Sequential Games with Finite Horizon and Turn Selection Process
Finite Strategy Sets Case

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Abstract: A class of models of sequential games is proposed where the turns of decision are random for all players. The models presented show different variations in this class of games. In spite of the random nature of the turn selection process, first the number of turns per player is fixed, and afterwards models without this property are considered, as well as some that allow changes in other components. For all the models, a series of results are obtained to guarantee the existence of Nash equilibria. A possible application is shown for drafting athletes in sports leagues.

1 INTRODUCTION

The focus of game theory is on studying situations in which several individual subjects make decisions that affect all of them in the form of a utility reward. Essentially, any interaction that can be roughly described as before can be studied as a game.

In the classical theory (Fudenberg and Tirole, 1991), (Tadelis, 2013), the situations that can be analysed are deterministic in their rules, which means that a particular structure is established before the players interact, and such structure has no random elements once the players are making decisions. That, in itself, provides a frame to study the game mathematically and come up with the decisions that have to be taken in order to maximize the utility of each player, subject to the fact that each player can only make his own decision.

Another important characteristic to take into account to study the situation is the time frame of the game. When the decisions are made in discrete time, games can be divided in two sets: if players make their decisions at the same time, or if they only know about the structure of the game, but are not aware of the decisions of the other players before making their own, it is a simultaneous game. If there is a certain order of turns to make a decision such that players in the future may know the choices previously made in order to adapt their behaviour, then we have a sequential game. In the latter one, the number of turns is called the horizon of the game.

The first approach to introduce random elements was made in sequential games, where a player called Nature was to make its decision before any other player made his own. Because of the structure of sequential games, one could decide whether players would be aware of the behaviour of the Nature player, which allows to study situations with random externalities.

Nevertheless, once the game was afoot, there were no more random elements, which led to the study of stochastic games, which allowed the possibility of random events occurring between decisions taken in each period of time (Neyman and Sorin, 2003). To do this, the theory reverted to studying simultaneous games. Therefore, the situation is modelled as a repeated simultaneous game, where at every turn, the game moves onto a different stage depending on a distribution that observes the action of every player and the current stage. At every period all players receive a utility, and at the end, the utility of each player is calculated as the discounted sum.

In recent years, this approach has been expanded, for example, by studying games where for every time period there is a generation of players, who play a noncooperative game, but in each successive time period, there is a descendant of exactly one member of the previous generation, and all these relatives have to play cooperatively at the same time (Balbus and Nowak, 2008), (Balbus et al., 2013), (Nowak, 2010), (Woźny and Growiec, 2012). Another line of study has been the modelling of altruism as a means of ob-

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taining utility in a different way, by taking into account that the decision also influences the future, or understanding altruism as something that benefits the future generations that play the game (Nowak, 2006) (Saez-Marti and Weibull, 2005). Besides, some advances made recently have to do with the change of stage and its effects on the other characteristics of the game, for example, by restricting the options available to each player through the use of penal codes, that is, by making players themselves punish each other according to the deviations incurred by some of them (Kitti, 2011). All of these models are within the context of repeated simultaneous games, and they compute the utilities as a discounted sum of winnings obtained in each repetition of the game.

The work presented here is an effort to model situations in a sequential frame, where individuals take decisions in order, but where the order in which such decisions are made is not given from the beginning, but rather decided at every period of the game. The decision of whose turn it is is made by a selection process which, in the eyes of every player, is random, so players can assign a probability distribution to the selection process. Individuals only receive a utility at the end of the game, depending on all the decisions taken by all the players. Given this model, the existence of an equilibrium will be proven, and the result will be extended to some other models: the first one allows changing the way each player models the selection process as he learns new information; the second one makes the turn selection process to be conditioned on the decisions made by the players at all previous turns; and a third one, where the strategy sets are conditioned on the decisions made in previous turns. As far as we know, the models and results presented here are a novel contribution to the theory of sequential games.

The structure of this note is as follows: section 2 defines several models in the class of sequential games with the turn selection process. The models in subsections 2.1 and 2.2 consider a fixed number of decisions made by each player during the game. In subsection 2.3 the base model is described and refined to obtain the models described in subsections 2.4–2.6.

In section 3, using the model of subsection 2.3 as a base, the results that guarantee the existence of Nash equilibria in the models are proved. The propositions and their proofs can be easily modified to adapt them for the other models presented before.

Finally, in section 4 an application is presented in a "draft" of athletes for a certain sports league, where the order of selection is not presented beforehand and it changes throughout the game, as decisions are being made.

2 SEQUENTIAL GAMES AND TURN SELECTION PROCESS

2.1 One Turn per Player

Several concepts used in this article are standard in game theory and can be consulted in a wide selection of books (Tadelis, 2013). Subscripts in elements denote the player attributed to each element, superscripts denote the period of time considered, and left-side subscripts are used for indexing sequences. A glossary of terms can be found in the appendix at the end of the article.

The main components of this model are:

- A set of **players** $\mathcal{N} = \{1,2,\ldots,N\}$, with $N \in \mathbb{N}$ fixed.
- For each player j, $\mathcal{S}_j$ is a finite set of **pure strategies**, which contains all the possible decisions that can be made throughout the game.
- A fixed $T \in \mathbb{N}$, which is the number of turns to be played or the **horizon** of the game.
- A **utility function** $u_j : \Sigma \to \mathbb{R}$ for each player j, where Σ is the cartesian product of $\mathcal{S}$ with itself T times, where $\mathcal{S} = \cup_{j=1}^{N} \mathcal{S}_j$.
- A probability density $p_j : \mathcal{N} \to [0,1]$ of the distribution that models the turn selection process according to each player j.

Furthermore, $\mathcal{M}_j$ will be defined as the set of **mixed strategies** for every player j obtained from the set of pure strategies $\mathcal{S}_j$.

The first thing to observe in any of the following models is that, since no player has knowledge of the turns at which they'll be deciding, a strategy plan for each of them must have an action for every turn, and on each turn ℓ the decision should be conditioned on the **scenario** $\mathbf{s} = (\mathrm{s}^{\ell-1}, \ldots, \mathrm{s}^1)$ made of the decisions taken in periods 1 through $\ell - 1$. Therefore, every player j decides on s_j a **plan of conditioned strategies** for each period ℓ and for each possible scenario of previously taken decisions. Now, the set of plans is denoted by C_j where only pure conditioned strategies are used when confronted by any previous decisions and the set of plans made of mixed conditioned strategies by D_j for each player j.

Finally, it is possible to build vectors of N components, each of which is the strategy of a player. These vectors are the **profiles of conditioned strategies** for the game, whose sets will be denoted by C if they consist completely of pure conditioned strategies, or D if they allow the use of mixed conditioned strategies.

In order to find a solution to this model, a way to evaluate each possible profile is needed, while allowing randomization in the turn selection process. Therefore, an expectation operator is defined for each player j, and each profile $x \in \mathsf{D}$, by building the product measure of the probability distribution of the turn selection process and the mixed strategy that is being evaluated.

In this model, exactly one decision will be allowed per player, even though, the period of time in which that decision will be made is not known beforehand. This way, the expected utility of each player j for a given $x = (x_1, x_2, \ldots, x_N) \in \mathsf{D}$ will be defined as

$$E_j(x) = \sum_{(n^1,\ldots,n^N)\in\mathcal{P}(\mathcal{N})} \sum_{s^1\in\mathcal{S}_{n^1}} \cdots \sum_{s^N\in\mathcal{S}_{n^N}} u_j(s^N,\ldots,s^1)$$
$$\times x_{n^N}(s^N \mid s^{N-1},\ldots,s^1)p_j(n^N)\cdots x_{n^1}(s^1)p_j(n^1),$$

where $\mathcal{P}(\mathcal{N})$ is the set of permutations of the set $\mathcal{N}$.

Based on this, a **Nash equilibrium** can be defined as a strategy profile $x^* \in \mathsf{D}$ such that, for every player $j \in \mathcal{N}$,

$$E_j(x^*) \geq E_j(x^*_{-j}, x_j)$$

for all $x_j \in \mathsf{D}_j$, where x_{-j} is the partial profile that considers the strategies of all players but j. This way, a profile (y_{-j}, z_j) means that the strategies of y are considered for every player but j, for whom the strategy given by z_j is used. This definition of Nash equilibrium will be used through all the models presented, where the respective expected utility function is considered.

2.2 Fixed Number of Turns per Player

Next the model will be generalized by allowing each player to make more than one decision. The number of decisions made, however, is fixed beforehand for each player, but as before, the periods in which these decisions are made are not known a priori.

If the number of decisions made in each period for each player is given by the vector $\mathbf{m} = (m_1, \ldots, m_N)$, then the expected utility for player j is defined as:

$$E_j(x) = \sum_{(n^1,\ldots,n^T)\in\mathcal{P}_{\mathbf{m}}(\mathcal{N})} \sum_{s^1\in\mathcal{S}_{n^1}} \cdots \sum_{s^T\in\mathcal{S}_{n^T}} u_j(s^T,\ldots,s^1)$$
$$\times x_{n^T}(s^T \mid s^{T-1},\ldots,s^1)p_j(n^T)\cdots x_{n^1}(s^1)p_j(n^1)$$

where $\mathcal{P}_{\mathbf{m}}(\mathcal{N})$ is the set of permutations of the set $\mathcal{N}$ with m_k repetitions of k, and $T = \sum_{j\in\mathcal{N}} m_j$.

2.3 Unknown Number of Turns per Player: Base Model

Now, the attention is shifted to the central model. Instead of having information of how many decisions a player will be making, that information will be hidden and all players will be allowed to potentially make as many decisions as possible in T periods of time. In this case, the expected utility of player j will be defined when facing the profile $x \in \mathsf{D}$ as:

$$E_j(x) = \sum_{n^1\in\mathcal{N}} \sum_{s^1\in\mathcal{S}_{n^1}} \cdots \sum_{n^T\in\mathcal{N}} \sum_{s^T\in\mathcal{S}_{n^T}} u_j(s^T,\ldots,s^1)$$
$$\times x_{n^T}(s^T \mid s^{T-1},\ldots,s^1)p_j(n^T)\cdots x_{n^1}(s^1)p_j(n^1).$$

2.4 Updated Models of the Turn Selection Process in each Period

In the base model of subsection 2.3, each player was thought to be making his predictions of the behaviour of the turn selection process from the beginning of the game and not changing thereafter. However, this a priori distribution may not be accurate throughout the game, or the player may learn new information of its behaviour by observing how players are selected at each turn. Therefore, a player has to be allowed to have different distributions for each period.

Instead of having a fixed probability density p_j, each player has a vector $(p_j^1, p_j^2, \ldots, p_j^T)$ of probability densities for each period. The expected utility for player j will be defined as

$$E_j(x) = \sum_{n^1\in\mathcal{N}} \sum_{s^1\in\mathcal{S}_{n^1}} \cdots \sum_{n^T\in\mathcal{N}} \sum_{s^T\in\mathcal{S}_{n^T}} u_j(s^T,\ldots s^1)$$
$$\times x_{n^T}(s^T \mid s^{T-1},\ldots,s^1)p_j^T(n^T)\cdots x_{n^1}(s^1)p_j^1(n^1).$$

2.5 Conditioned Turn Selections on Previous Decisions

In the base model, only the mixed strategies are dependent on the decisions made in the previous turns. To approach the usual modelling of games by stages every element should be allowed to change according to the decisions made previously. Though originally it was expected to be conditioned as a Markov-like structure, that is, the structure would depend only on the immediate previous decision made in the game, it is possible to generalize it, conditioning the turn selection process on every single decision taken so far. This also takes into account conditioning only the way the process itself has behaved, that is, changing the selection according to which players have been picked to make decisions through the game, and not their decisions.

In this case, the expected utility function for each player j is defined in the following way:

$$E_j(x) = \sum_{n^1 \in \mathcal{N}} \sum_{s^1 i \in \mathcal{S}_{n^1}} \cdots \sum_{n^T \in \mathcal{N}} \sum_{s^T \in \mathcal{S}_{n^T}} u(s^T, \ldots, s^1) \times x_{n^T}(s^T \mid s^{T-1}, \ldots s^1) p_j(n^T \mid s^{T-1}, \ldots, s^1) \cdots \times x_{n^1}(s^1) p_j(n^1).$$

2.6 Strategy Sets Changing According to Period

To approach the modelling per stages from a different perspective, the sets of strategies will be able to change according to the period of time in which the decision is made. This also can be modified to take into account the fact that strategy sets may change according to the previously made decisions.

For this model, the expected utility function for each player j is defined as follows:

$$E_j(x) = \sum_{n^1 \in \mathcal{N}} \sum_{s^1 \in \mathcal{S}^1_{n^1}} \cdots \sum_{n^T \in \mathcal{N}} \sum_{s^T \in \mathcal{S}^T_{n^T}} u(s^T, \ldots, s^1) \times x_{n^T}(s^T \mid s^{T-1}, \ldots s^1) p_j(n^T \mid s^{T-1}, \ldots, s^1) \cdots \times x_{n^1}(s^1) p_j(n^1)$$

or, accordingly, the strategy sets are conditioned by the previously made decisions, that is:

$$E_j(x) = \sum_{n^1 \in \mathcal{N}} \sum_{s^1 \in \mathcal{S}^1_{n^1}} \cdots \sum_{n^T \in \mathcal{N}} \sum_{s^T \in \mathcal{S}_{n^T}(s^{T-1}, \ldots, s^1)} u(s^T, \ldots, s^1) \times x_{n^T}(s^T \mid s^{T-1}, \ldots s^1) p_j(n^T \mid s^{T-1}, \ldots, s^1) \cdots \times x_{n^1}(s^1) p_j(n^1).$$

3 EXISTENCE OF NASH EQUILIBRIA IN THE MODELS

In this section a series of results is shown that ensures the existence of Nash equilibria in the base model of subsection 2.3. These results can also be adapted for each of the other models proposed, to prove the existence of equilibria in them as well.

To do this, it is necessary to prove the sufficient conditions of Kakutani's fixed-point theorem (Zeidler, 1986) for the best response correspondence associated with the expected utility function defined for each model. The proofs are similar in all the models, in some cases only needing to rewrite the elements in the expected utility function, which don't affect the proofs themselves.

Theorem 1. *The expected utility function is a continuous function in each player's plan of conditioned strategies.*

Proof. Observe that after expanding the sums in the expected utility function, each component is constant, linear, quadratic, cubic, etc., up to a T-ic function according to the periods in which player j chooses, with the rest of the components scaling the function. As such, each component is a continuous function, and the sum of all of them is also continuous. □

Theorem 2. *The set* D *is a non-empty, compact and convex subset of* $\mathbb{R}^q$ *for a suitable q.*

Proof. For each scenario $\mathbf{s} = (\mathrm{s}^{\ell-1}, \ldots, \mathrm{s}^1)$, the mixed strategy of player j given $\mathbf{s}$ lies in a a_i^ℓ-simplex, where a_i^ℓ is the number of strategies available to player j at period ℓ. For each player j, the set of plans is the cartesian product of these simplices for all the possible scenarios, and the set D is the cartesian product of the sets of plans for every player. As a cartesian product of simplices, it is a non-empty, compact and convex subset of $\mathbb{R}^q$. □

Define for each player j, $BR_j(x_{-j})$ his best response correspondence for the partial profile x_{-j}, which maps x_{-j} to the set of plans of mixed conditioned strategies $x'_j \in \mathsf{D}_j$ such that $E_j(x_{-j}, x'_j) \geq E_j(x_{-j}, y_j)$ for every $y_j \in \mathsf{D}_j$.

Theorem 3. *The best response correspondence* $BR\colon \mathsf{D} \to \mathsf{D}$, *given by*

$$BR(x^*) = (BR_1(x^*_{-1}), BR_2(x^*_{-2}), \ldots, BR_N(x^*_{-N})),$$

is a non-empty correspondence with a closed graph.

Proof. Since the expected utility function is a continuous function defined on a compact set, it must achieve its maximum at some point $\hat{x}_j \in \mathsf{D}_j$ for each $x_{-j} \in \mathsf{D}_{-j}$ for every player j. Therefore, BR_j is non-empty for every player j and every $x \in \mathsf{D}$, which implies that $BR(x)$ is a non-empty correspondence for every $x \in \mathsf{D}$.

Now, consider $({}_h x)_{h=1}^\infty$ as a sequence of strategy profiles and $({}_h x')_{h=1}^\infty$ as the associated sequence of the best responses, that is, ${}_h x' \in BR({}_h x)$. Let $x^* = \lim_{h\to\infty} {}_h x$ and $x'^* = \lim_{h\to\infty} {}_h x'$. Fix player j, then ${}_h x'_j \in BR_j({}_h x_{-j})$, which means that

$$E_j({}_h x_{-j}, {}_h x'_j) \geq E_j({}_h x_{-j}, y_j),$$

for any $y_j \in \mathsf{D}_j$. As the expected utility function is continuous in each player's plan of strategies, it is possible to take limits on both sides while preserving the inequality, which means that

$$\lim_{h\to\infty} E_j({}_h x_{-j}, {}_h x'_j) \geq \lim_{h\to\infty} E_j({}_h x_{-j}, y_j)$$

and, interchanging the order of limits and sums,

$$E_j(x^*_{-j}, x'^*_j) \geq E_j(x^*_{-j}, y_j)$$

for any $y_j \in \mathsf{D}_j$. This implies that $x'^*_j \in BR(x^*_{-j})$ for each player j, and, therefore, $x'^* \in BR(x^*)$. □

Finally, the convexity of the best response correspondence will require a few more arguments. Given a plan of strategies x_j, a **similar plan of strategies** y_j for scenario $\mathbf{s} = (\mathsf{s}^{\ell-1}, \ldots, \mathsf{s}^1)$ is defined as the plan in which all mixed strategies are the same in x_j and y_j, except for the one conditioned by $\mathbf{s}$, which is replaced by a pure conditioned strategy $s_j^\ell \in \operatorname{supp}(x_j, \mathbf{s})$, where $\operatorname{supp}(x_j, \mathbf{s}) = \{s_j^\ell \in \mathcal{S}_j \mid x_j(s_j^\ell \mid \mathbf{s}) > 0\}$ is the **support** of the conditioned strategy x_j for scenario $\mathbf{s}$. $\mathsf{SM}_j(x_j \mid \mathbf{s})$ denotes the set of plans of conditioned strategies similar to x_j of player j for scenario $\mathbf{s} = (\mathsf{s}^{\ell-1}, \ldots, \mathsf{s}^1)$.

With this in mind, it is possible to start our string of results.

Lemma 1. *Let $x \in \mathsf{D}$ be such that for player j, $x_j \in BR_j(x_{-j})$. Then, for any scenario $\mathbf{s} = (\mathsf{s}^{\ell-1}, \ldots, \mathsf{s}^1)$, and any two plans $y_j, z_j \in \mathsf{SM}_j(x_j \mid \mathbf{s})$:*

$$E_j(x_{-j}, y_j) = E_j(x_{-j}, z_j).$$

Proof. Assume that $E_j(x_{-j}, y_j) > E_j(x_{-j}, z_j)$ for some $y_j, z_j \in \mathsf{SM}(x_j \mid \mathbf{s})$. Observe that E_j can be partitioned in two components of the sums: those where the scenario $\mathbf{s}$ occurs in the first $\ell - 1$ periods and the player j is chosen to make the decision in period ℓ, and those where one of the previous conditions fails. The latter ones are not affected by replacing y_j with z_j, so the focus will be only on the former ones, where the inequality still holds. If this is the case, then in x_j the probability assigned to z_j could be reduced to zero, and the probability of y_j could be increased to $x_j(y_j) + x_j(z_j)$, which would increase the expected utility obtained. But this is a contradiction to the fact that x_j was a best response to x_{-j}. Therefore, the strict inequality cannot hold, and so equality is obtained. □

Lemma 2. *Let $x \in \mathsf{D}$ be such that for player j, $x_j \in BR_j(x_{-j})$. Then, for any scenario $\mathbf{s} = (\mathsf{s}^{\ell-1}, \ldots, \mathsf{s}^1)$, and any plan $y_j \in \mathsf{SM}_j(x_j \mid \mathbf{s})$,*

$$E_j(x) = E_j(x_{-j}, y_j).$$

Proof. As in the previous lemma, it is possible to partition the sum in two. The relevant part of the sums of $E_j(x)$ can be written as the weighted sum of the corresponding relevant parts of $E_j(x_{-j}, z_j)$ for all $z_j \in \mathsf{SM}_j(x_j \mid \mathbf{s})$, where the weights are the probabilities assigned to each strategy in $x_j(\cdot \mid \mathbf{s})$. But by lemma 1, all these parts have the same value, so it is possible to replace them by the relevant part of $E_j(x_{-j}, y_j)$. Being a convex combination of the same value, it amounts to exactly the relevant part of $E_j(x_{-j}, y_j)$, proving the equality. □

The following corollary is obtained as a consequence of the previous lemma.

Corollary 1. *Let $x \in \mathsf{D}$ be such that for player j, $x_j \in BR_j(x_{-j})$. Then, for any scenario $\mathbf{s} = (\mathsf{s}^{\ell-1}, \ldots, \mathsf{s}^1)$, the plan of strategies y_j obtained by substituting the conditioned strategy in scenario $\mathbf{s}$ for any mixed strategy which has a support that is a subset of the support of x_j, satisfies*

$$E_j(x) = E_j(x_{-j}, y_j).$$

Observe that if exactly two mixed strategies were changed by pure strategies, the equality would be obtained by applying lemma 1 twice, changing one by one each strategy. And it is possible to change two mixed strategies for any other two mixed strategies as well, and guarantee that the equality still holds, by applying lemma 2 to change the strategies one by one. It is possible to extend this process to any number of changes in the strategies. Given the previously described process, the following result is obtained which generalizes the previous lemmas.

Theorem 4. *Let $x \in \mathsf{D}$ be such that for player j, $x_j \in BR_j(x_{-j})$. Then, for any scenario $\mathbf{s} = (\mathsf{s}^{\ell-1}, \ldots, \mathsf{s}^1)$, and any two plans of strategies $y_j, z_j \in \mathsf{D}_j$ with their support a subset of the support of x_j, we have:*

$$E_j(x_{-j}, y_j) = E_j(x_{-j}, z_j)$$

Using the previous result, we can easily prove the last condition needed for the best response correspondence.

Theorem 5. *The best response correspondence BR is convex.*

Proof. Let $x \in \mathsf{D}$, fix $j \in \mathcal{N}$, and consider $x'_j, x''_j \in BR_j(x_{-j})$. Observe that any convex combination of x' and x'' is a plan of strategies in D_j. Since the convex combination of x' and x'' is a plan of strategies with its support, which is a subset of the support of plans, representing the best response, by theorem 4, the expected utility of the convex combination is the same as the expected utility for any best response. Therefore, the convex combination is also a best response. □

All the previous results give the conditions of Kakutani's fixed-point theorem for the best-response correspondence, which guarantees the existence of a fixed point; this fixed point is our Nash equilibrium. Therefore, the central theorem for the model is obtained.

Theorem 6. *Every sequential game with finite horizon and turn selection process and finite strategies sets has at least one Nash equilibrium.*

4 AN APPLICATION OF THE MODEL: PICKING TEAMMATES

In the following example an application of some of the models analysed in the previous sections is shown. To be precise, the example is a combination of two models, where the probabilities of the turn selection process, as well as the strategy sets change for each period conditioned on the decisions made before.

In a certain sports league, every year the two teams participating have to choose between certain college athletes to become part of their team. This year they have to pick between athlete A and athlete B. Both of them know that A, becoming a teammate, gives a utility of 1 whereas B gives a utility of 2. The utility obtained by each team after the picking is the sum of the utilities of the athletes they have picked.

However, the process involved goes like this: in each of the two picking periods, each team has a probability of being selected to make a choice. Since team 1 won last season, they get a probability of being selected in the first period of $1/3$, whereas team 2 gets a probability of $2/3$. If athlete A is selected first, the team that picked him has its probability of being selected for the second period reduced to half (which goes to the other team). If athlete B is selected first, the reduction in probability is to a third of the original probability. Then, with the new probabilities, a team is selected for the second pick, which automatically chooses the athlete not picked in the first round.

To solve the problem, first the probabilities in each instance are calculated. For the strategies sets, for team 1, there is $\mathcal{S}_1 = \{A_1, B_1\}$ and for team 2, $\mathcal{S}_2 = \{A_2, B_2\}$ (the subscripts are to identify who made the pick in the first round when conditioning). Since both players know the probabilities in the picking process they model it the same way, so the subscript of p_j is removed. This way, there are

$$\begin{array}{ll} p(1) = 1/3 & p(2) = 2/3 \\ p(1 \mid A_1) = 1/6 & p(2 \mid A_1) = 5/6 \\ p(1 \mid B_1) = 1/9 & p(2 \mid B_1) = 8/9 \\ p(1 \mid A_2) = 2/3 & p(2 \mid A_2) = 1/3 \\ p(1 \mid B_2) = 7/9 & p(2 \mid B_2) = 2/9 \end{array}$$

Now, the expected utility function is calculated for each player, inputting the probabilities for each possible scenario, and the utilities received in each case. Observe that since the second pick is automatic, $x_j(A_j \mid B_1)$, $x_j(A_j \mid B_2)$, $x_j(B_j \mid A_1)$, $x_j(B_j \mid A_2)$ are all equal to one, for $j \in \{1,2\}$, and the other mixed strategies are equal to zero. The expected utilities obtained are:

$$E_1(x) = \frac{25}{27} + \frac{x_1(A_1)}{27} + \frac{10x_2(A_2)}{27}$$

and

$$E_2(x) = \frac{16}{9} + \frac{7x_1(A_1)}{27} - \frac{10x_2(A_2)}{27}.$$

To maximize their expected utility functions, $x_1(A_1) = 1$ and $x_2(A_2) = 0$, which means that in case team 1 is chosen for the first round, they should choose athlete A, but if team 2 is chosen first, they should choose athlete B.

5 CONCLUSIONS

This work introduces a model for situations where the individuals may not know the moment in which they make a decision, or its placement according to the decisions of other players. It also introduces a model to create situations which allow a randomization process to equalize players, whether it is by being able to make decisions earlier, or by receiving information that is useful in later periods. It is also a good model for buying and selling scarce goods, or even goods of different utility value. The models introduced here are sequential, which is a more natural way of viewing situations that happen in real life, and they comprise a wide array of possibilities, by allowing various parts of the model to be changing throughout time, or even combining several of these changes at the same time.

However, the example presented has the basic elements to illustrate the model, but it is not made with larger sets of strategies, more players, or more periods of time, because, to solve the model, the calculations made become unbearable very quickly, and prone to errors if made by hand, which then requires a computer program to be created in order to analyze each situation. And this becomes the only method as the problem grows, since we start finding crossed products of different strategies, which become increasingly hard to solve.

For further work, the extension to infinite sets of strategies follows naturally. Also, possible ramifications include studying a way to change the problem into an optimization problem to find Nash equilibria, allowing an external source of noise to intervene in the decision process and to study the convergence of sequences of approximate solutions within each of these models.

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APPENDIX

- N is the number of players in the game.
- $\mathcal{N}$ is the set of players in the game.
- T is the horizon of the game, that is, the number of turns to be played in the game.
- $\mathcal{S}_j$ is the set of pure strategies for player j, which is considered to be finite.
- $\mathcal{S} = \cup_{j=1}^{N} \mathcal{S}_j$.
- $\mathcal{M}_j$ is the set of mixed strategies for player j, defined as the set of probability distributions that can be assigned to the set of pure strategies of player j.
- $\mathcal{M} = \cup_{j=1}^{N} \mathcal{M}_j$.
- x_{-j} is the strategy for all players except j obtained from a mixed strategy x by deleting the strategy played by j.
- $\mathbf{s} = (\mathsf{s}^{\ell-1}, \ldots, \mathsf{s}^1)$ is a scenario for period $\ell \in \{1, \cdots, T\}$, which is a vector consisting of the decisions $\mathsf{s}^1, \ldots \mathsf{s}^{\ell-1}$ that were made in periods 1 through $\ell - 1$.
- C_j is the set of plans made of conditioned pure strategies for player j. That is, for every possible scenario, the player makes his decision with a pure strategy, conditioned on the scenario that he is facing.
- D_j is the set of plans made of conditioned mixed strategies for player j, that is, the player assigns a mixed strategy to each possible scenario that can be faced in the game.
- $\mathsf{C} = \cup_{j=1}^{N} \mathsf{C}_j$.
- $\mathsf{D} = \cup_{j=1}^{N} \mathsf{D}_j$.
- $\mathcal{P}(\mathcal{N})$: the set of permutations of $\mathcal{N}$.
- $\mathcal{P}_{\mathbf{m}}(\mathcal{N})$ is the set of permutations of the set made of m_k repetitions of k for each $k \in \mathcal{N}$, if $\mathbf{m} = (m_1, \ldots, m_N)$.
- $BR_j(x_{-j})$ is the best response correspondence of player j for the partial profile x_{-j}, which maps x_{-j} to the set of plans of mixed conditioned strategies $x'_j \in \mathsf{D}_j$ such that $E_j(x_{-j}, x'_j) \geq E_j(x_{-j}, y_j)$ for every $y_j \in \mathsf{D}_j$.
- $BR(x) = (BR_1(x_{-1}), BR_2(x_{-2}), \ldots, BR_N(x_{-N}))$ is the best response correspondence for the mixed strategy $x \in \mathsf{D}$.
- $\mathsf{SM}_j(x_j \mid \mathbf{s})$ is the set of plans of conditioned strategies similar to x_j of player j for scenario $\mathbf{s}$.