A New Distance on a Specific Subset of Fuzzy Sets

Majid Amirfakhrian
Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran

Keywords: Fuzzy Sets, Fuzzy Numbers, Fuzzy LR Sets, Generalized *LR* Fuzzy Number, Value, Ambiguity.

Abstract: In this paper, first we propose a definition for fuzzy LR sets and then we present a method to assigning distance between these form of fuzzy sets. We show that this distance is a metric on the set of all trapezoidal fuzzy sets with the same height and all trapezoidal fuzzy numbers and is a pseudo-metric on the set of all fuzzy sets.

1 INTRODUCTION

There are lots of works that the authors investigated fuzzy sets, in order to find the nearest approximation of an arbitrary fuzzy numbers. Approximation of a fuzzy number can be done in three ways. Some authors assigned a single crisp number to a fuzzy number as a ranking method. In this case many information of the fuzzy number will be lost. The other method is using an interval as an approximations of a fuzzy number (Chanas, 2001; Grzegorzewski, 2002), But, in this case, the modal value (the core with height 1) of the fuzzy number will be lost. In some works such as (Abbasbandy and Asady, 2004; Delgado et al., 1998; Grzegorzewski and Mrǿwka, 2005), the authors tried to solve an optimization problem in order to obtain a trapezoidal fuzzy number as a nearest approximation. Some works were done on approximation of a fuzzy number (Anzilli et al., 2014; Ban et al., 2011; Cano et al., 2016). Some distances and their properties were done in (Abbasbandy and Amirfakhrian, 2006a; Abbasbandy and Amirfakhrian, 2006b).

In this work we introduce a fuzzy LR set and we present a distance to find the nearest trapezoidal fuzzy set to an arbitrary LR fuzzy set. The motivation behind this distance is trying to compare fuzzy sets of special format and the same height.

The structure of the this paper is as follows. In Section 2 the basic concepts of our work are introduced, then we introduce *LR* fuzzy set. In Section 3 we introduce a distance over all fuzzy LR sets with the same height and we named it *h*-source distance, which is a metric on the set of all trapezoidal *LR* fuzzy set, with the same height. In Section 4 the nearest trapezoidal fuzzy number to an arbitrary trapezoidal *LR* fuzzy set was introduced and a simple method for computing it, was presented. Section 5 contains some numerical examples.

2 PRELIMINARIES

Let $F(\mathbb{R})$ be the set of all normal and convex fuzzy numbers on the real line.

Definition 2.1. A generalized *LR* fuzzy number $\tilde{u}$ with the membership function $\mu_{\tilde{u}}(x), x \in \mathbb{R}$ can be defined as

$$\mu_{\tilde{u}}(x) = \begin{cases} l_{\tilde{u}}(x) & , \quad a \leq x \leq b, \\ 1 & , \quad b \leq x \leq c, \\ r_{\tilde{u}}(x) & , \quad c \leq x \leq d, \\ 0 & , \quad otherwise, \end{cases} \tag{2.1}$$

where $l_{\tilde{u}}(x)$ is the left membership function that is an increasing function on $[a,b]$ and $r_{\tilde{u}}(x)$ is the right membership function that is a decreasing function on $[c,d]$. Furthermore we want to have $l_{\tilde{u}}(a) = r_{\tilde{u}}(d) = 0$ and $l_{\tilde{u}}(b) = r_{\tilde{u}}(c) = 1$. In addition, if $l_{\tilde{u}}(x)$ and $r_{\tilde{u}}(x)$ are linear, then $\tilde{u}$ is a trapezoidal fuzzy number which is denoted by (a,b,c,d). If $b = c$, we denoted it by (a,c,d), which is a triangular fuzzy number.

For $0 < \alpha \leq 1$; α-cut of a fuzzy number $\tilde{u}$ is defined by,

$$[\tilde{u}]^{\alpha} = \{t \in \mathbb{R} \mid \mu_{\tilde{u}}(t) \geq \alpha\}. \tag{2.2}$$

Definition 2.2. (Voxman, 1998), A continuous function $s : [0,1] \longrightarrow [0,1]$ with the following properties is a regular reducing function :

1. $s(r)$ is increasing.
2. $s(0) = 0$,
3. $s(1) = 1$,
4. $\int_0^1 s(r)dr = \frac{1}{2}$.

Amirfakhrian, M.
A New Distance on a Specific Subset of Fuzzy Sets.
DOI: 10.5220/0006047900830087
In *Proceedings of the 8th International Joint Conference on Computational Intelligence (IJCCI 2016) - Volume 2: FCTA*, pages 83-87
ISBN: 978-989-758-201-1

In (Chong-Xin and Ming, 1991), the authors represented a fuzzy number $\tilde{u}$ by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$:

The *parametric form* of a fuzzy number is shown by $\tilde{v} = (\underline{v}(r), \overline{v}(r))$, where functions $\underline{v}(r)$ and $\overline{v}(r)$; $0 \leq r \leq 1$ satisfy the following requirements:

1. $\underline{v}(r)$ is monotonically increasing left continuous function.
2. $\overline{v}(r)$ is monotonically decreasing left continuous function.
3. $\underline{v}(r) \leq \overline{v}(r)$, $0 \leq r \leq 1$.
4. $\overline{v}(r) = \underline{v}(r) = 0$ for $r < 0$ or $r > 1$.

Definition 2.3. (Voxman, 1998), The *value* and *ambiguity* of a fuzzy number $\tilde{u}$ are defined by

$$Val(\tilde{u}) := \int_0^1 s(r)[\overline{u}(r) + \underline{u}(r)]dr, \tag{2.3}$$

and

$$Amb(\tilde{u}) := \int_0^1 s(r)[\overline{u}(r) - \underline{u}(r)]dr, \tag{2.4}$$

respectively.

Definition 2.4. A fuzzy set $\tilde{u}$ is a generalized *LR* fuzzy set, if there exist a positive number $h \in (0,1]$ such that

$$\mu_{\tilde{u}}(x) = \begin{cases} l(x), & a \leq x \leq b, \\ h, & b \leq x \leq c, \\ r(x), & c \leq x \leq d, \\ 0, & otherwise. \end{cases} \tag{2.5}$$

where $l(x)$ is nondecreasing on $[a,b]$ and $r(x)$ is nonincreasing on $[c,d]$ such that $l(a) = r(d) = 0$ and $l(b) = r(c) = h$. If $h = 1$ then $\tilde{u}$ is an *LR* fuzzy number.

In addition, if $l(x)$ and $r(x)$ are linear, then $\tilde{u}$ is a trapezoidal fuzzy set which is denoted by (a,b,c,d,h). In this case if $b = c$, we denote it by (a,b,d,h), which is a trapezoidal *LR* fuzzy set. Also if Let $TF(\mathbb{R})$ and $TF_h(\mathbb{R})$ be the set of all trapezoidal fuzzy numbers and all trapezoidal *LR* fuzzy set with height h on $\mathbb{R}$, respectively:

$$\begin{cases} TF(\mathbb{R}) = \{(a,b,c,d) : a < b \leq c < d\}, \\ TF_h(\mathbb{R}) = \{(a,b,c,d,h) : a < b \leq c < d\}. \end{cases} \tag{2.6}$$

Definition 2.5. A function $s \in C[0,h]$ with the following properties is a *source* function

1. $s(\alpha) \geq 0, \quad \alpha \in [0,h]$
2. $s(0) = 0$,
3. $s(h) = h$,
4. $\int_0^h s(\alpha)d\alpha = \frac{1}{2}h^2$.

For an *LR* fuzzy set $\tilde{u}$ and $\alpha \in [0,h]$ we define $\underline{u}(\alpha)$ and $\overline{u}(\alpha)$ as follows

$$\underline{u}(\alpha) = \inf\{x | \mu_{\tilde{u}}(x) \geq \alpha\}, \quad 0 \leq \alpha < h, \tag{2.7}$$
$$\overline{u}(\alpha) = sup\{x | \mu_{\tilde{u}}(x) \geq \alpha\}, \quad 0 \leq \alpha < h. \tag{2.8}$$

For a trapezoidal fuzzy set which is denoted by $\tilde{u} = (a,b,c,d,h)$, we have

$$\underline{u}(\alpha) = a + \tfrac{b-a}{h}\alpha, \tag{2.9}$$
$$\overline{u}(\alpha) = d - \tfrac{d-c}{h}\alpha. \tag{2.10}$$

Definition 2.6. We define *Value* and *Ambiguity* of an *LR* fuzzy set $\tilde{u}$ by the following relations:

1. $V_h(\tilde{u}) = \int_0^h s_h(\alpha)[\overline{u}(\alpha) + \underline{u}(\alpha)]d\alpha$,
2. $A_h(\tilde{u}) = \int_0^h s_h(\alpha)[\overline{u}(\alpha) - \underline{u}(\alpha)]d\alpha$.

Definition 2.7. Let s be a source function, then $I_{s,h}$ defined bellow is source number with respect to s.

$$I_{s,h} = \int_0^h s(\alpha)\alpha d\alpha. \tag{2.11}$$

Lemma 2.1. *For an arbitrary source function s over* $(0,h]$*, we have* $I_{s,h} < \frac{1}{2}h^3$.

Proof. By using Mid-point Theorem, the proof is straightforward. □

3 h-SOURCE DISTANCE BETWEEN FUZZY *LR* SETS

Definition 3.1. For two *LR* fuzzy sets $\tilde{u}$ and $\tilde{v}$, with same heights h, we define ***h*-source distance** D as follows,

$$D(\tilde{u},\tilde{v}) = \frac{1}{2}\{ |V_h(\tilde{u}) - V_h(\tilde{v})| + |A_h(\tilde{u}) - A_h(\tilde{v})| + h^3 d_H([\tilde{u}]^h, [\tilde{v}]^h) \}.$$

where d_H is Hausdorff metric, and $[\tilde{w}]^h = \{x | \mu_{\tilde{w}}(x) \geq h\}$ is the h-cut of fuzzy number $\tilde{w}$.

Theorem 3.1. *For* $\tilde{u}, \tilde{v}, \tilde{w}$ *in LR fuzzy sets, the h-source distance, D, satisfies the following properties:*

1. $D(\tilde{u},\tilde{u}) = 0$,
2. $D(\tilde{u},\tilde{v}) = D(\tilde{v},\tilde{u})$,
3. $D(\tilde{u},\tilde{w}) \leq D(\tilde{u},\tilde{v}) + D(\tilde{v},\tilde{w})$.

source distance between fuzzy numbers defined in (Abbasbandy and Amirfakhrian, 2006b) is a special case of h-source distance.

Example 3.1. *Let* $\mu_{\tilde{u}}(x) = \begin{cases} h & , \quad x = a, \\ 0 & , \quad otherwise, \end{cases}$, $\mu_{\tilde{v}}(x) = \begin{cases} h & , \quad x = b, \\ 0 & , \quad otherwise. \end{cases}$

$$D(\tilde{u},\tilde{v}) = \frac{1}{2}(h^3|a-b| + h^3|a-b|) = h^3|a-b|.$$

In this case if $\tilde{u}$ and $\tilde{v}$ are two crisp real numbers: $\mu_{\tilde{u}}(x) = \chi_{\{a\}}$, $\mu_{\tilde{v}}(x) = \chi_{\{b\}}$, *then*

$$D(\tilde{u},\tilde{v}) = |a-b|.$$

For the set of all LR fuzzy sets with the same height, we have the following theorem.

Theorem 3.2. *For $\tilde{u},\tilde{v},\tilde{u}',\tilde{v}' \in TF_h(\mathbb{R})$ and nonnegative real number k, h-source distance D satisfies the following properties:*

1. $D(k\tilde{u},k\tilde{v}) = kD(\tilde{u},\tilde{v})$,
2. $D(\tilde{u}+\tilde{v},\tilde{u}'+\tilde{v}') \leq D(\tilde{u},\tilde{u}') + D(\tilde{v},\tilde{v}')$.

4 NEAREST APPROXIMATION OF LR FUZZY SETS

In this section we use h-source distance to find the nearest approximation of an arbitrary fuzzy set. We start with a theorem on set of all fuzzy sets with the same height.

Theorem 4.1. *Let $\tilde{u},\tilde{v} \in TF_h(\mathbb{R})$, then $D(\tilde{u},\tilde{v}) = 0$, if and only if $\tilde{u} = \tilde{v}$.*

Proof. If $\tilde{u} = \tilde{v}$, from Theorem 3.1 we have $D(\tilde{u},\tilde{v}) = 0$. Let $\tilde{u} = (a_u,b_u,c_u,d_u,h)$ and $\tilde{v} = (a_v,b_v,c_v,d_v,h)$ are two trapezoidal fuzzy sets. If $D(\tilde{u},\tilde{v}) = 0$ then

$$\begin{cases} a) & \max\{h^3|c_u-c_v|, h^3|b_u-b_v|\} = 0, \\ b) & V_h(\tilde{u}) = V_h(\tilde{v}), \\ c) & A_h(\tilde{u}) = A_h(\tilde{v}). \end{cases} \tag{4.1}$$

From (a), we have $\max\{h^3|b_u-a_u|, h^3|b_v-a_v|\} = 0$ and hence $a_u = a_v$ and $b_u = b_v$. From (b) and (c)

$$\begin{aligned} V_h(\tilde{u}) + A_h(\tilde{u}) &= 2\int_0^h s(\alpha)\overline{u}(\alpha)d\alpha \\ &= 2\int_0^h s(\alpha)\overline{v}(\alpha)d\alpha \\ &= V_h(\tilde{v}) + A_h(\tilde{v}), \qquad (4.2) \\ V_h(\tilde{u}) - A_h(\tilde{u}) &= 2\int_0^h s(\alpha)\underline{u}(\alpha)d\alpha \\ &= 2\int_0^h s(\alpha)\underline{v}(\alpha)d\alpha \\ &= V_h(\tilde{v}) - A_h(\tilde{v}), \qquad (4.3) \end{aligned}$$

and hence

$$\begin{cases} \frac{d_u\, h^2}{2} - \frac{d_u-c_u}{h}I_{s,h} = \frac{d_v\, h^2}{2} - \frac{d_v-c_v}{h}I_{s,h}, \\ \frac{a_u\, h^2}{2} + \frac{b_u-a_u}{h}I_{s,h} = \frac{a_v\, h^2}{2} - \frac{b_v-a_v}{h}I_{s,h}. \end{cases} \tag{4.4}$$

By considering $\theta = h^3 - 2I_{s,h}$, the system 4.4 is equivalent to the following relations:

$$\begin{cases} hd_u\theta + 2c_uI_{s,h} = hd_v\theta + 2c_vI_{s,h}, \\ ha_u\theta + 2b_uI_{s,h} = ha_v\theta + 2b_vI_{s,h}. \end{cases} \tag{4.5}$$

Since $b_u = b_v$ and $c_u = c_v$, using Lemma 2.1, we have $a_u = a_v$ and $d_u = d_v$, hence $\tilde{u} = \tilde{v}$. □

Corollary 4.2. *h-source distance, D, is a metric on $TF_h(\mathbb{R})$, for a fixed height h.*

Proof. By Theorems 3.1 and 4.1 the proof is clear. □

Corollary 4.3. *Let $\tilde{u}$ be an arbitrary LR fuzzy set with height h. Let*

$$\underline{t} = \int_0^h \underline{u}(\alpha)s(\alpha)d\alpha, \quad \overline{t} = \int_0^h \overline{u}(\alpha)s(\alpha)d\alpha.$$

The nearest trapezoidal fuzzy set of $\tilde{u}$ is $\tilde{v} = (a_v,b_v,c_v,d_v,h)$, where

$$\begin{aligned} a_v &= \frac{1}{(h^3-2I_{s,h})}\left(-2b_v+2h\underline{t}\right), & (4.6) \\ b_v &= \underline{u}(h), & (4.7) \\ c_v &= \overline{u}(h), & (4.8) \\ d_v &= \frac{1}{(h^3-2I_{s,h})}\left(-2c_v+2h\overline{t}\right). & (4.9) \end{aligned}$$

Lemma 4.4. *For an arbitrary LR fuzzy set $\tilde{u}$ with height h, the nearest trapezoidal fuzzy set exists and it is unique.*

Proof. By Corollary 4.3 the proof is clear. □

5 NUMERICAL EXAMPLES

In this section we present some numerical examples using the proposed method.

Example 5.1.

Let $\tilde{u} = (e^r, e^{2-r})$, $\frac{1}{2}$ and $s(r) = r$. The nearest trapezoidal fuzzy set of $\tilde{u}$ is $\tilde{v} = \left(24-14\sqrt{e}, \sqrt{e}, e^{3/2}, 24e^2-38e^{3/2}\right)$. In Figure 1, $\tilde{u}$ and $\tilde{v}$ are shown by solid and dashed lines, respectively.

Example 5.2.

Let $\tilde{u} = (e^r, e^{2-r})$, and $s(r) = r$. The nearest triangular fuzzy number of $\tilde{u}$ is $\tilde{v} = (6-2e, e, 2e(3e-7))$. In Figure 2, $\tilde{u}$ and $\tilde{v}$ are shown by solid and dashed lines, respectively.

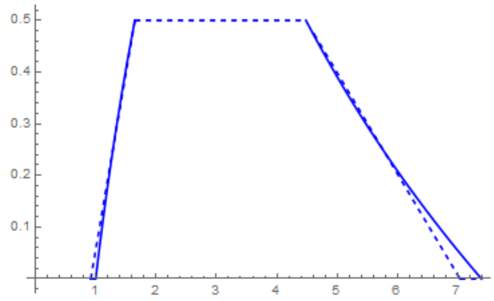


Figure 1: The nearest trapezoidal fuzzy set.

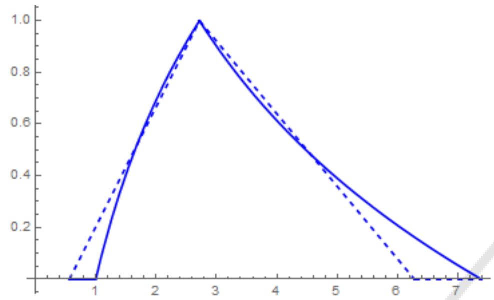


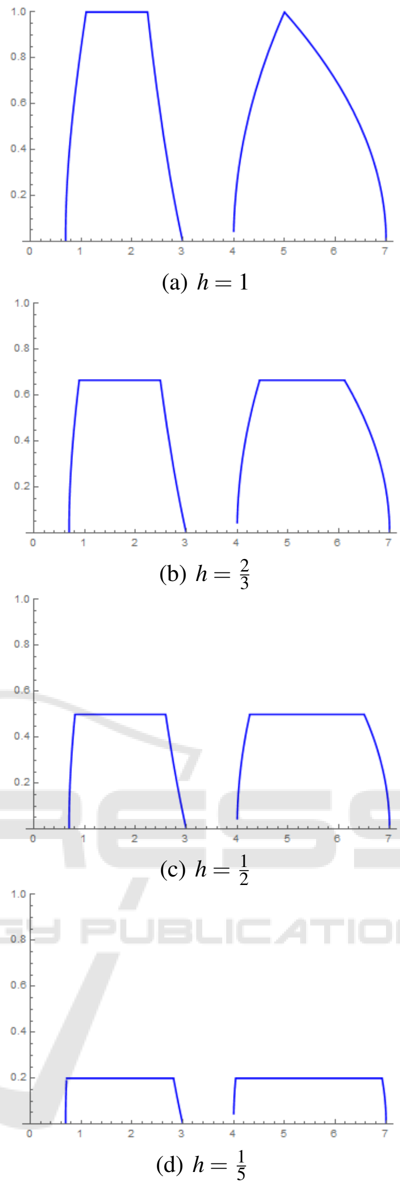
Figure 2: The nearest fuzzy number.

Example 5.3.

Let $\tilde{u}_h = (\ln(2+r^2, 4-\ln(1+r), h)$ and $\tilde{v}_h = (r^2+1, 5-2r^2, h)$. Using $s(r) = r$, the values of h-distance $D(\tilde{u}_h, \tilde{v}_h)$ between the fuzzy sets are shown in Table 1 for various values of h . See Figure 3.

Table 1: $\tilde{u}_h$ and $\tilde{v}_h$ for various values of h.

h	$D(\tilde{u}_h, \tilde{v}_h)$
1	3.95069
$\frac{2}{3}$	1.49782
$\frac{1}{2}$	0.76366
$\frac{1}{5}$	0.09769



(a) $h = 1$

(b) $h = \frac{2}{3}$

(c) $h = \frac{1}{2}$

(d) $h = \frac{1}{5}$

Figure 3: $\tilde{u}_h$ and $\tilde{v}_h$ for various values of h.

6 CONCLUSIONS

In this work we presented a method to find the nearest trapezoidal fuzzy set of an arbitrary LR fuzzy set with the same height. The nearest fuzzy set was found by using a new distance between fuzzy sets. Numerical examples shows that this method is acceptable.

REFERENCES

Abbasbandy, S. and Amirfakhrian, M. (2006a). The nearest approximation of a fuzzy quantity in parametric form. *Applied Mathematics and Computation*.

Abbasbandy, S. and Amirfakhrian, M. (2006b). The nearest trapezoidal form of a generalized left right fuzzy number. *Journal of Approximate Reasoning*.

Abbasbandy, S. and Asady, B. (2004). The nearest trapezoidal fuzzy number to a fuzzy quantity. *Applied Mathematics and Computation*.

Anzilli, L., Facchinetti, G., and Mastroleo, G. (2014).

A parametric approach to evaluate fuzzy quantities. *Fuzzy Sets and Systems*.

Ban, A. I., Coroianu, L., and Khastan, A. (2011). Conditioned weighted lr approximations of fuzzy numbers. *Fuzzy Sets and Systems*.

Cano, Y. C., Bez-Snchez, A. D., Romn-Flores, H., and Rojas-Medar, M. (2016). On the approximation of compact fuzzy sets. *Computers and Mathematics with Applications*.

Chanas, S. (2001). On the interval approximation of a fuzzy number. *Fuzzy Sets and Systems*.

Chong-Xin, W. and Ming, M. (1991). Embedding problem of of fuzzy number space: Part i. *Fuzzy Sets and Systems*.

Delgado, M., Vila, M. A., and Voxman, W. (1998). On a canonical representation of fuzzy numbers. *Fuzzy Sets and Systems*.

Grzegorzewski, P. (2002). Nearest interval approximation of a fuzzy number. *Fuzzy Sets and Systems*.

Grzegorzewski, P. and Mrówka, E. (2005). Trapezoidal approximations of fuzzy numbers. *Fuzzy Sets and Systems*.

Voxman, W. (1998). Some remarks on distance between fuzzy numbers. *Fuzzy Sets and Systems*.