A Shrinkage Factor-based Iteratively Reweighted Least Squares Shrinkage Algorithm for Image Reconstruction

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Abstract: In order to improve convergence speed and reconstruction precision of IRLS shrinkage algorithm (SIRLS), an improved iteratively reweighted least squares shrinkage algorithm (I-SIRLS) is proposed in this paper. A Shrinkage factor is brought in each iteration process of SIRLS to adjust the weight coefficient to approximate the optimal Lagrange Multiplier gradually. Put simply, the convergence speed is accelerated. The proposed algorithm needs less measurements. It can also get rid of falling into local optimal solution easily and the dependence on sparsity level. Simulations show that the I-SIRLS algorithm has faster convergence speed and higher reconstruction precision compared to the SIRLS.

1 INTRODUCTION

Nyquist sampling theorem requires that signal could be fully reconstructed only when the sampling rate is 2 times or more than 2 times the bandwidth. However, with the rapid increase of information demand, the bandwidth of the signal gets much wider, which has brought great pressure to signal processing. Compressed Sensing(CS) theory(Donoho D.L,2006) proposed by Donoho makes the sampling rate of signals exceed the Nyquist limit, which presents a new method to reconstruct the original signal from much fewer measurements using the prior knowledge, and compression and sampling are performed at the same time. How to use the limited sampling value to reconstruct the sparse signal with high accuracy, in recent years, makes more scholars pay attention to the sparse signal reconstruction(LC Jiao, SY Yang, F Liu, 2011). Until now, researchers have proposed a lot of algorithms to solve optimization problem(JH Wang, ZT Huang, YY Zhou, 2012) in signal reconstruction, including minimizing the norm(Van Den etc 2008, Needell D, 2009, Cai T T, 2009), minimizing the original signal norm(Chartrand R, 2007, Rodriguez P, 2006), matching pursuit algorithm and other algorithms of CS.

Minimizing the norm can be regard as one of convex optimization problems, thus making the NP-hard simplified into linear programming. But the norm-based signal reconstruction algorithm is still not able to effectively remove great redundancy between data. Chartrand R et.al(Chartrand R, 2007) put forward a method minimizing the non-convex norm of the original signal, which overcame the shortcomings of minimizing the norm by changing the nature of signal reconstruction. Not onlycan it better approximate to the original signal reconstruction andreduce data redundancy to a large extent, but also greatly reduce the number of observations which is needed to reconstruct the original signal accurately.

The IRLS(Chartrand R, 2008) is a typical non-convex of relaxation CS under RIP constraints, transforming norm minimization into norm with weights minimization(Daubechies I, 2010, Miosso C J, 2009, Ramani S, 2010, Daubechies I, 2010), it needs less measurements and it's convergence speed is often slow, and it is easy to fall into local optimal solution.To solve the above drawbacks of IRLS, this paper proposes an improved IRLS-based shrinkage algorithm, the feasibility and effectiveness of the improved algorithm is verified by experiments. Compared with IRLS-based shrinkage algorithm, the proposed method has a better reconstruction accuracy and convergence speed.

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2 IMPROVED ITERATIVELY REWEIGHTED LEAST SQUARES SHRINKAGE ALGORITHM

2.1 Compressed Sensing

In compressive sensing: the dimension measurement vector can be obtained from the original real-valued signal with the length:

$$y = \Phi x = \Phi \Psi s = \Theta s \tag{1}$$

Where Θ is a $N \times N$ basis matrix, and Φ is a $N \times N$ measurement matrix. Here exists two problems: (1) how to determine a stable basis Θ and measurement; (2) how to recover the N dimension vector $\mathbf{x}$ when the sparsity is K. The first problem can be solved when $M \geq K$.

That is, the matrix Φ must preserve the lengths of these particular K-sparse vectors. A related condition requires that the rows $\{\psi_j\}$ of Φ cannot sparsely represent the columns $\{\psi_j\}$ of Ψ and vice versa. The sparsest $\mathbf{x}$ is chosen when it meets the least nonzero. Zero norm of $\mathbf{x}$ is,

$$x = \arg\min_{x:Ax=b} \| x \|_0 \tag{2}$$

Obviously, it is a NP-hard problem. It costs so much and is impractical. So least square solution is considered. Based on the above consideration, $l_p (0 < p < 1)$ norm is chosen,

$$\tilde{x} = \arg\min_{x:Ax=b} \|x\|_p \tag{3}$$

2.2 Iteratively Reweighted Least Squares Shrinkage Algorithm

The basic idea of the improved algorithm is that the non-convex l_p norm is replaced by l_2 norm, then using the Lagrange multiplier method to seek the iteration algorithm of optimal solution for formula (3).In an iterative algorithm, given a current optimal solution x_{k-1}, set $X_{k-1} = diag\, | x_{k-1} |^q$, if X_{k-1} is reversible, then it satisfies the condition:

$$\|X_{k-1} x\|_2^2 = diag\, \|x_{k-1}\|_{2-2q}^{2-2q} \tag{4}$$

If $q = 1 - p/2$ is chosen, approximation optimization problem is converted to solving the l_p norm, meaning solving $\| x \|_p^p$. The equation is still valid if X_{k-1} in formula (4) is converted to its pseudo inverse matrix X_{k-1}^+. Following optimization problem is obtained:

$$\min_x \left\|X_{k-1}^+ x\right\|_2^2 \; s.t. \; b = Ax \tag{5}$$

To improve the sparseness, if there is a zero element in X_{k-1}, following formula can be obtained:

$$0.5\mathbf{A}(\mathbf{X}_{k-1})^2 \mathbf{A}^T \lambda = b \Rightarrow \lambda = 2(\mathbf{A}\mathbf{X}_{k-1}^2 \mathbf{A}^T)^{-1} b \tag{6}$$

The formula must solve the inverse matrix, so in the practical application, selecting the pseudo inverse matrix to instead the inverse matrix to decrease the computational complexity:

$$x_k = \mathbf{X}_{k-1}^2 \mathbf{A}^T (\mathbf{A}\mathbf{X}_{k-1}^2 \mathbf{A}^T)^+ \mathbf{b} \tag{7}$$

The SIRLS is in each iteration of IRLS, $\mathbf{A}$ and $\mathbf{A}^T$ is used as the intermediate amount, while shrinkage scalar is introduced minimize the objective function. The algorithm is very effective in solving the minimum problem of $f(x)$ in the formula (8):

$$f(x) = \lambda \|\mathbf{x}\|_p^p + \frac{1}{2}\|\mathbf{b} - \mathbf{A}x\|_2^2 \tag{8}$$

It is obvious that, choose the right P, we can obtain the global optimal solution. Replace $\| x \|_p^p$ with, $0.5\mathbf{x}^T W^{-1}(x) x$, $W(x)$ is a diagonal matrix, the diagonal value is $W(k,k) = 0.5x[k]^2 / p(x[k])$, This replacement is redundant for solving IRLS directly, but when it is introduced into the iterative shrinkage algorithm(like formula (13)), the result is desired:

$$f(x) = \frac{1}{2}\|b - \mathbf{A}x\|_2^2 + \lambda \|x\|_p^p = \frac{1}{2}\|b - \mathbf{A}x\|_2^2 + \lambda \mathbf{x}^T W^{-1}(x)\mathbf{x} \tag{9}$$

Come to solve:

$$\nabla f(x) = -\mathbf{A}^T (b - \mathbf{A}x) + \lambda W^{-1}(x) x = 0 \tag{10}$$

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By solving the inverse matrix to obtain a new update results. When W is fixed, updating $\mathbf{x}$, the approximation effect of the approximation solution

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in this process is relatively poor, especially in the treatment of high-dimensional signal. In order to make the algorithm deal with high-dimensional data, plus or minus to $c\mathbf{x}$ it. This is the shrinkage iterative algorithm based on the IRLS algorithm.In order to improve the convergence efficiency of the algorithm, a shrinkage factor c is brought in each iteration process of IRLS algorithm to minimize the objective function. Meanwhile, the iteration speed is improved and higher reconstruction precision is obtained. c ($c \geq 1$) is a relaxation constant, put it into equation (10)

$$-\mathbf{A}^T b + (\mathbf{A}^T\mathbf{A} - c\mathbf{I})x + (\lambda W^{-1}(x) + c\mathbf{I})x = 0 \quad (11)$$

Rebuilding the iterative algorithm with fixed-point iteration method, the result is:

$$\mathbf{A}^T b - (\mathbf{A}^T\mathbf{A} - c\mathbf{I})x_k = (\lambda W^{-1}(x_k) + c\mathbf{I})x_{k+1} \quad (12)$$

Obtaining a new iterative equation:

$$x_{k+1} = (\frac{\lambda}{c}\mathbf{W}^{-1}(x_k) + \mathbf{I})^{-1}(\frac{1}{c}\mathbf{A}^T b - \frac{1}{c}(\mathbf{A}^T\mathbf{A} - c\mathbf{I})x_k)$$

$$= S \cdot (\frac{1}{c}\mathbf{A}^T(\mathbf{b} - \mathbf{A}x_k) + \mathbf{x}_k) \quad (13)$$

Defining the diagonal matrix:

$$S = (\frac{\lambda}{c}\mathbf{W}^{-1}(x_k) + \mathbf{I})^{-1} = (\frac{\lambda}{c}\mathbf{I} + \mathbf{W}(x_k))^{-1}\mathbf{W}(x_k) \quad (14)$$

Applying the matrix to formula (17):

$$\frac{0.5x_k[i]^2 / \rho(x_k[i])}{\frac{\lambda}{c}0.5x_k[i]^2 / \rho(x_k[i])} = \frac{x_k[i]^2}{\frac{2\lambda}{c}x_k[i]^2 + x_k[i]^2} \quad (15)$$

It can be seen from the above equation, when $x_k[i]$ is very large, the equation (15) is close to 1, when $x_k[i]$ is very small, the above equation is close to 0, thus achieve the shrinkage effect. This is the most important improvement relative to the previous IRLS iteration shrinkage algorithm, this improvement speeds up the convergence speed, saves a lot of computation time, and gets higher re-construction precision. Similarly, this algorithm also requires the initial solution not be empty when initialize.

4 EXPERIMENTAL RESULTS AND DISCUSSION

4.1 Reconstruction of one-dimensional random signal

The experiment is carried out in MATLAB R2010a and it uses discrete-time signal with different lengths, the sparsity data is set to 5. Figure 2 and Figure 3 are results of signal with length 16 or 64; Average values of 1000 times experimental results with each length under the same conditions are shown in Table.3.

The I-SIRLS algorithm has better performance compared with the low measurement while reconstructing discrete signal, as shown in Figure 1 and Table 2. When the reconstruction probability is 1, the I-SIRLS algorithm requires less measurements than SIRLS algorithm. Meanwhile the reconstruction probability of the I-SIRLS algorithm can achieve 0.8, even though the number of samples is low, but the SIRLS algorithm probability can only get 0.6 or lower. When the two algorithms reach the maximum reconstruction probability, the I-SIRLS algorithm maintains a maximum reconstruction probability, but there is shocks in the performance of the SIRLS algorithm, it illustrates that the I-SIRLS algorithm has better shrinkage efficiency and stability. The target of the improved IRLS is to find the optimal solution x, making $f(x) = \lambda\|\mathbf{x}\|_p^p + \frac{1}{2}\|\mathbf{b} - \mathbf{A}x\|_2^2$ be the minimum. Algorithm realization after initializing ($k = 0$, $x_0 = 1$, $r_0 = b - Ax_k$) can be generalized in following steps:

(1) Back Projection: Calculate the residual $e = A^T r_{k-1}$

(2) Update shrinkage factor: Calculation of the value of the diagonal matrix $S(i,i) = x_k(i) / (\frac{2\lambda}{c}\rho(x_k[i]) + x_k[i]^2)$

(3) Shrinkage computing: Calculate $e_s = S(x_{k-1} + e / c)$

(4) Linear search: Select the appropriate μ to minimize the function $f(x_{k-1} + \mu(e_s - x_{k-1}))$

(5) Atomic map updater: Calculate $x_k = x_{k-1} + \mu(e_s - x_{k-1})$

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(6) Iteration stop condition: If $\left\|\mathbf{x}_k - \mathbf{x}_{k-1}\right\|_2^2$ Less than a given threshold value, Stop iteration; Otherwise, jump to the iterative process step (1)

(7) Output x_k

Table 1. Comparison of minimum number of measurements in I-SIRLS and SIRLS.

Signal length	Proposed method	SIRLS
16	8	12
64	15	18
256	21	25
1024	30	33

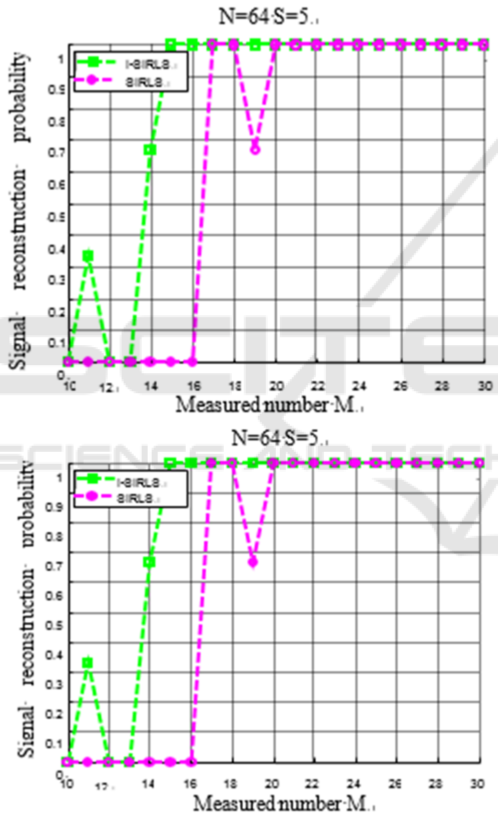

Figure 1. Contrast of reconstruction probability: (a) Signal length N=16; (b) Signal length N=64

4.2 Image Reconstruction

To further verify the reliability and effectiveness of the proposed algorithm, we select a test image: Lena. The original image is firstly transformed into DCT coefficients using DCT. The measurement matrix and the original im-age are shown in figure 2. Based on combination of CS theory, reconstructed images using the proposed algorithm with compression ratio is 0.3, 0.4 and 0.5, respectively. In the following, the results of evaluating the proposed algorithm compared with SIRLS algorithm are presented.

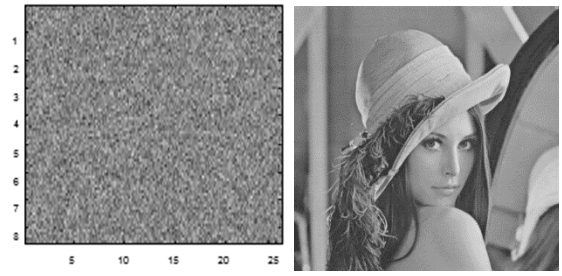

Figure 2. Measurement matrix and original Lena

According to results in Figure 3, Figure 4 and Table 3: the image reconstruction quality is worse with reducing sampling ratio, and the image even cannot be recognized. From the above table it can be seen that, the I-SIRLS algorithm has higher PSNR, it shows that the effect of reconstruction is better; the I-SIRLS algorithm has a smaller fluctuation range of PSNR, it shows that the stability of the proposed algorithm is superior to the SIRLS algorithm.

Figure 3.Reconstructed images of Lena: (a) SIRLS; (b) I-SIRLS in thecase of compression ratio respectively 0.4

Figure 4.Reconstructed images of Lena: (a) SIRLS (b) I-SIRLS in thecase of compression ratio respectively 0.5

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Table 2. Comparison of PSNR values in proposed reconstructed method and other methods.

Sampling rate	SIRLS	Proposed method
0.5	27.46	30.33
0.4	24.52	26.22
0.3	21.94	23.56

5 CONCLUSION

This article discusses the signal reconstruction based on compressed sensing theory, solving the problem that the convergence speed is not fast enough and the reconstruction accuracy is not high enough in IRLS-based shrinkage algorithm. Then the article presents an improved IRLS shrinkage algorithm. Each iteration process of SIRLS algorithm introduced a shrinkage factor, which makes the convergence rate and the reconstruction accuracy both better than the previous algorithm. The simulation results show that the improved algorithm has faster convergence rate and higher reconstruction precision compared to the previous algorithm.

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