0.15. By $L^2(-\infty, +\infty)$ we denote the Hilbert space of real functions that are square integrable in Lebesgue's sense:

$$L^2(\mathbb{R}) = \left\{ f : \int_{-\infty}^{+\infty} |f(x)|^2 dx < +\infty \right\}$$

with respect to the inner product and L^2-norm that, on $[-\pi, \pi]$, are

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx \qquad ||f|| = \sqrt{\langle f, f \rangle}$$

Given $f \in L^2(\mathbb{R})$ we denote by $\hat{f}$ the Fourier transform of f,

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx.$$

Parseval's equality states

$$\|f\|_2 = \|\hat{f}\|_2 \tag{1}$$

If f represents the signal, assuming that $f \in L^2(\mathbb{R})$ (the energy of the signal is finite), then f is said band-limited to $[-\pi, \pi]$ if $\hat{f}$ vanishes outside the set $[-\pi, \pi]$. The space of band-limited to $[-\pi, \pi]$ functions is the Paley-Wiener space, usually denoted by PW_π. It is defined by $\{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}), \ \text{supp}\hat{f} \subset [-\pi, \pi]\}$ and it follows, for instance, from its characterization by using the classical Paley-Wiener theorem (Young, 2001), p. 85, i.e.:

$$\{\text{f entire function} : |f(z)| \leqslant Ae^{\pi|z|},$$
$$z \in \mathbb{C}, \ f|_\mathbb{R} \in L^2(\mathbb{R})\} \tag{2}$$

The space PW_π play a significant role in signal processing applications (Higgins, 1985). As well known, any function $f \in PW_\pi$, can be expanded in terms of the orthonormal basis $\{e^{inx}\}_{n\in\mathbb{Z}}$ as
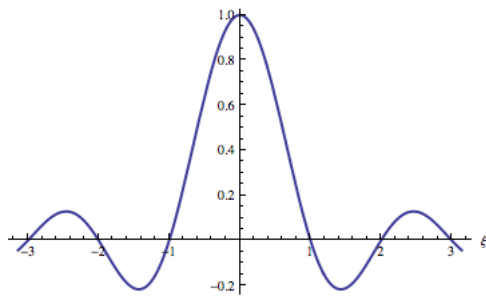
$$\hat{f}(x) = \sum_{n\in\mathbb{Z}} \langle \hat{f}, e^{inx} \rangle_{L^2(-\pi,\pi)} e^{inx} \tag{3}$$

where $\langle g, h \rangle_{L^2(-\pi,\pi)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\overline{h(x)}dx$. Taking the inverse Fourier transform in (3), we obtain the Whittaker-Kotelnikov-Shannon (WKS) sampling theorem,

$$f(x) = \sum_{n\in\mathbb{Z}} f(n)\text{sinc}(x - n), \quad x \in \mathbb{R} \tag{4}$$

with $\text{sinc}(x)$ the normalized sinc function commonly defined as

$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & x \neq 0, \\ 1 & x = 0, \end{cases} \tag{5}$$

Figure 1: Graph of sinc ξ.

whose graph is shown in figure (1).

The subject of recovery of band-limited signals from discrete data has its origins in the WKS sampling theorem, historically the first and simplest such recovery formula. It expresses the possibility of recovering a certain kind of signals from a sequence of regularly spaced samples. Without loss of generality, the formula (4) recovers a function with a frequency band of $[-\pi, \pi]$ given the functions values at the integers. But the WKS theorem has drawbacks. Foremost, the recovery formula does not converge given certain types of error in the sampled data, as Daubechies and De Vore mention in (Daubechies and DeVore, 2003). They use oversampling to derive an alternative recovery formula which does not have this defect. Furthermore, as already said, for the WKS theorem, the data nodes have to be equally spaced, and nonuniform sampling nodes are not allowed but, from many practical points of view it is necessary to develop sampling theorems for a sequence of samples taken with a nonuniform distribution along the real line.

As discussed in (Zayed, 2000), nonuniform sampling of band-limited functions has its roots in the work of Paley, Wiener, and Levinson. In fact, the first answer for this direction was given by Paley and Wiener (Paley and Wiener, 1934), and later an advanced result was presented by Levinson (Levinson, 1940). Their sampling formulae recover a function from nodes $\{\lambda_n\}_n$, where $\{e^{i\lambda_n x}\}_n$ forms a Riesz basis for $L^2[-\pi, \pi]$. The result is related with the perturbation of a Hilbert basis $\{e^{inx}\}_{n \in \mathbb{Z}}$ for the function space $L^2[-\pi, \pi]$ in such a way that the perturbed sequence $\{e^{i\lambda_n x}\}_n$ is also a Riesz basis for the same space. The maximum perturbation of the system $\{e^{inx}\}_n$ is found by Kadec, whose result is the celebrated *Kadec-1/4 theorem* (Kadec, 1964). This result plays a very important role in signal theory; it suffices to think for example, that the formula (3) expresses the fact that the $\hat{f}(x)$ can be seen as infinite sum of elementary contributions of exponential type complex. Modern digital data processing of functions (or signals or images) always uses a discretized version of the original sig-

nal f that is obtained by sampling f on a discrete set. The question then arises whether and how f can be recovered from its samples. Therefore, the objective of research on the sampling problem is twofold. The first goal is to quantify the conditions under which it is possible to recover particular classes of functions from different sets of discrete samples. The second goal is to use these analytical results to develop explicit reconstruction schemes for the analysis and processing of digital data. In particular, the results by Paley and Wiener, Kadec and others on the nonharmonic Fourier bases $\{e^{i\lambda_n x}\}_{n \in \mathbb{Z}}$ can be translated into results about nonuniform sampling and reconstruction of band-limited functions: (Benedetto, 1991), (Higgins, 1994), (Pavlov, 1979b), (Seip, 1995), (Zayed, 2000).

Our article concentrates on perturbation of regular sampling and are therefore similar in spirit to Kadec's result for band-limited functions, though it is based on a different point of view. Is the Riesz bases $\{\text{sinc}(x - n)\}_{n \in \mathbb{Z}}$ for the function space $L^2[-\pi, \pi]$ to being perturbed, in $\{\text{sinc}(x - \lambda_n)\}_{n \in \mathbb{Z}}$, not the complex exponentials. The result is also extended to $\{\text{sinc}(z - n)\}_{n \in \mathbb{Z}}$ for $PW_{[-\pi, \pi]}$, with $z \in \mathbb{C}$. The stability bound for the sinc basis on $L^2(-\pi, \pi)$ (or $PW_{[-\pi, \pi]}$) is greater than Kadec's stability bound (i.e. $1/4$); in some sense, the result obtained here for sinc basis can be seen as an improvement of Kadec's estimate.

Kadec theorem has been extensively generalized (see, for example (Avdonin, 1974), (Bailey, 2010), (Khrushchev, 1979), (Pavlov, 1979a), (Savchuk and Shkalikov, 2006), (Sun and Zhou, 1999), (Vellucci, 2014)) but to the best of our knowledge, there are no versions of this theorem for sinc bases. The paper is divided in two sections. For other contributions to exponential Riesz basis problem and Kadec's theorem see survey papers, as: (Ullrich, 1980), (Sedletskii, 2009). The first section contains small overview on the Lamb-Oseen constant and thereafter we revised know properties for sinc functions. The second section is devoted to $\{\text{sinc}(x - n)\}_{n \in Z}$.

1.1 Lambert Function W, Lamb-Oseen Constant

The Lambert function *W* (R. M. Corless, 1996), (Stewart, 2005), (Hayes, 2005), is defined by the equation

$$W(x)e^{W(x)} = x \qquad (6)$$

It is direct to find that the function $f(\xi) = \xi e^{\xi}$ for $\xi \in \mathbb{R}$ has a strict minimum point in $\xi = -1$. Indeed $\xi = -1$ is a strict relative point of minimum with
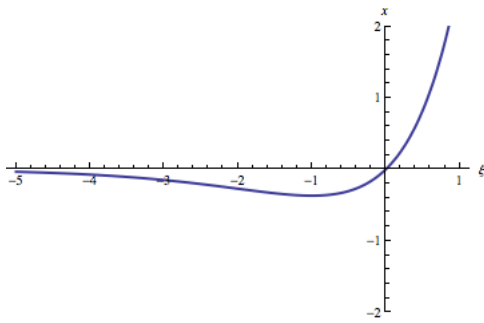
Figure 2: Diagram of $\xi e^{\xi} = f(\xi)$.

$f(-1) = -\frac{1}{e}$, and since

$$\lim_{\xi \to -\infty} \xi e^{\xi} = 0 \qquad \lim_{\xi \to +\infty} \xi e^{\xi} = +\infty,$$

the minimum is global, that is

$$\min_{\xi \in \mathbb{R}} \xi e^{\xi} = -\frac{1}{e}$$

We may draw the diagram of $\xi e^{\xi} = f(\xi)$. The figure (2) clarifies the behaviour of the function $f(\xi) = \xi e^{\xi}$. It is clear that the existence of the Lambert function $W(x)$ depends on the value of x.

More precisely, we have

i) if $x < -1/e$ the equation (6) has no real solutions;

ii) if $-1/e \leqslant x < 0$ the equation (6) admits two solutions;

iii) if $x \geqslant 0$ the equation (6) admits one solution.

The statements i), ii) and iii) are specified with the following properties of $f(\xi) = \xi e^{\xi}$, $\xi \in \mathbb{R}$.

i The function $f(\xi) = \xi e^{\xi}$ is strictly increasing in the interval $(-1, +\infty)$;

ii The function $f(\xi) = \xi e^{\xi}$ is strictly decreasing in the interval $(-\infty, -1)$.

It follows

Proposition 1.1. *The function $f(\xi) = \xi e^{\xi}$ has an increasing inverse in $(-1, +\infty)$, and a decreasing inverse in $(-\infty, -1)$.*

We consider $f(\xi) = \xi e^{\xi}$ restricted to the interval $(-\infty, -1]$ and we denote by W_{-1} its inverse. W_{-1} is defined in the interval $[-1/e, 0)$. We have two identities arising from the definition of W_{-1}:

$$W_{-1}(\xi e^{\xi}) = \xi,$$
$$\left[\Leftrightarrow W_{-1}[f(\xi)] = W_{-1}\left[\xi e^{\xi}\right] = \xi \right],$$
$$\forall \xi \in (-\infty, -1] \tag{7}$$

and

$$W_{-1}(\bar{x}) e^{W_{-1}(\bar{x})} = \bar{x} \quad [\Rightarrow f(W_{-1}(\bar{x})) = \bar{x}]$$
$$\forall \bar{x} \in [-\frac{1}{e}, 0) \tag{8}$$

Also we denote by W_0 the restriction to the interval $[-1/e, 0)$ of the increasing inverse of $f(\xi) = \xi e^{\xi}$. The two identities hold true:

$$W_0(\xi e^{\xi}) = \xi,$$
$$\left[\Leftrightarrow W_0[f(\xi)] = W_0\left[\xi e^{\xi}\right] = \xi \right],$$
$$\forall \xi \in [-1, 0) \tag{9}$$

and

$$W_0(\bar{x}) e^{W_0(\bar{x})} = \bar{x} [\Rightarrow f(W_0(\bar{x})) = \bar{x}]$$
$$\forall \bar{x} \in [-\frac{1}{e}, 0) \tag{10}$$

1.1.1 Numerical Values

Let us assume that $\bar{x}$ is a solution of our equation.

$$e^{\bar{x}} - 2\bar{x} = 1 \tag{11}$$

In order to use the Lambert function W, we observe that from (11) we get the equivalences

$$e^{\bar{x}} - 2\bar{x} = 1 \Leftrightarrow e^{e^{\bar{x}} - 2\bar{x}} = e$$

whence $-\frac{1}{2}e^{\bar{x}}e^{-\frac{1}{2}e^{\bar{x}}} = -\frac{1}{2}e^{-\frac{1}{2}}$. Therefore we can identifies $-\frac{1}{2}e^{\bar{x}}$ with $W\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)$. Since $-\frac{1}{e} < -\frac{1}{2}e^{-\frac{1}{2}} < 0$, the equation which defines the function W of Lambert, has two branches which verifies the same equation $W(x)e^{W(x)} = -\frac{1}{2}e^{-\frac{1}{2}}$ and we will have

$$-\frac{1}{2}e^{\bar{x}} = W_0\left(-\frac{1}{2}e^{-\frac{1}{2}}\right) \tag{12}$$

and

$$-\frac{1}{2}e^{\bar{x}} = W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right). \tag{13}$$

We call $\bar{x}_1$ the $\bar{x}$ solution of (12), and $\bar{x}_2$ the solution of (13).

We state easy that $\bar{x}_1 = 0$. In fact from (12) we have

$$-\frac{1}{2}e^{\bar{x}_1} = -\frac{1}{2}\left[= W_0\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)\right]$$

and from $e^{\bar{x}_1} = 1$, easily follows $\bar{x}_1 = 0$.

From (13), and the relation (8) we get $e^{\bar{x}_2} = -2W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)$, and so $\bar{x}_2$:

$$\ln\left(-2W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)\right) = -\ln\frac{1}{-2W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)}$$

Now we multiply numerator and denominator by $e^{\frac{1}{2}}$

$$= -\ln\left[\frac{-\frac{1}{2}e^{-\frac{1}{2}}}{W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)}e^{\frac{1}{2}}\right]$$

$$= -\frac{1}{2} - \ln\frac{-\frac{1}{2}e^{-\frac{1}{2}}}{W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)}$$

By (8) we have

$$\bar{x}_2 = -\frac{1}{2} - W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right).$$

The value $-\frac{1}{2} - W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)$ is called the parameter of Oseen, or Lamb-Oseen constant. It is often denoted by the Greek character α. Numerical estimates give

$$\alpha = 1.25643...$$

1.1.2 Properties of $\alpha = -\frac{1}{2} - W_{-1}\left(-\frac{1}{2}e^{-\frac{1}{2}}\right)$

We have introduced the Lambert function W in order to give an useful expression to the root of equation

$$e^{\alpha} = 2\alpha + 1. \qquad (14)$$

It is easy to prove:

Proposition 1.2. *The real number α is transcendental.*

Proof. Such thesis means that α is not root of an algebraic equation with rational coefficients. In fact, if α were algebraic, then algebraic should be the right side of (14). But Lindemann - Weierstrass theorem (Baker, 1990), state that e^{α} should be transcendental. This is a contradiction and the thesis follows. $\qquad\square$

1.2 Known Results for Whittakers Cardinal Series

Hardy who was referring to (4) - in literature also known as basis functions in Whittakers cardinal series - wrote: "It is odd that, although these functions occur repeatedly in analysis, especially in the theory of interpolation, it does not seem to have been remarked explicitly that they form an orthogonal system" (Hardy, 1941). See also: (Benedetto, 1998), (Butzer, 1983), (Whittaker, 1915).

Orthonormality is a fundamental property of the sinc-function. There is a well-known property (whose proof is given for the reader convenience), necessary for the orthonormality proof of the system.

Since for any $\lambda \in \mathbb{R}$, the function

$$f_{\lambda} = f_{\lambda}(\xi) = \begin{cases} 0 & |\xi| > \pi \\ e^{i\lambda\xi} & \text{otherwise,} \end{cases}$$

has Fourier transform $\mathcal{F}(f_{\lambda})(\tau) = \text{sinc}(\tau - \lambda)$. By Paley Wiener theorem

$$\text{Supp } g \subset [-\pi, \pi] \implies (\mathcal{F}g) \in PW_{\pi}, \qquad (15)$$

we have that $\text{sinc} \in PW_{\pi}$, as consequence of Parseval identity (1) we see that

$$\forall \lambda \in \mathbb{R} \;\; \text{sinc}(\tau - \lambda) \in L^2(\mathbb{R}) \Rightarrow$$
$$\Rightarrow \;\; \|\text{sinc}(\tau - \lambda)\|_{L^2(\mathbb{R})} = 1 \qquad (16)$$

Moreover:

Proposition 1.3. *If $f \in PW_{\pi}$, and*

$$\int_{\mathbb{R}} f(\tau)\text{sinc}(\tau - n)\,d\tau = 0, \quad \forall n \in \mathbb{Z},$$

then $f = 0$ a.e..

Proof. Indeed we may select $f \in PW_{\pi}$, then by definition of PW space $\exists\, g \in L^2(\mathbb{R})$, such that

$$\mathcal{F}f = g \qquad \text{and} \qquad \text{supp}(g) \subset [-\pi, \pi]. \qquad (17)$$

We now consider

$$\int_{\mathbb{R}} f(\tau)\text{sinc}(\tau - n)\,d\tau =$$

Applying Parseval equality (1) and changing τ in ξ we have

$$\frac{1}{2\pi}\int_{\mathbb{R}} f(\tau)\mathcal{F}(u_n)\,d\tau = \frac{1}{2\pi}\int_{\mathbb{R}} (\mathcal{F}f)(\xi)u_n(\xi)\,d\xi$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\xi)e^{in\xi}\,d\xi \qquad (18)$$

Then the assumption becomes

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\xi)e^{in\xi}\,d\xi = 0, \quad \forall n \in Z,$$

and the thesis follows since $\{e^{in\xi}\}_{n\in\mathbb{Z}}$ is a basis in $L^2(-\pi, \pi)$, therefore $g = 0$, and hence $f = 0$. $\qquad\square$

A useful property of sinc-functions, used in this paper, is the following well-known property whose proof is given for readers convenience

Proposition 1.4. *For any real numbers λ and ν, we have*

$$\int_{\mathbb{R}} \text{sinc}(\tau - \lambda)\text{sinc}(\tau - \nu)\,d\tau = \text{sinc}(\lambda - \nu). \qquad (19)$$

Proof. Consider the LHS multiplied by $4\pi^2$

$$4\pi^2\int_{\mathbb{R}} \text{sinc}(\tau - \lambda)\text{sinc}(\tau - \nu)\,d\tau =$$
$$= \int_{\mathbb{R}} 2\pi\text{sinc}(\tau - \lambda)\overline{2\pi\text{sinc}(\tau - \nu)}\,d\tau =$$
$$= \int_{\mathbb{R}} \mathcal{F}(u_{\lambda})(\xi)\overline{\mathcal{F}(u_{\nu})}(\xi)\,d\xi$$

which becomes, solving the Fourier transform

$$= 2\pi \int_{\mathbb{R}} u_\lambda(\xi)\overline{(u_\nu)}(\xi)d\xi = 2\pi \int_{-\pi}^{\pi} e^{i(\lambda-\nu)\xi}d\xi$$

$$= 4\pi^2 \frac{\sin\pi(\lambda-\nu)}{\pi(\lambda-\nu)} = 4\pi^2 \mathrm{sinc}(\lambda-\nu) \qquad (20)$$

$\square$

From the previous results follows that

$$\left\{ \mathrm{sinc}(x-n) \right\}_{n \in Z}$$

is an orthonormal basis in $L^2(\mathbb{R})$. By (Higgins, 1985), we have a generalization of this statement: the system of *sinc functions* $\{\mathrm{sinc}(z-n)\}_{n \in Z}$ is an orthonormal basis in PW_π. Shannon mentioned the orthogonality without proof (Shannon, 1949); Hardy, on the other hand, proceeds from the complete orthonormal character of $\{e^{inx}\}$ in $L^2(-\pi,\pi)$ via the Fourier transform, (Higgins, 1985) and (Hardy, 1941). Accordingly, since $\mathrm{sinc}(z-n)$, $z \in \mathbb{C}$, is the image of e^{-inx} under the inverse transform, the collection $\{\mathrm{sinc}(z-n)\}_{n \in \mathbb{Z}}$ is an orthonormal basis of PW_π (Eoff, 1995). There are some mathematical connections with signal theory. The modern approach to sampling techniques and, more generally, to signal theory is undoubtedly based on Hilbert-space formulation, which allows to reinterpret the original Shannons sampling procedure as an orthogonal projection onto the subspace of band-limited functions. In this mathematical representation, the continuous signal is considered a function of the continuous variable $x \in \mathbb{R}$. We recall, however, that very often we do not talk really functions, but rather of equivalence classes of functions, where two functions are equivalent if they differ in a set of measure zero. This is the case when we consider the elements of the spaces L^p with $1 \leqslant p \leqslant \infty$. Here, change the value of a function at a point does not change its equivalence class, since the points have measure zero.

We now assume that the input function f that we want to sample is in $L^2(-\pi,\pi)$, a space that is considerably larger than the usual subspace of band-limited functions, which we have called PW_π to indicate that we consider only functions defined in $[-\pi,\pi]$.

The orthonormality property previously expressed, greatly simplifies the implementation of the approximation process by which a function $f \in L^2$ is projected onto PW_π. Specifically, the orthogonal projection operator $P : L^2 \to PW_\pi$ can be rewritten as

$$P f = \sum_{n \in \mathbb{Z}} \langle f, \mathrm{sinc}(x-n) \rangle \, \mathrm{sinc}(x-n)$$

where the inner product represents the signal contribution along the direction specified by $\mathrm{sinc}(x-n)$ because of the orthogonality of the basis functions. This
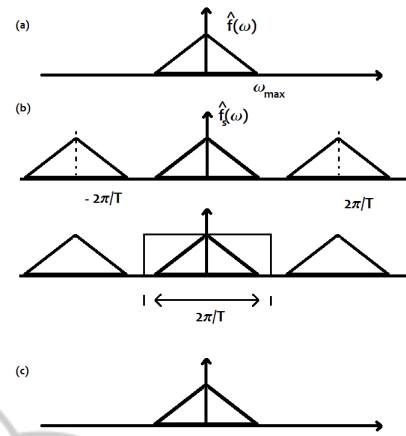

Figure 3: Frequency interpretation of the sampling theorem. (a) the Fourier transform of the signal and its bandwidth $B = \omega_{max}$; the signal is band-limited, i.e. $\hat{f}(x) = 0$ without the bandwidth (b) sampling in time means to replicate the Fourier transform in the frequencies, and (c) the analog signal is reconstructed by ideal low-pass filtering.
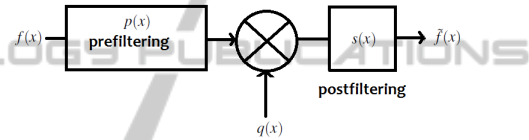

Figure 4: A single branch of modulation and filtering: the channel output is prefiltered by a filter with impulse response $p(x)$, modulated by a sequence $q(x)$, post-filtered by another filter of impulse response $s(x)$. In the traditional approach, the pre- and postfilters are both ideal low-pass: $p(x) = s(x) = \mathrm{sinc}(x)$. In the more modern schemes, the filters can be selected more freely under the constraint that they remain biorthogonal: $\langle p(x-k), s(x-l) \rangle = \delta_{k-l}$.

inner product computation is equivalent to first filtering the input function with the ideal low-pass filter and sampling thereafter, (Unser, 2000). See figure (3).

In sampling theory, the main goal is to reconstruct a continuous function $g(x)$ from its samples $g(x_i)$. If the sampling is uniform, i.e., the set of these samples lies on a uniform Cartesian grid (see figure (5)), then the function $g(x)$ can be recovered exactly from its samples as long as $g \in L^2(\mathbb{R})$ is band-limited and the grid-points density is larger than the Nyquist density (Shannon, 1949), (Aldroubi and Feichtinger, 1998). This is the meaning of WKS sampling theorem.

The classical result for non-uniform sampling is due to Paley and Wiener, as widely discussed in the introduction. See figure (4): optimal branch of modulation or sampling is employed, then mild perturbation of post-filtering uniform sampling sets does not degrade the sampled capacity. One general example was proved by Kadec (Kadec, 1964). Suppose that the sampling rate is f_s and the sampling set satisfies $|t_n - n/f_s| \leqslant f_s/4$. Then $\{e^{it_nx}\}_{n \in \mathbb{Z}}$ also forms a Riesz basis of $L^2(-\pi,\pi)$, thereby preserving infor-
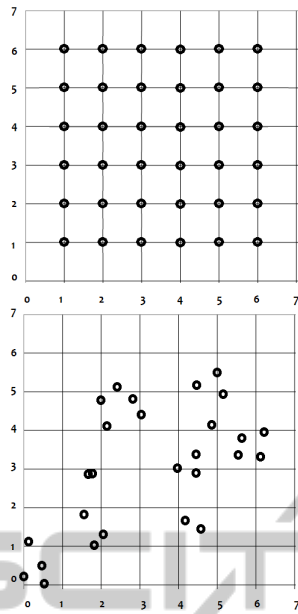
Figure 5: Sampling grids. Top: uniform cartesian sampling. Bottom: A typical nonuniform sampling set as encountered in various signal and image processing applications.
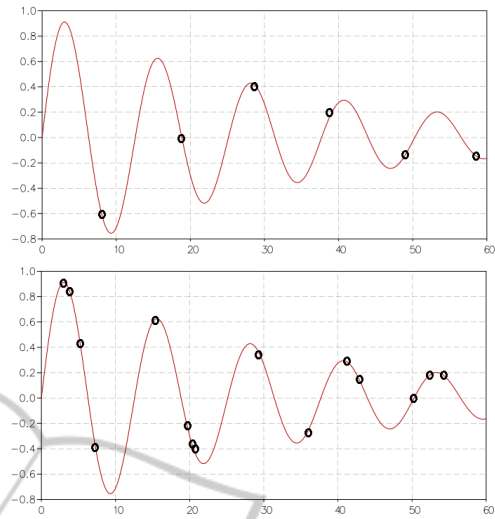

Figure 6: Top: A function f defined on $\mathbb{R}$ has been sampled on a uniformly spaced set. Bottom: The same function f has been sampled on a non-uniformly spaced set.

mation integrity. Note that the original Kadec's theorem has a similar formulation: $|t_n - n| \leqslant 1/4$, i.e. with sampling rate = 1. These nonuniform sampling and reconstruction schemes, while generally complicated to implement in practice, significantly broaden the class of sampling mechanisms that allow perfect reconstruction of band-limited signals, and indicate stability and robustness of the sampling sets. Kadecs result immediately implies that the sampled capacity is invariant under mild perturbation of the sampling sets, (Y. Chen, 2014).

We conclude this section, writing a different version of WSK theorem existing in literature, for complex functions, is the following

Theorem 1.5. *Let* $f \in PW_\pi$. *Then:*

$$f(z) = \sum_{n \in \mathbb{Z}} f(n) \, \mathrm{sinc}(z - n), \tag{21}$$

uniformly on compact subsets of $\mathbb{C}$.

(see, for example: (A. G. García, 1998), (Higgins, 1996)).

2 Kadec-type Estimate

Result is given by

Theorem 2.1. *If* $\{\lambda_n\}$ *is a sequence of complex numbers for which*

$$|\lambda_n - n| \leqq L < \frac{\alpha}{\pi}, \quad n = 0, \pm 1, \pm 2, \dots \tag{22}$$

then $\{\mathrm{sinc}(\lambda_n - t)\}$ *satisfies the Paley-Wiener criterion and so forms a Riesz basis for* $L^2(-\pi, \pi)$.

Proof. We show that (Paley-Wiener criterion, Young's book (Young, 2001), theorem 13 p. 35)

$$\left\| \sum_n c_n \left(\mathrm{sinc}(\lambda_n - t) - \mathrm{sinc}(n - t) \right) \right\| \leqslant \lambda < 1 \tag{23}$$

whenever $\sum_n |c_n|^2 \leqq 1$. We use the Taylor series of $\mathrm{sinc}(\lambda_n - t)$:

$$\mathrm{sinc}(n - t) + \sum_{k=1}^{+\infty} \frac{(\lambda_n - n)^k}{k!} \frac{d^k}{dz^k} \left(\mathrm{sinc}\, z \right)_{z = n - t}$$

Therefore we have, since $\sum_n |c_n|^2 \leqslant 1$,

$$\left\| \sum_n c_n \left(\mathrm{sinc}(\lambda_n - t) - \mathrm{sinc}(n - t) \right) \right\| =$$

$$= \left\| \sum_n c_n \sum_{k=1}^{+\infty} \frac{(\lambda_n - n)^k}{k!} \frac{d^k}{dz^k} \left(\mathrm{sinc}\, z \right)_{z = n - t} \right\|$$

$$\leqslant \sum_{k=1}^{\infty} \frac{L^k}{k!} \left| \frac{d^k}{dz^k} \mathrm{sinc}\, z \right|_{z = n - t} \tag{24}$$

On the other hand,

$$\mathrm{sinc}\, z = \frac{\sin \pi z}{\pi z} = \int_0^1 \cos(s \pi z) ds$$

and

$$\frac{d^k}{dz^k} \left\{ \frac{\sin \pi z}{\pi z} \right\} =$$

$$= \pi^k \int_0^1 s^k \cos\left(s \pi z + k \frac{\pi}{2} \right) ds \tag{25}$$

Putting $z = x + \iota y$ in the last formula, we get:

$$\frac{d^k}{dz^k}\left\{\frac{\sin \pi z}{\pi z}\right\} \leqslant$$

$$\leqslant \pi^k \int_0^1 |s^k|\left|\cos\left(s\pi x + k\frac{\pi}{2} + \iota s\pi y\right)\right|ds$$

$$\leqslant \pi^k \int_0^1 s^k \cosh(s\pi y)ds$$

$$= \frac{\pi^k}{k+1}\cosh \pi y \qquad (26)$$

Putting in formula (24): $z = n - t$, i.e. $x = n - t$ and $y = 0$, it result

$$\left\|\left\|\sum_n c_n\left(\mathrm{sinc}(\lambda_n - t) - \mathrm{sinc}(n-t)\right)\right\|\right\| \leqslant$$

$$\leqslant \sum_{k=1}^\infty \frac{(\pi L)^k}{(k+1)!} =: \lambda \qquad (27)$$

and we find $\lambda = \frac{1}{M}\left(e^M - M - 1\right)$ where $M = \pi L$. In order to get $\lambda < 1$, we solve, in a first moment, the equation $\lambda = 1$, that is

$$e^M = 2M + 1 \qquad (28)$$

We obtain same useful properties on the solutions of equation (28), using the Lambert Function W. Such function is defined by the equation $W(x)e^{W(x)} = x$ or, for the true meaning, by $\xi e^\xi = x$. From previous section we obtain the thesis. $\qquad\square$

The proof of the following theorem is the same as the previous theorem, and we will omit it.

Theorem 2.2. *If* $\{\lambda_n\}$ *is a sequence of complex numbers for which*

$$|\lambda_n - n| \leqq L < \frac{\alpha}{\pi}, \;\; n = 0, \pm 1, \pm 2, ... \qquad (29)$$

then $\{\mathrm{sinc}(\lambda_n - z)\}$ *satisfies the Paley-Wiener criterion and so forms a Riesz basis for* PW_π.

3 CONCLUSIONS AND FUTURE DEVELOPMENTS

It is possible to generalize the approach of the sampling theorem to other classes of functions, thanks to a greater abstractness earned using Hilbert spaces and projection operators. This is achieved by simply replacing $\mathrm{sinc}\,x$ by a more general $\phi(x)$, called *generating function*. Consequently, we specify the approximation space V as

$$V(\phi) = \left\{\sum_{k \in \mathbb{Z}} c_k \phi(x-k) : \{c_n\}_{n \in \mathbb{Z}} \in \ell^2\right\}$$

where

$$\ell^2 = \left\{\{x_n\}_{n \in \mathbb{Z}} \;\middle|\; \sum_{k \in \mathbb{Z}} |x_k|^2 < \infty\right\}$$

is the space of square-summable sequences. This means that any function $s(x) \in V(\phi)$, continuous and characterized by a sequence of coefficients c_k is the discrete representation of the signal in the signal processing (note that not necessarily c_k are the samples of the signal, and ϕ can be significantly different from the $\mathrm{sinc}(x)$). Indeed, one of our motivations is to discover functions that are simpler to handle numerically and have a much faster decay. Though we need some mathematical safeguards:

- The coefficients $\{c_n\}_{n \in \mathbb{Z}} \in \ell^2$.

- Second, the representation should be stable1 and unambiguously defined. In other words, the family of functions $\{\phi(x-k) : k \in \mathbb{Z}\}$ should form a Riesz basis of $V(\phi)$.

In this view, and before listing the developments of this work, we try to point out a possible practical implication of the theorem 2. Assume the existence of a formula for reconstruction, like the (4) and that, taking the Fourier transform in this formula, we could get

$$\hat{f}(x) = \sum_{n \in \mathbb{Z}} \langle \hat{f}, \mathrm{sinc}(x-n)\rangle_{L^2(-\pi, \pi)} \mathrm{sinc}(x-n)$$

Thus, reconstruction by means of this formula is equivalent to the fact that the set $\{\mathrm{sinc}(x - \lambda_n), n \in \mathbb{Z}\}$ formed an orthonormal basis for $L^2(-\pi, \pi)$. According to this interpretation, our theorem could state that if we have sampling set

$$\{\lambda_n \in \mathbb{R} : |\lambda_n - n| \leqslant L < \alpha/\pi\} \qquad (30)$$

for all $k \in \mathbb{Z}$, then the set $\{\mathrm{sinc}(x - \lambda_n), n \in \mathbb{Z}\}$ is a Riesz basis for $L^2(-\pi, \pi)$ and, on the other hand, that $\{\mathrm{sinc}(x - \lambda_n), n \in \mathbb{Z}\}$ is the image of an orthonormal basis for $L^2(-\pi, \pi)$ under a bounded and invertible operator from $L^2(-\pi, \pi)$ in itself. So the problem to recover band-limited function from its samples merit a deeper investigation. These questions will be investigated elsewhere.

REFERENCES

A. G. García, J. M. (1998). On the distributional fourier duality and its applications. *J. Math. Anal. Appl.*

Aldroubi, A. and Feichtinger, H. (1998). Exact iterative reconstruction algorithm for multivariate irregularly sampled functions in spline-like spaces: The lp theory. *Proc. Amer. Math.*

Avdonin, S. (1974). On the question of riesz bases of exponential functions in l2. *Vestnik Leningrad. Univ..*

Bailey, B. (2010). Sampling and recovery of multidimensional bandlimited functions via frames. *J. Math. Anal. Appl.*

Baker, A. (1990). *Transcendental Number Theory*. Cambridge University Press.

Benedetto, J. (1991). *Irregular sampling and frames, in: C. Chui (Ed.), Wavelets A Tutorial in Theory and Applications.* Academic Press.

Benedetto, J. (1998). *Frames, sampling, and seizure prediction.* Springer-Verlag.

Butzer, P. L. (1983). A survey of the whittakershannon sampling theorem and some of its extensions. *J. Math. Res. Exposit.*

Daubechies, I. and DeVore, R. (2003). Approximating a bandlimited function using very coursely quantized data: a family of stable sigma-delta modulators of arbitrary order. *Ann. of Math.*

Eoff, C. (1995). The discrete nature of the paley-wiener spaces. *Proc. Amer. Math. Soc.*

Hardy, G. H. (1941). Notes on special systems of orthogonal functions, iv: The orthogonal functions of whittakerys cardinal series. *Proc. Cambridge Philos. Soc.*

Hayes, B. (2005). Why w? *American Scientist*.

Higgins, J. (1985). Five short stories about the cardinal series. *Bull. Amer. Math. Soc.*

Higgins, J. (1994). Sampling theory for paley-wiener spaces in the riesz basis settings. *Proc. Roy. Irish Acad. Sect.*

Higgins, J. R. (1996). *Sampling Theory in Fourier and Signal Analysis: Foundations.* Claredon Press, Oxford.

Kadec, M. (1964). The exact value of the paleywiener constant. *Sov. Math. Dokl.*

Khrushchev, S. V. (1979). Perturbation theorems for exponential bases and the mackenhoupt condition. *Dokl. Akad. Nauk.*

Levinson, N. (1940). *Gap and Density Theorems.*, volume 26. Amer. Math. Soc. Colloquium Pubblication.

Oseen, C. W. (1912). Uber wirbelbewegung in einer reibenden flussigkeit. *Arkiv fr matematik, astronomi och fysik*.

Paley, R. and Wiener, N. (1934). *Fourier transforms in the complex domain.*, volume 19. Amer. Math. Soc. Colloquium Publications.

Pavlov, B. (1979a). Basicity of an exponential system and muckenhoupt's condition. *Sov. Math. Dokl.*

Pavlov, B. (1979b). The basis property of a system of exponentials and the condition of muckenhoupt. *Dokl. Akad. Nauk.*

R. M. Corless, G. H. G. e. a. (1996). On the lambert w function. *Advances in Computational Mathematics.*

Savchuk, A. M. and Shkalikov, A. A. (2006). On the eigenvalues of the sturmliouville operator with potentials from sobolev spaces. *Math. Notes.*

Sedletskii, A. M. (2009). Nonharmonic analysis. *J. Math. Sc.*

Seip, K. (1995). On the connection between exponential bases and certain related sequences in $l^2[-\pi, \pi]$. *J. Funct. Anal.*

Shannon, C. (1949). Communications in the presence of noise. *Proc. IRE.*

Stewart, S. M. (2005). A new elementary function for our curricula? *Australian Senior Mathematics Journal.*

Sun, W. and Zhou, X. (1999). On the stability of multivariate trigonometric systems. *J. Math. Anal. Appl.*

Ullrich, D. (1980). Divided differences and systems of nonharmonic fourier series. *Proc. Amer. Math. Soc.*

Unser, M. (2000). Sampling - 50 years after shannon. *Proceedings of the IEEE.*

Vellucci, P. (2014). A simple pointview for kadec-1/4 theorem in the complex case. *Ricerche di Matematica.*

Whittaker, J. (1915). On the functions which are represented by the expansion of interpolating theory. *Proc. R. Soc. Edinburgh.*

Y. Chen, A. J. Goldsmith, Y. C. E. (2014). Channel capacity under sub-nyquist nonuniform sampling. *IEEE Transactions on Information Theory.*

Young, R. (2001). *An introduction to non-harmonic Fourier series.* Academic Press, Inc.

Zayed, A. (2000). *Advances in Shannon's Sampling Theory.* CRC Press.