Learning Plaintext in Galbraith's LWE Cryptosystem

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Abstract: Unlike many widely used cryptosytems, Learning with Errors (LWE) - based cryptosystems are known to be invulnerable to quantum computers. Galbraith's Binary LWE (GB-LWE) was proposed to reduce the large key size of the original LWE scheme by over two orders of magnitude. In GB-LWE, recovering the plaintext from the ciphertext involves solving for the binary vector $\boldsymbol{x}$ in the equation $\boldsymbol{xA} = \boldsymbol{b}$ ($\boldsymbol{A}$, a 640×256 binary matrix and $\boldsymbol{b}$, a 256 element integer vector are knowns). Previously, lattice-based attacks on binary matrices larger than 400×256 were found to be infeasible. Linear programming was proposed and shown to handle significantly larger matrices but its success rate for 640×256 matrices was found to be negligible. Our strategy involves identification of regimes L, M and H within the output (based on LP relaxation) where the mis-prediction rates are low, medium or high respectively. Bits in the output vector are guessed and removed to create and solve a reduced instance. We report extensive experimental results on prediction accuracy and success probability as a function of number of bits removed in L, M and H. We identify trade-offs between lower execution time and greater probability of success. Our success probability is much higher than previous efforts and its execution time of 1 day with 150 cores is a partial response to the challenge posed in (Galbraith, 2013) to solve a random 640×256 instance using "current computing facilities in less than a year".

1 INTRODUCTION

Introduced by Regev (Regev, 2005) in 2005, Learning with Errors (LWE) is a problem in machine learning and is as hard to solve as certain worst-case lattice problems. Unlike most widely used cryptographic algorithms, it is known to be invulnerable to quantum computers. It is the basis of many cryptographic constructions including identity-based encryption (Cash et al., 2010), (Agrawal et al., 2010), oblivious transfer protocols (Peikert et al., 2008), homomorphic encryption (Brakerski and Vaikuntanathan, 2014), (Brakerski et al., 2014) and many more.

The LWE cryptosystem performs bit by bit encryption. The private key, $\boldsymbol{s}$, is a vector of length n where each element of $\boldsymbol{s}$ is randomly chosen over $\mathbf{Z}_q$, q prime. The corresponding public key has two components. The first is a random $m \times n$ matrix, $\boldsymbol{A}$, with elements over $\mathbf{Z}_q$ and with rows denoted $\boldsymbol{a}_i^T$. The second component is a vector, $\boldsymbol{b}$, of length m where the i^{th} element of $\boldsymbol{b}$ is $\boldsymbol{a}_i^T \boldsymbol{s} + e_i \pmod q$. The e_i's are drawn from a discretized normal distribution with mean 0 and standard deviation σ.

To encrypt a bit, x, a random binary vector (nonce), $\boldsymbol{u}$, of length m is chosen. This is a per-message ephemeral secret. The ciphertext is ($\boldsymbol{c}_1$, c_2) where $\boldsymbol{c}_1^T = \boldsymbol{u}^T \boldsymbol{A} \pmod q$ and $c_2 = \boldsymbol{u}^T \boldsymbol{b} + x\lfloor q/2 \rfloor \pmod q$. A received message is decrypted to 0 or 1 depending on whether $c_2 - \boldsymbol{c}_1 \boldsymbol{s}$ is closer to 0 or $\lfloor q/2 \rfloor$.

To thwart various lattice-based attacks, Lindner et al. (Lindner and Peikert, 2011) suggested the values of 256, 640 and 4093 respectively for n, m and q leading to a public key of size approximately 250 Kbytes. This unacceptable storage overhead for resource-constrained devices motivated consideration of binary values in matrix $\boldsymbol{A}$.

Galbraith (Galbraith, 2013) studied a ciphertext-only attack on the GB-LWE scheme to recover the plaintext. Given $\boldsymbol{c}_1 = \boldsymbol{u}^T \boldsymbol{A}$ and $\boldsymbol{A}$, the challenge is to obtain $\boldsymbol{u}^T$. Once $\boldsymbol{u}^T$ is known, the plaintext x can be easily computed from c_2. Because $\boldsymbol{u}^T$ is binary, obtaining its value is equivalent to finding the rows of $\boldsymbol{A}$ that sum to $\boldsymbol{c}_1$, i.e. the Vector Subset Sum (VSS) problem. (Galbraith, 2013) studied lattice-based attacks on GB-LWE and concluded that such attacks were infeasible for $m > 400$.

(Galbraith, 2013) also posed two challenges. The first of these was to be completed on an ordinary *PC* in one day and involved computing $\boldsymbol{u}^T$ given a random 400×256 binary matrix $\boldsymbol{A}$ and $\boldsymbol{c}_1^T = \boldsymbol{u}^T \boldsymbol{A}$. The second problem was the same but with a random 640×256 binary matrix to be solved in one year with "cur-

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rent computing facilities". Herold et al. (Herold and May, 2017) responded to the first challenge successfully by formulating VSS as a problem in Linear Programming (LP) and/or Integer Linear Programming (ILP). However, for m>490, the success rate of this approach dropped greatly. Our goal is to address the second of these two challenges with a much higher success rate and execution time of roughly 1 day.

The main contribution of this paper is the outline of a strategy wherein we attempt to solve a given instance of the VSS problem by guessing some of the bits in the unknown vector $\boldsymbol{u}^T$. We remove these to create and solve a reduced instance or sub-instance of the original problem. Using LP as in (Herold and May, 2017), we obtain fractional solutions for m>2n in particular. Rounding the solutions to 0 or 1 results in a mis-prediction rate of about 20% on average for $m = 640$. We identify regimes L, M and H within the predicted solution vector where the mis-prediction rates are low, medium or high respectively. It is necessary to carefully choose the bits in $\boldsymbol{u}^T$ for removal since a valid sub-instance can only be created by removal of bits that are all correctly guessed. This suggests choosing bits in regime L. However, extensive experiments conducted by us led to the conclusion that the resulting sub-instance is hard to solve even if the size of the sub-instance is greatly reduced. Indeed it is best to remove bits in H but the correctness of their predicted values is hardest to guarantee.

We report results on both, prediction accuracy and success probability as a function of number of bits removed in L, M and H. We also experimented with simultaneously removing bits in multiple regimes. Finally, we categorize instances into classes A, B and C based merely on the LP output and with no knowledge of the ephemeral secret. A Class A instance will, in general, have a lower mis-prediction rate and can be solved with far less computational effort.

Our convention is to represent a vector in lowercase bold and a matrix in uppercase bold. A row vector, $\boldsymbol{v}$, is represented as $\boldsymbol{v}^T$.

The rest of the paper is organized as follows. Section 2 contains a brief background including related work. Section 3 outlines our approach and contains results of experiments on mis-prediction rates and success probabilities obtained by removing bits in different regimes. Section 4 includes an explanation of our main results, suggests certain optimizations and presents a classification of instances. Section 5 contains the conclusion.

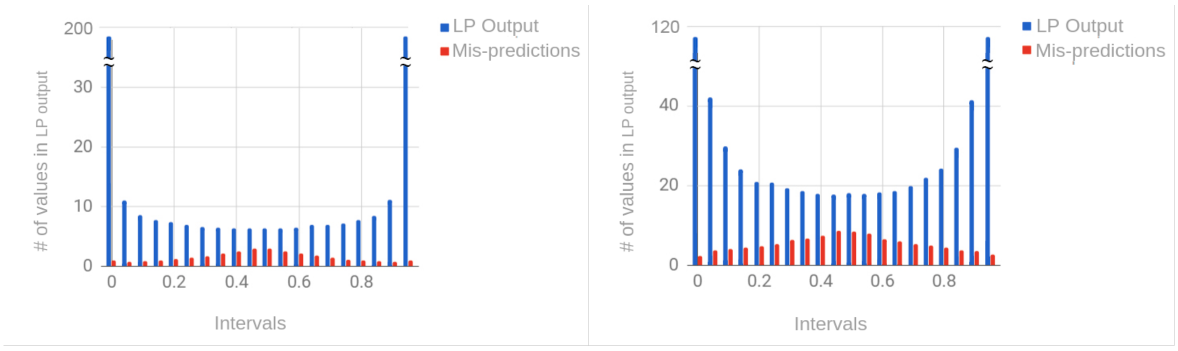
2 BACKGROUND AND RELATED WORK

(Regev, 2005) proposed LWE - a lattice based hard problem. To reduce the key size in the original LWE problem, the ring LWE cryptosystem was studied in (Lyubashevsky et al., 2010) and was also shown to be as hard as worst case lattice problems. Another variant of the LWE problem known as Binary-LWE (Micciancio and Peikert, 2013) was later proposed. Here, the secret is a binary string instead of a string of integers modulo q. It results in smaller cryptographic keys without compromising theoretical security guarantees. Hardness of B-LWE problem was proved theoretically in (Micciancio and Peikert, 2013). (Bai and Galbraith, 2014) concluded that a key size of 440 is sufficient to make B-LWE as secure as the LWE problem with key size 256. Later another variant of LWE problem, GB-LWE was studied in (Galbraith, 2013) for devices with low storage capacity. In this case, both the matrix $\boldsymbol{A}$ and secret vector $\boldsymbol{s}$ are binary which further reduces the size of the public key.

For m = 640, n = 256 and q = 4093, the LWE crypto scheme provides security equivalent to AES-128 (Lindner and Peikert, 2011). With these parameters, the total size of the public key is $640(256+1)log_2(4093) = 246.7$ Kbytes. However, this is far higher than the size of the RSA or ECC public keys which are less than 1 Kbyte. In GB-LWE, the binary matrix $\boldsymbol{A}$ is not stored but is generated on the fly using a **PRNG** with a 256 bit seed (Coron et al., 2012). In this case, we just need to store the seed and the vector $\boldsymbol{b}$, which require $256+640log_2(4093) = 7935$ bits - a considerable reduction over LWE with non-binary $\boldsymbol{A}$.

A lattice-based attack on GB-LWE was launched in (Galbraith, 2013) wherein an m-dimensional lattice was constructed with the columns of $\boldsymbol{A}$ as its basis vectors. The problem of finding $\boldsymbol{u}^T$ was mapped to the Closest Vector Problem with target vector, $\boldsymbol{b}$. It was concluded that for m >400, the attack was infeasible.

(Herold and May, 2017) studied the application of LP and ILP to obtain $\boldsymbol{u}^T$. They obtained results for n = 256 and m ranging from 400 to 640 for 1000 instances. The execution of a particular instance using ILP was aborted if it failed to obtain a solution within 10 seconds. The success probability dropped from 100% at m = 490 to 1% at m = 590. Under certain mild assumptions, they also proved that the solution with LP relaxation for $m \leq 2n$ is unique. For any given instance they computed a score which quantifies the search space for the ILP . 2^{19} instances of GB-LWE were generated for $m = 640$. From this ensemble, 271 weak instances were identified. 16 of these were solved within half an hour each.



(a) 256×512 (b) 256×640

Figure 1: Distribution of LP output values and corresponding number of mis-predictions(averaged over 1000 instances).

3 OUR APPROACH

Our goal is to reduce the size of a given problem instance by making accurate estimates of various bits in $\boldsymbol{u}^T$ and then creating a smaller sub-instance by removing those bits from $\boldsymbol{u}^T$, removing the corresponding rows of $\boldsymbol{A}$ and re-computing the new value of $\boldsymbol{c}_1$. Formally, if the values of the guessed bits are $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ in positions $i_1, i_2, \ldots, i_r$, then those bits are removed from $\boldsymbol{u}^T$, row vectors, $\boldsymbol{a}_{i_1}, \boldsymbol{a}_{i_2}, \ldots, \boldsymbol{a}_{i_r}$ in $\boldsymbol{A}$ are removed and the new value of $\boldsymbol{c}_1$ is computed as

$$\boldsymbol{c}_1' = \boldsymbol{c}_1 - \sum_{i=i_1,i_2,\ldots,i_r} \boldsymbol{x}_i * \boldsymbol{a}_i \qquad (1)$$

We then use either LP or ILP to solve for the remaining bits in $\boldsymbol{u}^T$.

In (Herold and May, 2017) , the VSS problem was formulated as an LP problem with the following constraints

$$\boldsymbol{A}^T\boldsymbol{u} \leq \boldsymbol{c}_1$$
$$-\boldsymbol{A}^T\boldsymbol{u} \leq -\boldsymbol{c}_1$$
$$u_i \leq 1,\ 1 \leq i \leq m$$
$$u_i \geq 0,\ 1 \leq i \leq m$$

Using information-theoretic arguments, (Herold and May, 2017) concluded that the above constraints resulted in a unique integral solution for $m \leq 1500$.

Using the above formulation, we experimented with over 1000 instances each for values of m equal to 512 and 640. Our experiments were conducted on an Intel i5 3.5 GHz quad core system running Ubuntu 16.04. All the programs were written in MATLAB 2015 and its inbuilt functions for LP and ILP solver were used. For m = 512, the LP solver provided the correct integral solution for $\boldsymbol{u}^T$ about 50% of the time. In the remaining 505 instances, the LP solver provided fractional outputs which were rounded to obtain a prediction of each element in $\boldsymbol{u}^T$. Since the correct value of $\boldsymbol{u}^T$ was not obtained, we used the ILP solver to obtain a solution. Of the 505 instances, 395 were solved within 30 seconds. Of the remaining, 67 took less than 10 minutes and 23 took between 10 minutes and 2 hours. The remaining 20 could not be solved by ILP even with maximum time configured to be 2 hours. For $m = 640$, neither LP nor ILP gave a correct solution for even a single instance.

Figure 1(a) shows the average number of values in the LP output lying in each interval of width 0.05 between 0 and 1 for m = 512. About 75% of the values lie in the intervals $[0, 0.05]$ or $[0.95, 1]$. Figure 1(a) also shows the number of bits incorrectly predicted from the LP output. Note that the percentage of incorrectly predicted bits increases greatly as the LP output value tends to 0.5. Figure 1(b) shows the corresponding plot for m = 640. Note that the general trend of higher number of LP output values at the extreme ends of the spectrum and an increasing percentage of mis-predictions towards 0.5 remain similar. However, the average percentage of mis-predictions is much larger for m = 640 in every interval. Overall, the average percentage of incorrectly predicted bits is 5% with $m = 512$ versus 20% with $m = 640$.

Our goal is to remove bits in $\boldsymbol{u}^T$ to create a smaller sub-instance via Equation. 1. Based on error probability alone (Figure 1), it should seem that removing bits most strongly predicted to be either 0 or 1 is the right strategy. However, another crucial factor is the effectiveness of bit removal as a function of the LP output value. For ease of explanation, we re-arrange the bits in $\boldsymbol{u}^T$ in increasing order of the proximity of the LP output value to 0.5 (thus the rightmost bits are closest to 0.5). Under consideration are removal of bits in three different regimes - L, M and H (i.e. Low error, Moderate error and High error regimes).

The elements to be removed are split into clusters - each cluster, typically, resides in one of the regimes

L, M or H. A regime may also contain multiple clusters. A cluster is characterized by a common mis-prediction rate for each of its elements. If there are s elements in a cluster with mis-prediction rate e, then there are likely to be about $e' = \lceil s * e \rceil$ mis-predicted elements in that cluster.

The predicted values of the elements in a cluster may be laid out in an s-bit binary string. The *actual* value of the binary string is likely to be within a Hamming distance of e' from the string. Thus it is expected that there are $\sum_{i=0}^{e'} \binom{s}{i}$ possibilities for the actual value of that string. In a similar manner, we can estimate the number of possibilities for values in the other clusters under consideration. The true value of the bits proposed to be removed would likely be contained in the Cartesian product of the sets of strings derived for each cluster. For each value of the removed bits in the Cartesian product, we create a separate sub-instance and attempt to solve it using the ILP solver subject to a maximum stipulated time. We proceed this way using each combination of values in the Cartesian product until we obtain a solution or until all values in the Cartesian product have been processed.

Note that the above procedure does not guarantee that the actual values of the removed bits is contained in the Cartesian product. This is because the estimated mis-prediction rate is an experimentally determined average over thousands of randomly generated instances and specific instances may have higher mis-prediction rates in the chosen sets together with local variations.

We experimented with creating sub-instances by removing bits in different regimes of $\boldsymbol{u}^T$ and solved them within a stipulated time. We consider an experiment a success if application of ILP on the sub-instance resulted in obtaining the correct values of all bits in the reduced $\boldsymbol{u}^T$ within a stipulated time. Tables 1, 2, 3 show the success probability as a function of number of bits removed and the specific regime(s) involved, with 1000 instances.

Table 1 shows that removing the rightmost 50 bits in H succeeds in creating a solvable instance about 20% of the time but removing 50 bits starting at position 300 from the left of $\boldsymbol{u}^T$ (in Regime M) resulted in average success rate of only 2%. Removing the leftmost 50 bits in $\boldsymbol{u}^T$ (in Regime L) failed to create a solvable sub-instance in every case. To achieve 20% success, we need to remove about 70 bits in M or 220 bits in L. For a given number of bits, it is most effective to remove bits in H, less effective to remove bits in M and least effective to remove bits in L.

Table 2 shows the effect of removing bits in multiple regimes. Removing 25 bits in H resulted in 1% success (Table 1). The same effect can be obtained by removing only 10 bits in H together with 30 bits in M or 5 bits in H together with 20 bits in M and 120 bits in L. Finally, Table 3 shows the effect of applying LP rather than ILP to a sub-instance. In general, ILP is far more likely to solve a sub-instance, although its execution time is considerably higher.

Table 1: Success Probability as a function of number of bits removed in a single regime ($m = 640$, ILP).

L	Succ. Prob. %	M	Succ. Prob. %	H	Succ. Prob. %
170	0.7	30	0.6	20	0.5
180	1.2	40	1.2	25	1.0
190	2.3	50	2	30	1.8
200	4.9	60	4.6	40	5
220	20.2	80	26.0	50	20.3
250	75.9	100	72.8	70	68.1
280	99.7	150	100	120	100

Table 2: Success Probability as a function of number of bits removed in multiple regimes ($m = 640$, ILP).

L	M	H	Succ. Prob. %
140	20	-	1.2
140	40	-	7.6
160	20	-	3.5
160	40	-	27.4
180	-	5	2.7
200	-	5	6.4
220	-	5	38.7
220	-	10	56.8
-	40	5	1.3
-	30	10	1
-	40	10	2.2
120	20	5	1
120	20	10	2.3

Table 3: Success Probability as a function of number of bits removed in a single regime ($m = 640$, LP).

L	Succ. Prob. %	M	Succ. Prob. %	H	Succ. Prob. %
210	1.0	60	1.0	35	1.0
230	5.1	85	5.0	55	5.2
240	11.3	90	12.2	60	12.4
250	24.5	100	26.5	65	24.3
320	100	160	100	150	100

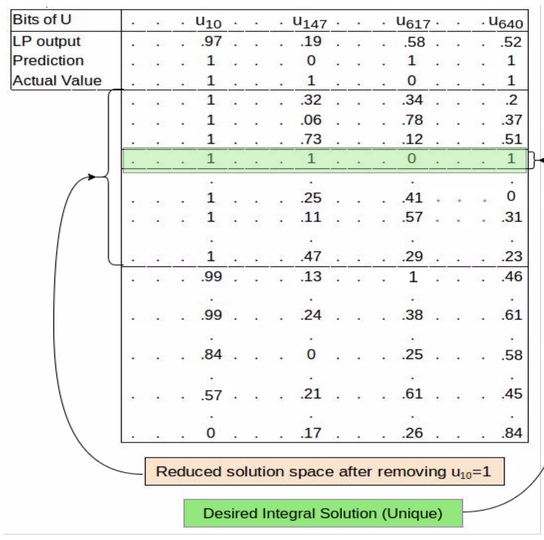


Figure 2: Illustrating solution space reduction by removing a bit of $\boldsymbol{u}^T$.

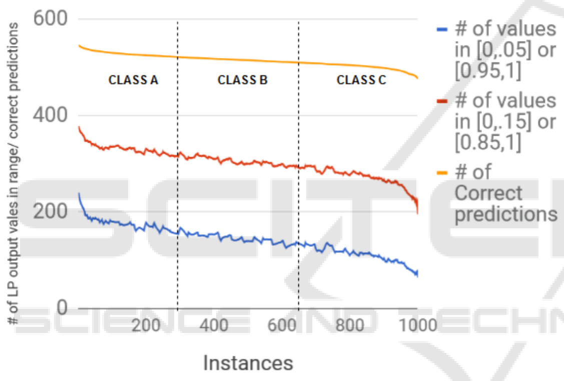


Figure 3: Instance Classification with 3 metrics.

4 RESULTS, ANALYSIS AND DISCUSSION

We first present our results, then suggest metrics to classify instances and finally propose an optimization.

4.1 Results and Interpretation

For $m = 512$, of the 1000 instances considered, the ILP solver was unable to provide a solution to only 43 instances within 10 minutes. In each of these cases, we removed the 5 rightmost bits in Regime H. These bits are likely to be 0 or 1 with nearly equal probability. So, we iterated over all $2^5 = 32$ combinations of those bits, i.e., we removed each combination of those 5 bits and applied ILP to the resulting sub-instance setting the timeout period to 10 seconds. We succeeded in 23 of the 43 cases with average execution time less than 3 minutes. Of the remaining 20 cases, 18 succeeded by removing the rightmost 10 bits while two required the removal of 15 bits.

In the case of $m = 640$, the ILP solver did not succeed in even a single case. As in the case of $m = 512$, we experimented with removal of the rightmost bits in Regime H. For each instance, we first removed 20 bits iterating over all 2^{20} combinations of them. We configured the ILP package to time out after 30 seconds. Five instances out of 1000 yielded the correct value of $\boldsymbol{u}^T$ for a success probability of 0.5%. On average, the solution (if it could be found) would be obtained after 2^{19} iterations. So, the execution time on 150 cores is estimated to be

$$\frac{2^{19}\ \mathit{iterations} \times 30\ \mathit{sec}}{3600\ \frac{\mathit{sec}}{\mathit{hour}} \times 24\ \frac{\mathit{hours}}{\mathit{day}} \times 150\ \mathit{cores}} \sim 1\ \mathit{day}$$

If 30 bits at the rightmost end of Regime H were removed, the success rate would increase to nearly 2% at the expense of a greatly increased execution time of 50 days with 3000 cores.

Table 2 suggests that removing bits in multiple regimes greatly improves the success rate. However, in general, we require removal of larger sets of bits in M and a much larger set of bits in L. Despite the smaller mis-prediction rates, the total number of bits in error is large. This requires iterating over a large number of combinations of values. Since the total number of possible values is multiplicative in the cardinalities of the chosen sets of strings, we end up with prohibitive execution times.

We next present an explanation of our results based on a possible interpretation of the LP output. There are an infinite number of solutions to $\boldsymbol{u}^T\boldsymbol{A} = \boldsymbol{c}_1$ in the interval [0, 1] though the LP solver outputs only a single solution. It follows from the linearity of this equation that the average of any set of solution vectors is also a valid solution. Because of the way the LP solver works, its output may be thought of as a reasonable approximation to the average of all solutions.

One can consider a finite subset of all solutions by rounding to say 10 decimal places. Figure 2 enumerates these solutions. The first three rows show the LP output before and after rounding and the actual values of the elements in $\boldsymbol{u}^T$. The elements in $\boldsymbol{u}^T$ (columns) are arranged in increasing order of their proximity to 0.5 while the solution vectors (rows) are listed in decreasing order of the value of u_{10}. The value of a bit, u_i in $\boldsymbol{u}^T$ obtained from the LP solver can be thought of as the approximate average of the values in that column.

If a sub-instance is created by removing bit u_{10}, the solution space now contains only those vectors

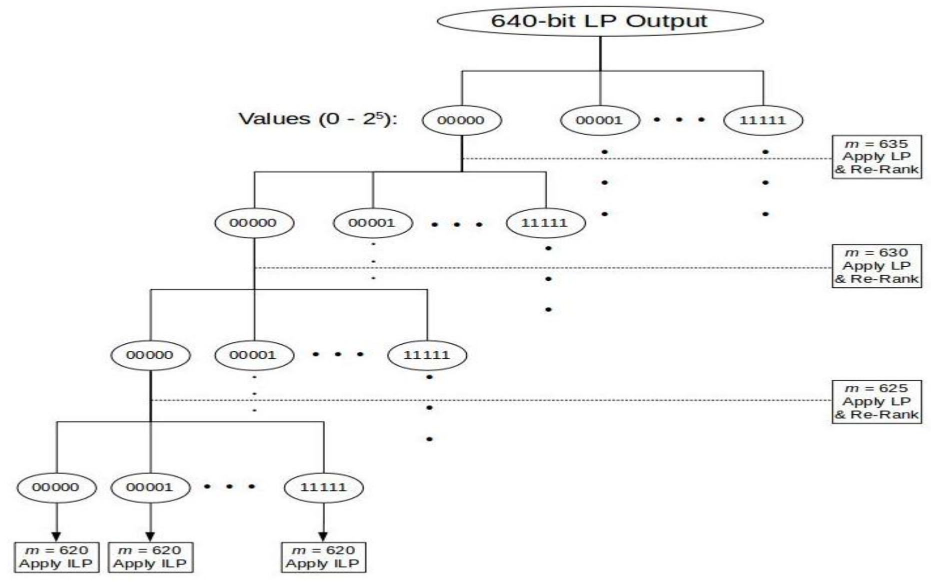

640-bit LP Output
Values (0 - 2⁵):
00000
00001
11111
m = 635 Apply LP & Re-Rank
m = 630 Apply LP & Re-Rank
m = 625 Apply LP & Re-Rank
m = 620 Apply ILP

Figure 4: Removing 20 bits in $\boldsymbol{u}^T$ in blocks of size 5.

(rows) with $u_{10} = 1$. Because the value of u_{10} in the LP output is 0.9, this column will have a preponderance of high values (closer to 1), so removal of this bit will shrink the solution space but much less so compared to that caused by removal of a bit in the rightmost section of the table. As an example of the latter, the LP output of u_{640} is 0.52, so the high values (close to 1) and low values (close to 0) would tend to balance out. In the computation of the average, there would be fewer values with 1, so creating a sub-instance by removing this bit would shrink the space to a much larger extent. In general, removing multiple bits at the rightmost end would considerably reduce the solution space vis-a-vis removing bits from the left of the vector. This explains why the success rate for removal of bits from Regime H is so much higher than that from regime L (Table 1).

In Figure 2 the unique binary solution is highlighted. Finally, the LP value of u_{147} is 0.19, so our prediction is that its value is 0. However, its actual value is 1. If a sub-instance could be created by first correcting and then removing this bit, we would expect the solution space to greatly shrink. Removing such bits, especially in Regime L, appears to be an attractive option but identifying them is non-trivial.

4.2 Instance Classification and Optimization

Our estimates of execution time to obtain the secret, $\boldsymbol{u}^T$, vary greatly across instances. One metric to classify instances is the number of mis-predicted values in $\boldsymbol{u}^T$. However, this requires knowledge of $\boldsymbol{u}^T$. With knowledge of only $\boldsymbol{A}$ and $\boldsymbol{b}$, is it possible to deduce the class affiliation of an instance? (Class A contains the weak/easy instances and Class C contains the hardest).

For a given instance, we count the number of values in the output of the LP solver within specified ranges. Figure 3 shows the count of the values greater than 0.95 or less than 0.05. Also included for comparison are the counts of values greater than 0.85 or less than 0.15. The instances (along the X axis) are arranged in decreasing order of the number of correct predictions. These metrics appear to be consistent and could be used as a basis for instance classification (Figure 3). As supporting evidence for the validity of this classification scheme, we note that the successful instances reported in Tables 1 and 2 are overwhelmingly in Class A.

In our experiments on removing r bits in Regime H, we created sub-instances of size $m-r$ bits by removing all r bits in one go. Another option is to create sub-instances of decreasing sizes, $m-b$, $m-2b$, ..., $m-r$, with b, the block size, as selectable parameter. As before, the LP solver would be initially applied on the given instance but now b rightmost bits (in Regime H) would be removed to create 2^b sub-instances of size $m-b$, one per combination of values of the b bits (Figure 4). The LP solver would be applied to each sub-instance and the LP output sorted anew in increasing order of proximity to 0.5. The rightmost b bits (in Regime H) would be removed to create instances of size $m-2b$ and so on. Only after all r bits were removed would ILP be applied on the resulting sub-instances.

A key feature of this approach is that LP is used repeatedly and the bits to be removed are dynamically determined by subsequent applications of LP. This leads to a greater shrinkage of the solution space. As proof of concept, we considered the removal of 50 rightmost bits in Regime H in two steps - removal of 30 bits first followed by the removal of 20 bits. The success probability increased from 20% for a one-shot removal to 26% in the 2-step case. In the 5-step case (removal of $10+10+10+10+10$ bits), the success probability increased to 32%. In the case of removal of just 20 bits in Regime H, the success probability increased by 20% for a 4-step removal ($5+5+5+5$ bits) over a single step removal.

One possible advantage of multi-step removal is that it may be likely to prune the search tree in Figure 4. For example, it may be possible to rank the different sub-instances based on presumed probability of success heuristically computed from the LP outputs. Root-to-leaf paths deemed to have greater success probability could be explored first resulting in much reduced execution time.

5 CONCLUSION

We addressed the challenge posed by (Galbraith, 2013) to obtain the plaintext in a ciphertext only attack for $m = 640$. We were able to solve the challenge for 5 instances out of 1000 (in 1 day with 150 cores) and for 10 instances (in 2 days with 2400 cores). We applied LP/ILP on reduced instances by removing bits in different regimes - L, M and H. We found that it was most effective to remove bits in H. A sub-instance of size 550 can be solved with 97% success probability by removing just 90 bits in H. We performed an optimization wherein we removed bits in smaller blocks rather than in one go and obtained significant improvement in success rate. While our initial results are based on experiments with 1000 random instances, we generated and tested another 1000 random instances and our conclusions are nearly identical. Finally, we outlined a very simple way to categorize instances into classes A, B and C where instances in A are easiest to solve while instances in C are hardest.

The approach and experiments reported here, while simple, were partially successful in trying to address Galbraith's challenge. Through insightful refinements and optimizations, we feel it may be possible to greatly increase the success rate while decreasing the execution time.

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