Comparison of Fuzzy Extent Analysis Technique and its Extensions with Original Eigen Vector Approach

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Abstract: Fuzzy set theory has been extensively incorporated in the original Analytical Hierarchical Process (AHP) with an aim to better represent human judgments in comparison matrices. One of the most popular technique in the domain of Fuzzy AHP is Fuzzy Extent Analysis method which utilizes the concept of extent analysis combined with degree of possibility to calculate weights from fuzzy comparison matrices. In original AHP, where the comparison matrices are composed of crisp numbers, Satty proposed that Eigen Vector of these comparison matrices estimate the required weights. In this research we perform a comparison analysis of these two approaches based on a data set of matrices with varying level of inconsistency. Furthermore, for the case of FEA, in addition to degree of possibility, we use centroid defuzzification and defuzzification by using the mid number of triangular fuzzy number to rank the final weights calculated from fuzzy comparison matrices.

1 INTRODUCTION

Analytical Hierarchy Process (AHP) proposed by (Saaty, 1980) is a methodology for structuring, measurement and synthesis (Forman and Gass, 2001) which utilizes pairwise comparisons to derive ratio scales indicating the preferences of the decision makers among different alternatives and associated criteria. These comparisons are recorded in a comparison matrix and processed to determine the corresponding weights of the given criterion as well as available alternatives. The normalized weighted sum provides a weight associated with each available alternatives and thus help decision maker to choose the best decision.

In the literature, two different scales are used to record pairwise comparisons i.e. scale based on crisp numbers (scale of 1-9) and scale based on fuzzy numbers. The original method uses the scale of 1-9 in which decision maker preferences of weak, normal and strong are represented by some number in the given scale and recorded in comparison matrices. Afterwards, weights are calculated from well-defined mathematical structure of consistent matrices and their associated eigenvectors ability to generate true or approximate weights (Saaty, 1980).

However, this approach has found some criticism on the premises that crisp numbers disregards the vagueness of human language thus implying that vague linguistic variables (i.e., weak, strong, etc.) cannot be represented with a ratio scale based on crisp numbers and may lead to wrong decisions in the decision analysis process (Tsaur et al., 2002).

Fuzzy set theory introduced by (Zadeh, 1965) has been used frequently in the literature to represent vagueness of human thought. It represents the belongingness of an object to a set by means of membership functions which ranges from zero to one and has found many applications over the past many years. Some of the fields which utilize fuzzy sets include health care (Kilic et al., 2004), system modeling (Uncu et al., 2004; Uncu et al., 2003), supplier selection (Kahraman et al., 2003), control theory (Takagi and Sugeno, 1985), capital investment (Tang et al., 2005) etc.

Fuzzy numbers are introduced as part of fuzzy set theory, which takes the form of a set of real numbers with a convex and continuous membership function of bounded support. These numbers can be used to accurately represent linguistic scales incorporating vagueness and uncertainties of human mind. Fuzzy AHP is simply an extension of the original AHP with human preferences recorded in the form of fuzzy numbers and hence the resulting comparison matrix formed is also composed of fuzzy numbers. However, to extract weights from these fuzzy comparison matrices require additional treatment as the arithmetic operations of fuzzy numbers is different from crisp numbers. To address this issue, various algorithms have

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been proposed over the years with an aim to process these fuzzy comparison matrix and extract weights. Some of the most popular algorithms are given by (Van Laarhoven and Pedrycz, 1983; Boender et al., 1989; Buckley, 1985; Deng, 1999). Readers are referred to (Büyüközkan et al., 2004) and (Ataei et al., 2012) for a comprehensive review of literature on FAHP algorithms and its applications.

Among these FAHP algorithms, Fuzzy Extent Analysis (FEA) method (Chang, 1996) is the most frequently used FAHP algorithm (Ding et al., 2008). It utilizes the concept of extent analysis combined with degree of possibility to calculate weights from fuzzy comparison matrices. However, this method has been criticized (Wang et al., 2006) mainly due to the way fuzzy numbers are compared using degree of possibility. Over the last five years around hundred research articles have been published on how to compare fuzzy numbers which shows that there is no general consensus on a single method to rank and order fuzzy numbers (Zhü, 2014). Therefore, in our analysis in addition to the original method of degree of possibility, we will also use Centroid Defuzzification (Ross, 1995) and defuzzification by simply using the mid number of the triangular fuzzy number.

Rest of the paper is organized as follows. In Section 2, a brief overview of fuzzy arithmetic and FEA method will be provided. In section 3 the set up used for the experimental analysis will be discussed. Later in section 4, the results of the experimental analysis will be presented. Paper will be finalized with some concluding remarks as well as future research directions in section 5.

2 FUZZY EXTENT ANALYSIS

As discussed before, one of the major challenges faced in AHP is to employ a weighing scale which accurately represents expert opinions in the form of comparison ratios while taking into account the inherent vagueness of human thought. Note that this vagueness is neither random nor stochastic (Ataei et al., 2012) and fuzzy numbers are helpful in capturing this imprecision. The construction of fuzzy numbers are such that it represents the linguistic variables with a set of possible values each having its own membership degree and thus aids in capturing this vagueness. There are many different types of fuzzy numbers, however in this paper we will use a triangular fuzzy number which is represented through $[l\ m\ u]$ and membership function μ_M defined as follows and graphically illustrated in Figure 1;

$$\mu_M(x) = \begin{cases} \frac{x}{m-l} - \frac{l}{m-l}, & x \in [l\ m] \\ \frac{x}{m-u} - \frac{u}{m-u}, & x \in [m\ u] \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

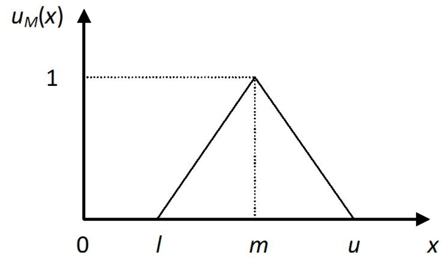

Figure 1: Membership function of Triangular Fuzzy Number.

Let $(l_1\ m_1\ u_1)$ and $(l_2\ m_2\ u_2)$ then the basic fuzzy arithmetic operations are summarized as follows;

- Addition:
  $(l_1\ m_1\ u_1) \oplus (l_2\ m_2\ u_2) = (l_1 + l_2 \quad m_1 + m_2 \quad u_1 + u_2)$
- Multiplication:
  $(l_1\ m_1\ u_1) \odot (l_2\ m_2\ u_2) = (l_1.l_2 \quad m_1.m_2 \quad u_1.u_2)$
- Scalar Multiplication:
  $(\lambda\ \lambda\ \lambda) \odot (l_1\ m_1\ u_1) = (\lambda.l_1 \quad \lambda.m_1 \quad \lambda.u_1)$
- Inverse:
  $(l_1\ m_1\ u_1)^{-1} \approx (1/u_1 \quad 1/m_1 \quad 1/l_1)$

Fuzzy Extent Analysis (FEA) proposed by (Chang, 1996) is one of the most popular technique in the literature to calculate weights from fuzzy comparison matrices. In the original Extent Analysis method, provided we have $X = \{x_1, x_2, \cdots, x_n\}$ as an object set and $G = \{g_1, g_2, \cdots, g_n\}$ as a goal set, then for each object, extent analysis for each goal g_i is performed. Applying this theory in fuzzy comparison matrix, one can calculate the value of fuzzy synthetic extent with respect to the i^{th} object as follows;

$$S_i = \sum_{j=1}^{m} M_{g_i}^j \otimes \left[\sum_{i=1}^{n} \sum_{j=1}^{m} M_{g_i}^j \right]^{-1} \tag{2}$$

Where

$$\sum_{j=1}^{m} M_{g_i}^j = \left(\sum_{j=1}^{m} l_j, \sum_{j=1}^{m} m_j, \sum_{j=1}^{m} u_j \right) \tag{3}$$

In the original AHP, while using the scale of 1-9, we can calculate the final weights through the process explained above. However, for the case where fuzzy triangular numbers are employed in the judgment scale, the result would be a fuzzy triangular weight value as indicated in Equation 3.

As opposed to the straight forward ordering of crisp numbers, ordering of the fuzzy numbers are not that simple. In fact, over the last couple of years many articles have been published discussing this issue and as of today there is no widely accepted technique (Zhü, 2014) to rank and order fuzzy numbers. In the FEA technique proposed by (Chang, 1996), a method known as degree of possibility is proposed for ordering as well as defuzzifying weights calculated from Equation 3

In this approach, a pair wise comparison is carried out for every fuzzy weight with other fuzzy weights and the corresponding degree of possibility of being greater than other fuzzy weights is determined. The minimum of these possibilities is used as the overall score for each criterion i. That is to say by applying the comparison of the fuzzy numbers, the degree of possibility is obtained for each pair wise comparison as follows:

$$V(M_2 \geq M_1) = hgt(M_1 \cap M_2) = \mu_{M_2}(d) =$$

$$\begin{cases} 1, & \text{if } m_2 \geq m_1 \\ 0, & \text{if } l_1 \geq u_2 \\ \frac{l_1 - u_2}{(m_2-u_2)-(m_1-l_1)}, & \text{otherwise.} \end{cases}$$

The same is illustrated in the Figure 2.

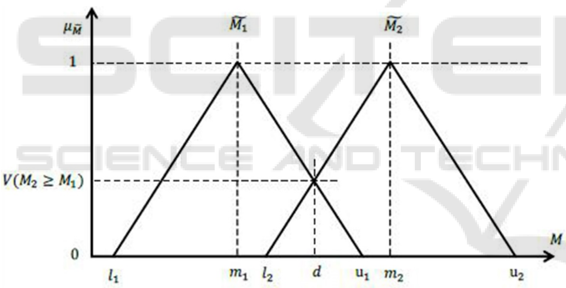

Figure 2: Degree of possibility.

Note that, degree of possibility for a convex fuzzy number to be greater than k convex fuzzy numbers is given by;

$$V(M \geq M_1, M_2, \cdots, M_k) = V[(M \geq M_1) \text{ and }$$

$$(M \geq M_2), \cdots, (M \geq M_k)]$$

$$= \min V(M \geq M_i), \quad i = 1, 2, \cdots, k$$

Assuming that $w'_i = \min V(M_i \geq M_k)$ then weight vector is given by

$$W' = w'_1, w'_2, \cdots, w'_n$$

Normalizing the above weights gives us the final priority vector $w_1, w_2, \cdots, w_n$.

Subsequent research on this methodology has proposed some modifications. for example, (Wang et al., 2006) in its review of the normalization processes in fuzzy systems proposed that row sums should be normalized by Equation 4 in order to calculate fuzzy synthetic extent values. This modification will also be part of our analysis.

$$S_i = \frac{\sum_{j=1}^n l_{ij}}{\sum_{j=1}^n l_{ij} + \sum_{k=1,k\neq i}^n \sum_{j=1}^n u_{kj}}, \frac{\sum_{j=1}^m m_{ij}}{\sum_{k=1}^n \sum_{j=1}^n m_{kj}}, \frac{\sum_{j=1}^n u_{ij}}{\sum_{j=1}^n u_{ij} + \sum_{k=1,k\neq i}^n \sum_{j=1}^n l_{kj}} \quad (4)$$

Therefore, our experimental analysis will include in total following five techniques;

- FEA with degree of possibility (Original Method)
- FEA with degree of possibility including modification to normalization (Wang et al., 2006)
- FEA with Centroid Defuzzification (Ross, 1995)
- FEA with defuzzification using mid number of the triangular fuzzy number
- Original eigen vector method (Saaty, 1980)

3 RESEARCH METHODOLOGY

Three major control parameters are employed in the experimental analysis to evaluate the performance of the techniques discussed in the paper. These parameters include level of fuzziness (α), inconsistency of the decision maker (β), and size of the comparison matrices (n). By varying the levels of these control parameters, we generate a set of matrices on which we apply the above mentioned techniques. Note that even though there exist some algorithms in the literature (Golany and Kress, 1993) which provides a methodology to generate comparison matrices with various levels of consistency levels, however these technique are limited only for original AHP and thus cannot be replicated for comparison matrices consisting of fuzzy numbers. Therefore, in this research a novel framework is proposed through which random fuzzy comparison matrices can be generated for various control parameters as required by the experimental set up.

This algorithm is step by step explained as follows;

Step 1: Assuming we have n criterion, we randomly generate crisp weights $w_1, w_2, \cdots, w_n$ and normalize them.

Step 2: Through these weights we can generate a perfectly consistent comparison matrix as follows

$$W = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & \ddots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{pmatrix}$$

Step 3: Once the comparison matrix is generated, each element of the matrix is converted into a triangular fuzzy number $[l' \ m' \ u']$ with a fuzzification parameter α such that $l' = w_i/w_j - \alpha$, $m' = w_i/w_j$ and $u' = w_i/w_j + \alpha$.

Step 4: As stated before, in reality human judgments are rarely consistent and thus comparison matrices formed through these judgments are also not consistent. Therefore, we introduce different levels of inconsistency in the matrices through the inconsistency parameter β. Depending on this parameter, an interval $[a \ b]$ is generated for each l' of the triangular fuzzy number such that $a = l' - l'(\beta)$ and $b = l' + l'(\beta)$. Same procedure is followed to create inconsistency intervals for m' and u'. Afterwards, a number is randomly selected from each one of these intervals and is correspondingly assigned as the lower, modal and upper value of the triangular fuzzy number i:e., $[l \ m \ u]$. However, once inconsistency parameter is increased, there is a possibility that the interval $[a \ b]$ generated for each element of the triangular fuzzy number intersects and the numbers are randomly chosen in such a way that they violates the condition $l < m < u$. We address this issue as follows; Whenever the inconsistency intervals intersect, they are shrunk in such a way that for each lower value of the triangular fuzzy number, the right endpoint of the interval is readjusted such that it is the mid point of the right end point of the interval of lower value and the left end point of the interval generated for modular number. Similarly, both end points of the inconsistency interval of modular number are readjusted and the left endpoint of the inconsistency interval of upper number is readjusted. Numbers randomly chosen from these intervals will always satisfy the condition of $l < m < u$. This part of the algorithm is graphically explained below for clarity.

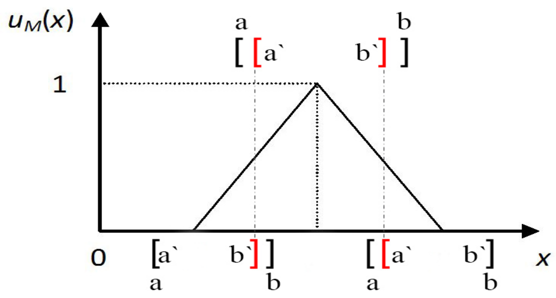

Figure 3: Interval formation.

Previous comparative analysis of methodologies in the original AHP shows that level of inconsistency and size of the matrix are two important criteria which directly affects the performance of a certain technique. In fuzzy AHP, the weighing scale is composed of fuzzy numbers and thus we add a third performance evaluation criteria which is level of fuzziness. Therefore, the aim of our analysis will be to not only investigate performance measure of each algorithm in general but also change in performance as we change these three parameters.

4 RESULTS AND DISCUSSIONS

The framework of our experimental analysis includes three variables α, β and n. For the fuzzification parameter (α) three fuzzification levels are used (0.05, 0.1 and 0.15) and decision analysts can set the fuzzification level himself and conduct FAHP accordingly as this parameter is not inherent to the problem. The inconsistency parameter (β) indicates to the level of inconsistency of the decision maker. For this analysis we used five different levels for the inconsistency parameter (0, 0.5, 1, 1.5 and 2). In addition, four different matrix sizes were considered (3, 7, 11 and 15). Matrices having dimension 3×3 can be regarded as the representative of a small sized problems while matrices of dimensions 7×7 and 11×11 are representatives for medium sized problems and 15×15 size matrix can be considered for larger cases.

Therefore, our experimental analysis consists of 60 different experimental conditions and for each condition 10 replications are created randomly. In total the data set consists of six hundred matrices with varying parameters of fuzzification, inconsistency and size of the matrix. The error terms are calculated as the root mean squared difference between the resulting weights calculated from the 5 different techniques discussed in this paper and the initial weights used to construct comparison matrices.

Through this experimental study, we can conclude that utilizing just the mid number of a triangular fuzzy weight can give us more accurate results (Figure 4). While the original method of degree of possibility as well as the Eigen Vector approach along with FEA method with other defuzzification techniques performed inferior compared to using mid number of the triangular fuzzy number.

Figure 4 shows that increase in size of the matrix has a significant effect on the performance of all the algorithms. However, this improved performance is due to the fact that as we increase the size of the matrix, the values of the starting normalized weights are

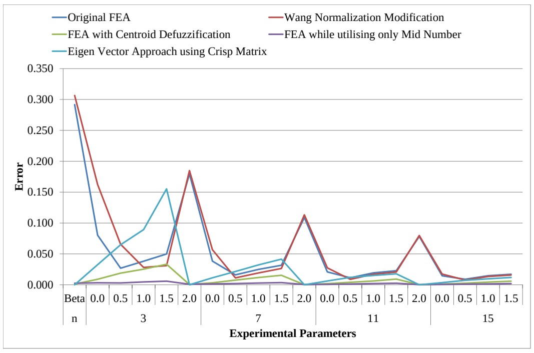

Figure 4: Experimental Analysis.

decreased and hence the final error term is also low which depicts improvement in performance. Therefore, this improved performance cannot be associated with any of the FAHP algorithm.

As we increase the inconsistency factor, performance of most of the algorithms is decreased except for original FEA method and FEA method with modified normalization for which performance increases as we increase the inconsistency. However, this increase in performance is not enough and even at high inconsistency levels, FEA with defuzzification using mid number is the best performing algorithm.

5 CONCLUSIONS

In this paper, we introduced a novel experimental analysis framework through which performance of various FAHP technique can be analyzed. The analysis revealed that the FEA method with defuzzification using mid number outperformed the other techniques in almost all experimental conditions.

Review of the existing literature on FAHP reveal that there are many different algorithms proposed in this domain. However, there is no throughout analysis of these techniques which measure their performance for different experimental conditions. Such a comparison would be invaluable for the researchers and the practitioners of the field since it will hint which technique might be more suitable for the problem that they are facing.

In future we plan to conduct similar performance analysis for other FAHP algorithms and through experimental analysis such as this, we plan to investigate differences between conventional AHP techniques with Fuzzy AHP techniques.

REFERENCES

Ataei, M., Mikaeil, R., Hoseinie, S. H., and Hosseini, S. M. (2012). Fuzzy analytical hierarchy process approach for ranking the sawability of carbonate rock. *International Journal of Rock Mechanics and Mining Sciences*, 50:83–93.

Boender, C., De Graan, J., and Lootsma, F. (1989). Multi-criteria decision analysis with fuzzy pairwise comparisons. *Fuzzy sets and Systems*, 29(2):133–143.

Buckley, J. J. (1985). Fuzzy hierarchical analysis. *Fuzzy sets and systems*, 17(3):233–247.

Büyüközkan, G., Kahraman, C., and Ruan, D. (2004). A fuzzy multi-criteria decision approach for software development strategy selection. *International Journal of General Systems*, 33(2-3):259–280.

Chang, D.-Y. (1996). Applications of the extent analysis method on fuzzy ahp. *European journal of operational research*, 95(3):649–655.

Deng, H. (1999). Multicriteria analysis with fuzzy pairwise comparison. *International Journal of Approximate Reasoning*, 21(3):215–231.

Ding, Y., Yuan, Z., and Li, Y. (2008). Performance evaluation model for transportation corridor based on fuzzy-ahp approach. In *Fuzzy Systems and Knowledge Discovery, 2008. FSKD'08. Fifth International Conference on*, volume 3, pages 608–612. IEEE.

Forman, E. H. and Gass, S. I. (2001). The analytic hierarchy process-an exposition. *Operations research*, 49(4):469–486.

Golany, B. and Kress, M. (1993). A multicriteria evaluation of methods for obtaining weights from ratio-scale matrices. *European Journal of Operational Research*, 69(2):210–220.

Kahraman, C., Cebeci, U., and Ulukan, Z. (2003). Multi-criteria supplier selection using fuzzy ahp. *Logistics information management*, 16(6):382–394.

Kilic, K., Sproule, B. A., Türksen, I. B., and Naranjo, C. A. (2004). Pharmacokinetic application of fuzzy structure identification and reasoning. *Information Sciences*, 162(2):121–137.

Ross, T. J. (1995). *Fuzzy Logic With Engineering Applications*. Mcgraw-Hill College, first edition edition.

Saaty, T. L. (1980). *The Analytic Hierarchy Process: Planning, Priority Setting, Resource Allocation (Decision Making Series)*. Mcgraw-Hill (Tx).

Takagi, T. and Sugeno, M. (1985). Fuzzy identification of systems and its applications to modeling and control. *Systems, Man and Cybernetics, IEEE Transactions on*, (1):116–132.

Tang, Y.-C., Beynon, M. J., et al. (2005). Application and development of a fuzzy analytic hierarchy process within a capital investment study. *Journal of Economics and Management*, 1(2):207–230.

Tsaur, S.-H., Chang, T.-Y., and Yen, C.-H. (2002). The evaluation of airline service quality by fuzzy mcdm. *Tourism management*, 23(2):107–115.

Uncu, O., Kilic, K., and Turksen, I. (2004). A new fuzzy inference approach based on mamdani inference using discrete type 2 fuzzy sets. In *Systems, Man and*

Cybernetics, 2004 IEEE International Conference on, volume 3, pages 2272–2277. IEEE.

Uncu, O., Turksen, I., and Kilic, K. (2003). Localm-fsm: A new fuzzy system modeling approach using a two-step fuzzy inference mechanism based on local fuzziness level. In *Proceedings of international fuzzy systems association world congress*, pages 191–194.

Van Laarhoven, P. and Pedrycz, W. (1983). A fuzzy extension of saaty's priority theory. *Fuzzy sets and Systems*, 11(1):199–227.

Wang, Y.-M., Elhag, T., and Hua, Z. (2006). A modified fuzzy logarithmic least squares method for fuzzy analytic hierarchy process. *Fuzzy Sets and Systems*, 157(23):3055–3071.

Zadeh, L. (1965). Fuzzy sets. *Information and Control*, 8(3):338 – 353.

Zhü, K. (2014). Fuzzy analytic hierarchy process: Fallacy of the popular methods. *European Journal of Operational Research*, 236(1):209–217.