Superposition of Qualitative Rectangles using a Quantitative Model

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Abstract: This paper describes an approach to qualitative problem-solving using the quantitative method in spatial reasoning. We consider the superposition of two objects, such that pre-specified parts of the objects are visible. First, we qualify an object to create a model. It is expressed as a matrix of tiles, which are either black or white depending on the visibility requirement. We use this to determine the location of two objects. This process involved quantitative treatment. We describe a sound and complete algorithm that provides quantitative solutions and implemented it as a system with a graphical user interface. Then, we extend this algorithm so that we may search for a better solution considering a qualitatively equivalent model of the objects; that is, the topological relationships between the black and white regions are identical. This approach is useful for analyzing or designing a projection of three-dimensional objects onto a two-dimensional plane, because it not only reduces the computational expense but also provides a better fit with common sense and human reasoning.

1 INTRODUCTION

Many problems involve locating multiple objects in a finite area with superposition or folding, where the geometry may be more complicated than simply a two-dimensional plane. For example, we may wish to allocate multiple windows on a screen of limited area (Konishi and Takahashi, 2012), determine the location of gates or codes in designing logical circuits (Lapaugh, 1996), place labels of building names in maps (Freeman, 1991; Li et al., 1998), or arrange an attractive display for goods. These problems are related to so-called "packing problems," which represent a class of optimization problems of pack objects into containers. Many efficient algorithms have been developed for packing problems (Lodi et al., 2002; Birgin et al., 2010), the aim of which is usually either to pack a single container as densely as possible, or to pack all objects using as few containers as possible. However, in the above kinds of problems, the aspect of visibility is accompanied; that is, an object may have a part that should be visible and another part that may not be; and the optimal solution is determined depending not only on the density or the number of containers but also on where specific parts are located. This can be reduced to a problem of superposition, in which the former part is visible and the latter part is hidden. With such problems, it is easier to understand how to determine approximately the positions of the connected parts or superposed parts of objects at an early stage of the design process than to compute this using precise data. More efficient methods for handling these kinds of problem are required with the increase in the number of occasions on which three-dimensional data are analyzed or designed.

Previously, we reported a qualitative method that handles objects consisting of should-be-visible parts and may-be-hidden parts (Konishi and Takahashi, 2012; Ghourabi and Takahashi, 2015a; Ghourabi and Takahashi, 2015b). The method looks for a superposition that satisfies these constraints. Here the term "qualitative" means representation and reasoning without using precise data. In these studies, we use a rectangle in which the size and ratio of the edges are variable as a target object, and consider objects as rectangles consisting of several tiles. This is represented in matrix form, where each element requires the attribute of visibility. We define the relative direction and distance of two tiles as a basic relationship, and perform reasoning on this representation.

Consider the superposition of the object shown in Figure 1(b) onto the object shown in Figure 1(a) without rotation, so that the white parts are visible. We obtain five solutions, as shown in Figures 1(c)–(g),

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which are qualitatively different. For example, (c) is obtained by placing the black region around the bottom left corner of (b) on the black region around the top right corner in (a).

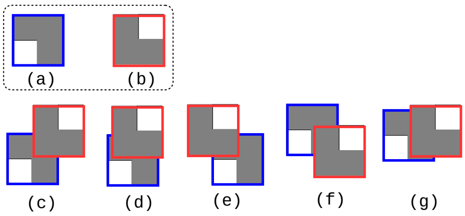

Figure 1: Qualitatively different solutions of a superposition.

We formalized an algorithm for this superposition method using the proof assistant Isabelle/HOL (Ghourabi and Takahashi, 2015a). This algorithm maps the relative direction and distance of a pair of tiles from the lower rectangle to those in the upper rectangle. This algorithm is sound but incomplete; that is, solutions may exist that cannot be found in some cases. In this paper, the algorithm is revised to preserve completeness and is implemented as a system with a visual interface.

Moreover, we extend this algorithm so that it can generate a "better" solution by regarding objects qualitatively. See Figure 2. If we superpose pattern (b) onto pattern (a) without rotation, so that white parts are visible, we obtain (c) as one solution, by placing (b) in the position indicated by the red frame; whereas if we superpose pattern (d) onto pattern (a), we obtain (e) as a solution. Here, the patterns shown in (b) and (d) are qualitatively equivalent, because their configurations are the same; that is, the topological relationships between the black and white regions are identical. In this case, (e) may be a better solution than (c), as more of the black parts of (a) are hidden.

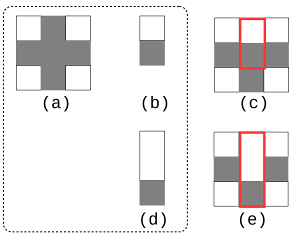

Figure 2: Superposition based on regarding objects qualitatively.

The principle idea of our approach is that we first determine the existence of a solution for a pair of rectangles by considering their configurations. This is solved quantitatively by using a matrix representation. If a solution exists, then a better solution is sought using another rectangle with the same configuration.

The remainder of this paper is organized as follows. In Section 2, we describe our qualitative reasoning and tiling approach to create qualitative rectangles, and also define a matrix representation along with the associated computations. In Section 3, we describe a sound and complete algorithm for the superposition. In Section 4, the method is expanded to find a better solution by regarding objects qualitatively. Section 5 concludes the paper.

2 PRELIMINARIES

2.1 Qualitative Spatial Treatment

Numerical data such as coordinates or sizes are typically used when handling spatial data such as figures, images, and animations. This leads to a significant burden in terms of data storage and processing. Considering the current growth in handling big spatial data, an efficient approach is required. Qualitative spatial reasoning (QSR) is one such approach to representing spatial data without numerical data. QSR enables reasoning via a symbolic representation; thus, it not only reduces computational expense, it also provides a better fit with common sense and human reasoning.

There have been a number of studies of QSR in the field of artificial intelligence (Stock, 1997; Cohn and Hazarika, 2001; Cohn and Renz, 2007; Chen et al., 2013). There are various types of QSR depending on which aspect one is interested in; for example, direction, size, and mereology (i.e., part-whole relationships). Spatial occlusion is a concept closely related to superposition, but studies of spatial occlusion do not consider the location of the occluded part of an object (e.g., (Randell et al., 2001)). In our approach to the superposition of qualitative rectangles, we focus on the relative direction between the tiles used to construct an object. There are several existing QSR systems that focus on the relative directions of two objects (Goyal and Egenhofer, 2001; Renz and Mitra, 2004; Skiadopoulos and Koubarakis, 2005). Of these, the tiling approach considers the shapes of objects and divides the plane into rectangular tiles (Goyal and Egenhofer, 2001; Li and Liu, 2015). We use a variation of the tiling approach that expresses directional relations between objects with better precision.

2.2 Tiling Approach

The target object is a closed finite connected region, which may include holes, and where the should-be-visible parts and may-be-hidden parts are indicated in advance[2].

We assume that objects are rectangles, which have should-be-visible parts and may-be-hidden parts; each edge of these parts is parallel either with the x- or y-axes. We extend these edges to divide the entire rectangle into $r*c$ tiles $(1 \leq r, c)$, as shown in Figure 3, where the should-be-visible part is shown in white and the may-be-hidden part in black.

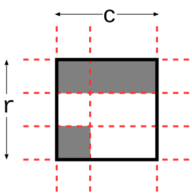

Figure 3: Tiling for an object: rectangle.

If an object cannot be described in such a simple form, we take a minimum bounding rectangle, which is an expression of the maximum extents of an object within a Cartesian coordinate system. As shown in Figure 4(a), where the green area shows a should-be-visible part, we approximate the border lines of objects as well as the border lines of the should-be-visible part so that each of them is parallel with either the x- or y-axes (Figure 4(b)). We now obtain a rectangle consisting of should-be-visible parts and may-be-hidden parts, where each of the edges of these parts are parallel with either the x- or y-axes. Any part not occupied by an object is regarded as a may-be-hidden part (Figure 4(c)). We then perform a similar process, as shown in Figure 3.

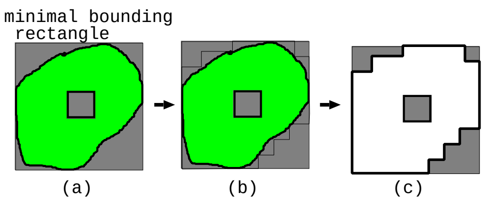

Figure 4: Preprocessing for tiling an object of any shape.

For any object, we can obtain a rectangle consisting of $r*c$ tiles $(1 \leq r, c)$ that includes no identical patterns of black and white in each adjacent row or column. This rectangle is called a *unit* and has the configuration of the target figure in the least refined form. If additional lines are drawn between the adjacent red parallel dotted lines in Figure 3, a rectangle consisting of more tiles is got, which is a more refined model of the object. For a pair of models, if their topological configuration of black and white regions are identical, then they are said to be *qualitatively equivalent*. All models of an object are qualitatively equivalent.

For example, all figures in Figure 5 are qualitatively equivalent, but they are not quantitatively equivalent. In this case, Figure 5(a) is taken as a unit. For any object, there exists a unique unit. If a tile is occupied by a should-be-visible part of an object, then it is called a *white tile*; otherwise it is called a *black tile*.

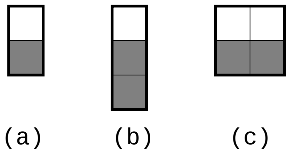

Figure 5: Qualitatively equivalent models.

2.3 Expression for a Unit Configuration

For a unit, each tile is referred as t_{ij}, where the top left of the unit is t_{00} and the value of i increases in the top down direction and the value of j increases in the left-to-right direction.

Definition 2.1 (granularity). *For a unit in which a row is r and a column is c, $r*c$ is called* the granularity *of the unit*[3]. *Let U_1 be a unit with granularity $r_1 * c_1$, and let U_2 be a unit with granularity $r_2 * c_2$. If either $r_1 \leq r_2, c_1 < c_2$ or $r_1 < r_2, c_1 \leq c_2$, then it is said that U_2* is more granular than U_1.

The tiles of a unit are classified into black and white tiles. Let T be a set of tiles of a unit, and let $\mathcal{B}(T)$ and $\mathcal{W}(T)$ be sets of the black and white tiles of T, respectively. $\mathcal{B}(T) \cup \mathcal{W}(T) = T$ and $\mathcal{B}(T) \cap \mathcal{W}(T) = \emptyset$ hold.

Figure 6 shows a unit with granularity $2*3$. Let T be a set of tiles of this unit. $\mathcal{B}(T) = \{t_{00}, t_{02}, t_{10}\}$, and $\mathcal{W}(T) = \{t_{01}, t_{11}, t_{12}\}$.

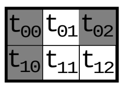

Figure 6: Unit expression.

Let t_{ij} and t_{pq} be tiles of a unit with granularity $r*c$ $(0 \leq i, p < r, 0 \leq j, q < c)$. We define function *dir* that computes the direction of target tile t_{pq} wrt reference tile t_{ij} as follows, where $k = |i-p|, h = |j-q|$:

[2] In previous reports, we assumed that a given object is a rectangle; however, in principle we may consider arbitrary geometries.

[3] Note that the meaning of "granularity" here differs from that used in (Ghourabi and Takahashi, 2015b).

$$dir(t_{ij},t_{pq}) = \begin{cases} same & \text{if } p=i, q=j \\ up_k & \text{if } p<i, q=j \\ down_k & \text{if } p>i, q=j \\ left_h & \text{if } p=i, q<j \\ right_h & \text{if } p=i, q>j \\ up_k \circ left_h & \text{if } p<i, q<j \\ up_k \circ right_h & \text{if } p<i, q>j \\ down_k \circ left_h & \text{if } p>i, q<j \\ down_k \circ right_h & \text{if } p>i, q>j \end{cases}$$

For example, in Figure 6, $dir(t_{02},t_{12}) = down_1$ and $dir(t_{02},t_{10}) = down_1 \circ left_2$.

For a specific reference tile, we define a set of relative directions in which black/white tiles are located.

Definition 2.2 (Direction of B/W-tiles). *For a specific tile t, $\mathcal{D}_{\mathcal{B}}(t) = \cup_{t' \in \mathcal{B}(T)}\{dir(t,t')\}$ is called* the set of directions of B-tile wrt t, *and $\mathcal{D}_{\mathcal{W}}(t) = \cup_{t' \in \mathcal{W}(T)}\{dir(t,t')\}$ is called* the set of directions of W-tile wrt t.

For example, in Figure 6, $\mathcal{D}_{\mathcal{B}}(t_{02}) = \{same, left_2, down_1 \circ left_2\}$ and $\mathcal{D}_{\mathcal{W}}(t_{02}) = \{left_1, down_1 \circ left_1, down_1\}$.

3 ALGORITHM FOR SUPERPOSITION

3.1 Superposition of a Unit

Here superposition refers to placing a *foreground unit* denoted by U_f onto a *background unit* denoted by U_b, in such a way that a tile of U_f is placed on a tile of U_b.

We define the success of superposition using directions.

Definition 3.1 (Success of Superposition.). *Let T_f and T_b be sets of tiles of a foreground unit U_f and a background unit U_b, respectively. For $t_f \in T_f$, $t_b \in T_b$, if $\mathcal{D}_{\mathcal{W}}(t_b) \cap (\mathcal{D}_{\mathcal{W}}(t_f) \cup \mathcal{D}_{\mathcal{B}}(t_f)) = \emptyset$, then it is said that* superposition of t_f on t_b succeeds. *If there exists such a pair t_f and t_b, then it is said that* superposition of U_f on U_b succeeds.

When a superposition of t_f on t_b succeeds, for all $d \in \mathcal{D}_{\mathcal{W}}(t_b)$, there does not exist $t' \in T_f$ such that $dir(t_f,t') = d$. Intuitively, there is no tile in direction d wrt t_f on U_f, where d is the direction of a white tile wrt t_b on U_b.

3.2 Computing Superposition

For a specific black tile t_b of T_b, we generate $\mathcal{D}_{\mathcal{W}}(t_b)$, a set of directions of all white tiles wrt to t_b. For each element d of the set, we define a function *atiles* that generates a set of tiles of T_f an element of which can be put on t_f without hiding a white tile in the direction of d [4]. That is, $atiled(d)$ is a set of tiles that does not have any tiles in the direction of d.

Let $r*c$ be a granularity of U_f.

$atiles(same) = \emptyset$

$atiles(up_k) =$
$\{t_{ij} | 0 \le i < min(k,r), 0 \le j < c\}$

$atiles(down_k) =$
$\{t_{ij} | max(0, r-k) \le i < r, 0 \le j < c\}$

$atiles(left_k) =$
$\{t_{ij} | 0 \le i < r, 0 \le j < min(k,c)\}$

$atiles(right_k) =$
$\{t_{ij} | 0 \le i < r, max(0, c-k) \le j < c\}$

$atiles(dir \circ dir') = atiles(dir) \cup atiles(dir')$

We then take the intersection of the obtained sets of tiles for each d:

$$puton(t_b) = \bigcap_{d \in \mathcal{D}_{\mathcal{W}}(t_b)} atiles(d) \subseteq T_f.$$

If $puton(t_b) \neq \emptyset$, the superposition succeeds and we obtain the set of solutions:

$$sol(T_b,T_f) = \{(t_b,t_f) \mid t_f \in puton(t_b)\}.$$

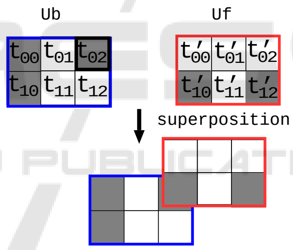

Figure 7: Superposition.

For example, in Figure 7, let $T_f = \{t'_{00}, t'_{01}, t'_{02}, t'_{10}, t'_{11}, t'_{12}\}$ and $T_b = \{t_{00}, t_{01}, t_{02}, t_{10}, t_{11}, t_{12}\}$. $\mathcal{B}(T_b) = \{t_{00}, t_{02}, t_{10}\}$. Let t_{02} be a tile that is to be superposed. Then $\mathcal{D}_{\mathcal{W}}(t_{02}) = \{left_1, down_1 \circ left_1, down_1\}$. In this case, $atiles(left_1)$ is the set of tiles in U_f to put on t_{02} without hiding t_{01}, and we obtain $\{t'_{00}, t'_{10}\}$. Similarly, $atiles(down_1 \circ left_1) = \{t'_{00}, t'_{10}, t'_{11}, t'_{12}\}$ and $atiles(down_1) = \{t'_{10}, t'_{11}, t'_{12}\}$. Therefore, $puton(t_{02}) = atiles(left_1) \cap atiles(down_1 \circ left_1) \cap atiles(down_1) = \{t'_{10}\} \neq \emptyset$, and hence the superposition of t'_{10} on t_{02} succeeds.

The figures obtained by these superpositions may be identical, even if the solutions in $sol(T_b,T_f)$ are distinct. For example, in Figure 8, solutions (t_{00}, t'_{02}) and (t_{10}, t'_{12}) both generate the same figure.

[4]The name *atiles* indicates the tiles that are “allowed” to be placed.

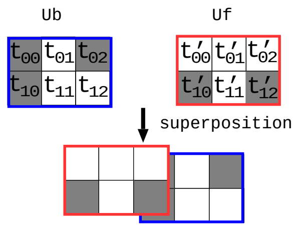

Figure 8: Different solutions give the same result.

3.3 Soundness and Completeness

Here we prove the soundness and completeness of the algorithm.

3.3.1 Soundness

The following lemma indicates that, if there exists a tile that is allowed to be placed, then no tile exists that hides a white tile.

Lemma 3.1. *Let T be a set of tiles of a unit with granularity $r * c$. For any direction d, if there exists $t_{ij} \in T$ such that $t_{ij} \in atiles(d)$, then there does not exist $t_{pq} \in T$ such that $dir(t_{ij}, t_{pq}) = d$.*

Proof. Assume that there exists $t_{pq} \in T$ such that $dir(t_{ij}, t_{pq}) = d$ ($0 \leq p < r$, $0 \leq q < c$). Here, we show a proof for the case whereby $d = up_k$. In this case, $k = i - p$ from the definition of dir. If $k > r$, then $0 \leq i < min(k,r) = r$, since $t_{ij} \in atiles(up_k)$. Therefore, $p = i - k < i - r < 0$ holds, which contradicts $0 \leq p$. If $k \leq r$, then $0 \leq i < min(k,r) = k$, since $t_{ij} \in atiles(up_k)$. Therefore, $p = i - k < 0$ holds, which contradicts $0 \leq p$. For other directions, the proof proceeds similarly.

Theorem 3.1. *Let $t_f \in T_f$ and $t_b \in T_b$. For all $(t_b, t_f) \in sol(T_b, T_f)$, superposition t_f on t_b succeeds.*

Proof. Assume that there exists $(t_b, t_f) \in sol(T_b, T_f)$ such that superposition t_f on t_b does not succeed. Then there exists d such that $d \in \mathcal{D}_\mathcal{W}(t_b) \cap (\mathcal{D}_\mathcal{W}(t_f) \cup \mathcal{D}_\mathcal{B}(t_f))$; that is, $d \in \mathcal{D}_\mathcal{W}(t_f) \cup \mathcal{D}_\mathcal{B}(t_f)$. Therefore, there exists $t'_f \in T_f$ such that $dir(t_f, t'_f) = d$. On the other hand, $t_f \in atiles(d)$ for all $d \in \mathcal{D}_\mathcal{W}(t_b)$, from $t_f \in puton(t_b)$. There does not exist t'_f that satisfies $dir(t_f, t'_f) = d$ for any $d \in \mathcal{D}_\mathcal{W}(t_b)$, from Lemma 3.1. It is a contradiction. Therefore, for all $(t_b, t_f) \in sol(T_b, T_f)$, the superposition of t_f on t_b succeeds.

3.3.2 Completeness

The following lemma indicates that if there exists a tile that cannot be placed, there must exist a tile that hides a white tile.

Lemma 3.2. *Let T be a set of tiles of a unit with granularity $r * c$. For any direction d, if there exists $t_{ij} \in T$ such that $t_{ij} \notin atiles(d)$, then there exists $t_{pq} \in T$ such that $dir(t_{ij}, t_{pq}) = d$.*

Proof. Here, we show a proof for the case whereby $d = up_k$. If we assume $k > r$, then for all $t_{ij} \in T$, t_{ij} is included in $atiles(d)$, which contradicts the condition; therefore, $k \leq r$. $k \leq i < r$ holds, since $t_{ij} \notin atiles(up_k)$. It follows $0 \leq i - k < r - k < r$. Therefore, if we take $p = i - k, q = j$, then $t_{pq} \in T$ and $dir(t_{ij}, t_{pq}) = up_k$ hold. For other directions, the proof proceeds similarly.

Theorem 3.2. *Let $t_f \in T_f$ and $t_b \in T_b$. If superposition t_f on t_b succeeds, then $(t_b, t_f) \in sol(T_b, T_f)$.*

Proof. Assume that $(t_b, t_f) \notin sol(T_b, T_f)$. Then, there exists $d \in \mathcal{D}_\mathcal{W}(t_b)$ such that $t_f \notin atiles(d)$ holds, and there exists t'_f such that $dir(t_f, t'_f) = d$ from Lemme 3.2. $dir(t_f, t'_f) \in \mathcal{D}_\mathcal{W}(t_f) \cup \mathcal{D}_\mathcal{B}(t_f)$, and thus $d \in \mathcal{D}_\mathcal{W}(t_f) \cup \mathcal{D}_\mathcal{B}(t_f)$. Hence, $\mathcal{D}_\mathcal{W}(t_b) \cap (\mathcal{D}_\mathcal{W}(t_f) \cup \mathcal{D}_\mathcal{B}(t_f)) \neq \emptyset$, and it follows that the superposition of t_f on t_b fails. It is a contradiction. Therefore, $(t_b, t_f) \in sol(T_b, T_f)$.

3.4 Effectiveness

There may be multiple solutions of the superposition problem. Here, we take the number of black tiles hidden by the superposition as a measure of the evaluation, because effective usage of a finite space is often considered as a requirement.

Hereafter, we refer to a solution for the superposition described so far as a "quantitative solution" to distinguish it from a "qualitative solution," which will be introduced later.

Definition 3.2 (Quantitative Solution). *Let T_f and T_b be tiles of U_f and U_b, respectively. An element of $sol(T_b, T_f)$ is called* a quantitative solution for the superposition of U_f on U_b.

Definition 3.3 (Covered Tile). *Let $(t_b, t_f) \in sol(T_b, T_f)$. We call the black tiles hidden by the superposition* covered tiles. *The number of the covered tiles is defined as follows:*

$$N_{cov}(t_b, t_f) = |\mathcal{D}_\mathcal{B}(t_b) \cap \bigcup_{t'_f \in T_f} dir(t_f, t'_f)|$$

It is clear that $N_{cov}(t_b, t_f)$ is less than or equal to the smaller value of $|\mathcal{B}(T_b)|$ and $|T_f|$.

Definition 3.4 (Quantitative Optimal Solution). *Let T_f and T_b be sets of tiles of U_f and U_b, respectively. For $(s, s'), (t, t') \in sol(T_b, T_f)$, if $N_{cov}(s, s') < N_{cov}(t, t')$ it is said that (t, t') is* a better solution *than*

(s,s'). *If there exists no* $(t,t') \in sol(T_b,T_f)$ *such that* $N_{cov}(s,s') < N_{cov}(t,t')$ *holds, then* (s,s') *is called* a quantitative optimal solution for the superposition of U_f on U_b.

For example, compare Figures 7 and 8, which show the superpositions of the same pair of U_f and U_b. $N_{cov}(t_{02},t'_{10}) = 1$ in Figure 7, whereas $N_{cov}(t_{00},t'_{02}) = 2$ in Figure 8. Therefore, the latter is a better solution.

3.5 Implementation

We implemented a system based on this method in Java (Figure 9). The system initially shows a field consisting of $5 * 5$ cells for each rectangle by default. A user inputs patterns of pairs of black and white rectangular tiles, and then the system makes a unit and solves the problem. All quantitative optimal solutions that are found are displayed, and if a user selects one of them, the resizing and moving processes of the superposition are shown as an animation, which provides a visual aid to help the user's understanding.

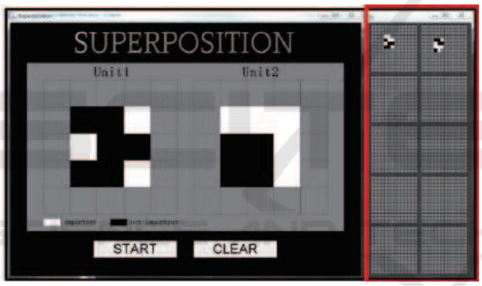

Figure 9: A screen shot of the system.

4 SYSTEM EXTENSION

4.1 Maximal Black Rectangle

A unit produced by a tiling approach represents a configuration; that is, a set of topological relationships between black and white regions. A superposition is defined as the manner in which we place a tile onto another tile. Each tile corresponds to a subregion of an original target object, and the exact size is ignored when creating a unit. This implies that there may be another solution whereby a greater number of tiles are covered if we admit superposing one tile onto more than one other tile.

For example, in Figure 10, when superposing U_f onto U_b, (a) is the quantitative optimal solution when superposing one tile onto a single tile, whereas (b) and (c) show the solutions if allowed to place a tile onto multiple tiles. In these cases, we have a larger number of covered tiles. Such a phenomenon occurs when the granularity of two units differ, and indicates that we must take into account such qualitative factors in merging two representations.

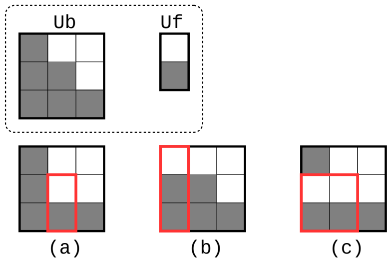

Figure 10: Superpositions of two units with different granularity.

To this end, we extend the representation of a unit while preserving the condition whereby we place a single tile onto another single tile. We fix a background unit and extend only a foreground unit, since the number of the black tiles is used as an evaluation measure. A maximal black rectangle, corresponding to the limit of the extension of the background unit, is determined. Intuitively, a maximal black rectangle is the connected set of black tiles of which the shape is a rectangle.

Definition 4.1 (Maximal Black Rectangle)**.** *Let T be a set of tiles of a unit with granularity $r * c$. A black rectangle $R(x,y,r',c')$ included in the unit where $0 \leq x < r - r'$, $0 \leq y < c - c'$ is defined as follows:*
$R(x,y,r',c') =$
$\{t_{ij} \mid t_{ij} \in \mathcal{B}(T), x \leq i < x+r', y \leq j < y+c'\}$.

$R(x,y,r',c')$ *is called a* maximal black rectangle *of the unit, if none of the following conditions are satisfied:*

1. $y \neq 0$ *and* $\forall t_{ij};\ x \leq i < x+r', j = y-1, t_{ij} \in \mathcal{B}(T)$.
2. $y \neq c - c'$ *and* $\forall t_{ij};\ x \leq i < x+r', j = y+c', t_{ij} \in \mathcal{B}(T)$.
3. $x \neq 0$ *and* $\forall t_{ij};\ i = x-1, y \leq j < y+c', t_{ij} \in \mathcal{B}(T)$.
4. $x \neq r - r'$ *and* $\forall t_{ij};\ i = x+r', y \leq j < y+c', t_{ij} \in \mathcal{B}(T)$.

$r' * c'$ *is called its* granularity.

For example, for a unit in Figure 11, one of the maximal black rectangles is highlighted with the green frame.

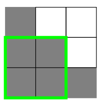

Figure 11: A maximal black rectangle.

4.2 Extension of a Unit

Assume that U_b is more granular than U_f. If there exists a maximal black rectangle of U_b in which the covered tiles in a quantitative optimal solution are properly included, then *an extension of* U_f is performed to the extent of the maximal black rectangle. The extended unit is qualitatively equivalent to the original unit.

Let U_f be a unit with granularity $r*c$, and let $R(x,y,r',c')$ be a maximal black rectangle of U_b. *An extension of* U_f, denoted by U'_f, is a unit with granularity $r'*c'$, and attributes (black/white) of its tiles are defined as follows:
Let T and T' be sets of tiles of U_f and U'_f, respectively.

- $t'_{ij} \in \mathcal{B}(T')$ iff

$$\begin{cases} t_{ij} \in \mathcal{B}(T) & (0 \le i < r, 0 \le j < c) \\ t_{i\ c-1} \in \mathcal{B}(T) & (0 \le i < r, c \le j < c') \\ t_{r-1\ j} \in \mathcal{B}(T) & (r \le i < r', 0 \le j < c) \\ t_{r-1\ c-1} \in \mathcal{B}(T) & (r \le i < r', c \le j < c') \end{cases}$$

- $t'_{ij} \in \mathcal{W}(T')$ otherwise.

It is the number of tiles and not the attribute of tiles of U_f that affects the number of covered tiles. Therefore, we are not concerned about the attributes of added tiles but hold that U_f and U'_f are qualitatively equivalent. Here, with vertical extension, the tiles in new rows have the same attributes as those in the last row of the original matrix, and those with horizontal extension are defined similarly.

Figure 12(a) shows an example of an extension from unit U_f, with granularity $2*2$ (Figure 12(b)) to U'_f and with granularity $4*4$ (Figure 12(c)). Note that U_f and U'_f are qualitatively equivalent, but quantitatively different.

Definition 4.2 (Qualitative Solution)**.** *Let* U'_f *be an extension of* U_f. *A solution for superposing* U'_f *on* U_b *is called a* qualitative solution for the superposition of U_f on U_b.

For example, consider the superposition of U_f onto U_b shown in Figure 13. A maximal black rectangle of U_b is highlighted as the green frame. Figure 13(a) shows a quantitative optimal solution for a superposition of U_f on U_b, and Figure 13(b) is a qualitative solution for a superposition of U_f on U_b.

4.3 Properties

The following properties immediately hold from the definition of extension of a unit and the completeness of the superposition algorithm shown in section 3.2.

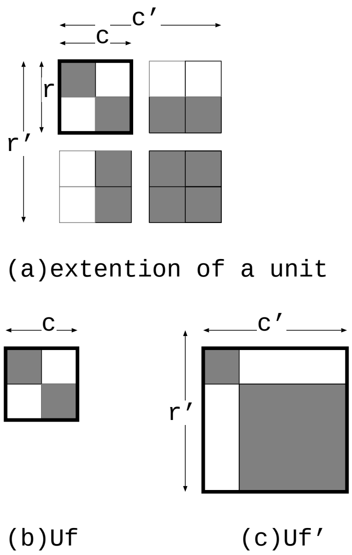

Figure 12: Definition of extension of a unit.

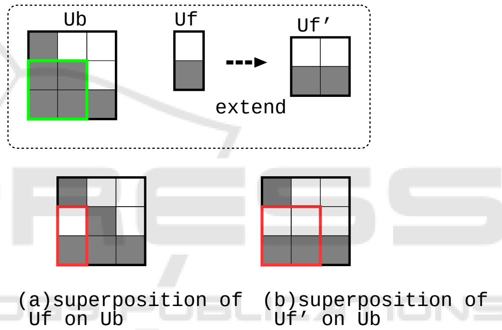

Figure 13: (a) Quantitative solution and (b) qualitative solution.

Proposition 4.1. *For a superposition of a unit* U_f *on* U_b*, if all covered tiles for the quantitative optimal solution are properly included in some maximal black rectangle, then a qualitative solution in which the number of covered tiles is the granularity of the maximal black rectangle can be obtained.*

Proposition 4.2. *If no quantitative solution exists for a superposition of* U_f *on* U_b*, then there is no qualitative solution for a superposition of* U_f *on* U_b.

5 CONCLUSION

We have described a qualitative treatment of two-dimensional superposed objects consisting of should-be-visible parts and may-be-hidden parts. We consider this as the problem of superposing rectangles while maintaining some pre-specified parts visible.

Our approach is as follows. First, we qualify an object to make a least refined model and represent it in matrix form. We find a solution with a quantitative computation using the matrix, then search for a better solution considering a qualitatively equivalent model of the object.

The main contributions of this work are as follows:

- We presented a sound and complete algorithm to provide quantitative solutions, and implemented it as a system with a graphical user interface.
- We extended the algorithm so that a qualitative solution can be obtained when the granularities of units differ.

The method described is useful for analyzing or designing a projection of three-dimensional objects.

An evaluation measure on the obtained solutions is determined based on the user's purpose. When superposing more than two units, the black tiles that are not hidden provide room for a third unit to be placed. Therefore, it is not always true that fewer covered tiles offer a better solution. The location of a black tile of a resulting figure is also an evaluation measure candidate.

As part of future work, we plan to evaluate the obtained solutions to handle superpositions of more than two units.

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