# Local Lyapunov Functions for Nonlinear Stochastic Differential Equations by Linearization


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Abstract: We present a rigid estimate of the domain, on which a Lyapunov function for the linearization of a nonlinear stochastic differential equation is a Lyapunov function for the original system. By using this estimate the demanding task of computing a lower bound on the $\gamma$-basin of attraction for a nonlinear stochastic systems is greatly simplified and the application of a resent numerical method for the same purpose facilitated.


## 1 INTRODUCTION

When analysing the stability of an equilibrium of a nonlinear deterministic system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}^d$, one often resorts to linearization around the equilibrium. Assuming, without restriction of generality, that the equilibrium in question is at the origin, then one analyzes the stability of the origin for the system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A := D\mathbf{f}(\mathbf{0})$ is the Jacobian of $\mathbf{f}$ at the origin. Now, if the matrix $A$ is Hurwitz, i.e. the real-parts of the eigenvalues of $A$ are all strictly negative, then one can solve the Lyapunov equation $A^\top P + PA = -Q$, where $Q \in \mathbb{R}^{d\times d}$ is an arbitrary symmetric and positive definite matrix. The solution $P \in \mathbb{R}^{d\times d}$ is then symmetric and positive definite and $V(\mathbf{x}) = \mathbf{x}^\top P\mathbf{x}$ is a Lyapunov function for the system, i.e. $V$ has a minimum at the equilibrium at the origin and the derivative of $V$ along solution trajectories of the linearized system fulfills

$$\nabla V(\mathbf{x}) \bullet A\mathbf{x} = -\mathbf{x}^\top Q\mathbf{x}$$

and is thus negative on $\mathbb{R}^d \setminus \{\mathbf{0}\}$. The function $V$ will also be a Lyapunov function for the original nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ on a neighbourhood $\mathcal{N}$ of the origin where

$$V'(\mathbf{x}) = \nabla V(\mathbf{x}) \bullet \mathbf{f}(\mathbf{x}) < 0 \quad \text{for } \mathbf{x} \in \mathcal{N} \setminus \{\mathbf{0}\}.$$

Here $V'$ denotes the orbital derivative of the system. The size of the set $\mathcal{N}$ is of great importance because compact sublevel sets of $V$ that are within $\mathcal{N}$ are lower bounds on the equilibrium's basin of attraction, i.e. the set of points which converge to the equilibrium as time goes to infinity. Explicit bounds for the size of $\mathcal{N}$ are quite easily derived, cf. e.g. (Hafstein, 2004). In this paper we will derive such an estimate, but for the considerably more demanding case of stochastic differential equations.

**Notation:** We denote by $\|\mathbf{x}\|$ the Euclidian norm of a vector $\mathbf{x} \in \mathbb{R}^d$ and for $A \in \mathbb{R}^{d\times d}$ by $\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$ the matrix norm induced by the Euclidian vector norm. Vectors are assumed to be column vectors.We denote by $\kappa(A) := \|A\|\|A^{-1}\|$ the condition number with respect to the $\|\cdot\|$ norm of the nonsingular matrix $A \in \mathbb{R}^{d\times d}$. For a symmetric and positive definite $Q \in \mathbb{R}^{d\times d}$ we define the energetic norm $\|\mathbf{x}\|_Q := \sqrt{\mathbf{x}^\top Q\mathbf{x}}$ and the corresponding induced matrix norm $\|A\|_Q := \max_{\|\mathbf{x}\|_Q=1} \|A\mathbf{x}\|_Q$. Recall that a symmetric and positive definite $Q \in \mathbb{R}^{d\times d}$ can be factorized as $Q = ODO^\top$ where $O \in \mathbb{R}^{d\times d}$ is orthogonal, i.e. $O^\top O = O^\top O = I$ and $D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{R}^{d\times d}$ is a diagonal matrix with $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_d$. For every $a \in \mathbb{R}$ we define the matrix $Q^a = O\,\mathrm{diag}(\lambda_1^a, \lambda_2^a, \ldots, \lambda_d^a)O^\top$. It is not difficult to see that for $a > 0$ we have $\|Q^a\| = \lambda_d^a$ and $\|Q^{-a}\| = \lambda_1^{-a}$. Further,

$$\begin{aligned}\|Q^{-\frac{1}{2}}\|^{-1}\|\mathbf{x}\| \le \|\mathbf{x}\|_Q &= \sqrt{\mathbf{x}^\top Q\mathbf{x}} \\ &= \|Q^{\frac{1}{2}}\mathbf{x}\| \le \|Q^{\frac{1}{2}}\|\|\mathbf{x}\|.\end{aligned}$$

We consider $d$-dimensional systems and in all sums where the upper and lower bounds of the sum are omitted they are assumed to be 1 and $d$ respectively, i.e. $\sum_i := \sum_{i=1}^d$, $\sum_{i,j} := \sum_{i,j=1}^d$ etc.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}_\infty$




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if it is continuous, monotonically increasing, $\alpha(0) = 0$, and $\lim_{x\to\infty} \alpha(x) = \infty$.

We write $\mathbb{P}$ and $\mathbb{E}$ for probability and expectation respectively. The underlying probability spaces should always be clear from the context. The abbreviation *a.s.* stands for *almost surely*, i.e. with probability one, and $\overset{\text{a.s.}}{=}$ means equal a.s.

## 2 THE PROBLEM SETTING

We give a short discussion of the setup and the problem at hand. For a more detailed discussion of the setup see (Gudmundsson and Hafstein, 2018, §2). The general $d$-dimensional stochastic differential equation (SDE) of Itô type we consider is of the form:

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}(\mathbf{X}(t))\mathrm{d}t + \mathbf{g}(\mathbf{X}(t)) \cdot \mathrm{d}\mathbf{W}(t) \qquad (1)$$

or equivalently

$$\mathrm{d}\mathbf{X}_i(t) = \mathbf{f}_i(\mathbf{X}(t))\mathrm{d}t + \sum_{u=1}^{U} \mathbf{g}^u(\mathbf{X}(t)) \cdot \mathrm{d}W_u(t)$$

for $i = 1,2,\dots,d$. Thus $\mathbf{f} = (f_1, f_2, \dots, f_d)^\top$, $\mathbf{g} = (\mathbf{g}^1, \mathbf{g}^2, \dots, \mathbf{g}^U)$, and $\mathbf{g}^u = (g_1^u, g_2^u, \dots, g_d^u)^\top$, where $f_i, g_i^u : \mathbb{R}^d \to \mathbb{R}$. We assume that the origin is an equilibrium of the system, i.e. $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{g}^\mathbf{u}(\mathbf{0}) = \mathbf{0}$ for $u = 1,2,\dots,U$ and we consider *strong solutions* to (1). For deterministic initial value solutions, i.e. $\mathbf{X}(0) = \mathbf{x} \in \mathbb{R}^d$ a.s., we write $\mathbf{X}^\mathbf{x}$ for the solution, i.e.

$$\mathbf{X}^\mathbf{x}(t) = \mathbf{x} + \int_0^t \mathbf{f}(\mathbf{X}(s))\mathrm{d}s + \int_0^t \mathbf{g}(\mathbf{X}(s))\mathrm{d}\mathbf{W}(s),$$

where the second integral is interpreted in the Itô sense. As shown in (Mao, 2008) it suffices to consider deterministic initial value solutions when studying the stability of an equilibrium.

Numerous concepts are in use concerning the stability of equilibria of SDEs. Here we will be concerned with the so-called *asymptotic stability in probability* of the zero solution (Khasminskii, 2012, (5.15)), also referred to as *stochastic asymptotic stability* (Mao, 2008, Definition 4.2.1). For a more detailed discussion of the stability of SDEs see the books by Khasminskii (Khasminskii, 2012) or Mao (Mao, 2008). We recall a few definitions:

**Definition 2.1** (Stability in Probability (*SiP*)). *The null solution* $\mathbf{X}(t) \overset{a.s.}{=} \mathbf{0}$ *to the SDE* (1) *is said to be* stable in probability *(SiP) if for every* $r > 0$ *and* $0 < \varepsilon < 1$ *there exists a* $\delta > 0$ *such that*:

$$\|\mathbf{x}\| \leq \delta \textit{ implies } \mathbb{P}\left\{\sup_{t\geq 0} \|\mathbf{X}^\mathbf{x}(t)\| \leq r\right\} \geq 1 - \varepsilon. \quad \square$$

**Definition 2.2** (Asymptotic Stability in Probability (*ASiP*)). *The null solution* $\mathbf{X}(t) \overset{a.s.}{=} \mathbf{0}$ *to the SDE* (1) *is said to be* asymptotically stable in probability *(ASiP) if it is SiP and in addition for every* $0 < \varepsilon < 1$ *there exists a* $\delta > 0$ *such that*:

$$\|\mathbf{x}\| \leq \delta \textit{ implies } \mathbb{P}\left\{\lim_{t\to\infty} \|\mathbf{X}^\mathbf{x}(t)\| = 0\right\} \geq 1 - \varepsilon.$$

$\square$

Our definitions of SiP and ASiP are equivalent to the more common

$$\lim_{\|\mathbf{x}\|\to 0} \mathbb{P}\left\{\sup_{t>0} \|\mathbf{X}^\mathbf{x}(t)\| \leq r\right\} = 1 \quad \text{for all } r > 0$$

for SiP and additionally

$$\lim_{\|\mathbf{x}\|\to 0} \mathbb{P}\left\{\limsup_{t\to\infty} \|\mathbf{X}^\mathbf{x}(t)\| = 0\right\} = 1$$

for ASiP, which can be seen by fixing $r > 0$ and writing down the definition of a limit: *for every* $\varepsilon > 0$ *there exists a* $\delta > 0$.

The reason for our formulation is that we want to look at a more practical concept related to such stability, namely a stochastic analog of the basin of attraction (BOA) in the stability theory for deterministic systems, cf. (Gudmundsson and Hafstein, 2018). Instead of the limit $\|\mathbf{x}\| \to 0$ we consider: Given some *confidence* $0 < \gamma \leq 1$ how far from the origin can sample paths start and still approach the equilibrium as $t \to \infty$ with probability greater than or equal to $\gamma$. This is the motivation for the next definition.

**Definition 2.3** ($\gamma$-Basin Of Attraction ($\gamma$-BOA)). *Consider the system* (1) *and let* $0 < \gamma \leq 1$. *We refer to the set*

$$\left\{\mathbf{x} \in \mathbb{R}^d : \mathbb{P}\left\{\lim_{t\to\infty} \|\mathbf{X}^\mathbf{x}(t)\| = 0\right\} \geq \gamma\right\} \quad (\gamma\text{-}BOA)$$

*as the* $\gamma$-basin of attraction*, or short* $\gamma$*-BOA, of the equilibrium at the origin.*

$\square$

Note that a 1-BOA corresponds to the usual BOA for deterministic systems.

For the SDE (1) the associated *generator* is given by

$$LV(\mathbf{x}) := \qquad (2)$$
$$\nabla V(\mathbf{x}) \bullet \mathbf{f}(\mathbf{x}) + \frac{1}{2}\sum_{i,j} \left[\mathbf{g}(\mathbf{x})\mathbf{g}(\mathbf{x})^\top\right]_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}(\mathbf{x})$$

for some appropriately differentiable $V : \mathcal{U} \to \mathbb{R}$ with $\mathcal{U} \subset \mathbb{R}^d$. Notice that this is just the drift term in the expression for the stochastic differential of the process $t \mapsto V(\mathbf{X}(t))$. The generator for a stochastic system corresponds to the orbital derivative of a deterministic system.

**Definition 2.4** (Local Lyapunov function)**.** *Consider the system* (1)*. A function $V \in C(\mathcal{U}) \cap C^2(\mathcal{U} \setminus \{\mathbf{0}\})$, where $\mathbf{0} \in \mathcal{U} \subset \mathbb{R}^d$ is a domain, is called a (local) Lyapunov function for the the system* (1) *if there are functions $\mu_1, \mu_2, \mu_3 \in \mathcal{K}_\infty$, such that V fulfills the properties:*

*(i)* $\mu_1(\|\mathbf{x}\|) \le V(\mathbf{x}) \le \mu_2(\|\mathbf{x}\|)$ *for all* $\mathbf{x} \in \mathcal{U}$

*(ii)* $LV(\mathbf{x}) \le -\mu_3(\|\mathbf{x}\|)$ *for all* $\mathbf{x} \in \mathcal{U} \setminus \{\mathbf{0}\}$

**Remark 2.5.** *It is of vital importance that V is not necessarily differentiable at the equilibrium, because otherwise a large number of systems with an ASiP null solution do not possess a Lyapunov function, cf. (Khasminskii, 2012, Remark 5.5).*

The following theorem provides the first centerpiece of Lyapunov stability theory for our application, cf. (Khasminskii, 2012, Theorem 5.5 and Corollary 5.1):

**Theorem 2.6.** *If there exists a local Lyapunov functions as in Definition 2.4 for the system* (1)*, then the null solution is ASiP. Further, let $V_{\max} > 0$ and assume that $V^{-1}([0, V_{\max}])$ is a compact subset of $\mathcal{U}$. Then, for every $0 < \beta < 1$ the set $V^{-1}([0, \beta V_{\max}])$ is a subset of the $(1-\beta)$-BOA of the origin.*

This concludes our discussion of the setup. In the next section we discuss Lyapunov functions for the linearization of (1) and prove the main contribution of this paper, a lower bound on the area where a Lyapunov function for the linearization is also a Lyapunov function for the nonlinear system.

## 3 MAIN RESULTS

We now consider the linearization of system (1). A Lyapunov function for the linearized system can then be constructed, e.g. with the method form (Hafstein et al., 2018), much more easily than for the nonlinear system (1). In addition to **f** and **g** satisfying the usual sufficient SDE solution-theory conditions *locally Lipschitz and the linear-growth conditions*, cf. e.g. (Mao, 2008, §2.3) or (Kallenberg, 2002, §21), we assume **f** and **g** are $C^2$ on a convex neighbourhood $\mathcal{U} \subset \mathbb{R}^d$ of the origin. The second order Taylor expansion for the components $f_i$ of **f** at $\mathbf{x} \in \mathcal{U}$ reads

$$\begin{aligned} f_i(\mathbf{x}) &= \sum_j x_j F_{ij} + \frac{1}{2} \sum_{j,k} x_j x_k R^i_{jk}(\mathbf{x}) \\ &= \big(F\mathbf{x}\big)_i + \frac{1}{2} \mathbf{x}^\top R^i(\mathbf{x})\, \mathbf{x}, \end{aligned}$$

and the components $g_i^u$ of $\mathbf{g}^u$,

$$\begin{aligned} g_i^u(\mathbf{x}) &= \sum_j x_j G^u_{ij} + \frac{1}{2} \sum_{j,k} x_j x_k R^{ui}_{jk}(\mathbf{x}) \\ &= \big(G^u\mathbf{x}\big)_i + \frac{1}{2} \mathbf{x}^\top R^{ui}(\mathbf{x})\, \mathbf{x} \end{aligned}$$

Here

$$F = \big(F_{ij}\big)_{i,j} \in \mathbb{R}^{d\times d} \text{ with } F_{ij} = \partial_j f_i(\mathbf{0})$$

and

$$G^u = \big(G^u_{ij}\big)_{i,j} \in \mathbb{R}^{d\times d} \text{ with } G^u_{ij} = \partial_j g^u_i(\mathbf{0})$$

and the matrices $R^i(\mathbf{x})$ and $R^{ui}(\mathbf{x})$ are the Taylor remainders

$$\begin{aligned} R^i(\mathbf{x}) &= \Big(R^i_{jk}(\mathbf{x})\Big)_{j,k} \in \mathbb{R}^{d\times d} \quad \text{and} \\ R^{ui}(\mathbf{x}) &= \Big(R^{ui}_{jk}(\mathbf{x})\Big)_{j,k} \in \mathbb{R}^{d\times d}. \end{aligned}$$

By abuse of notation we define the elements of upper bound matrices $R^i = \Big(R^i_{jk}\Big)_{j,k} \in \mathbb{R}^{d\times d}$ and $R^{ui} = \Big(R^{ui}_{jk}\Big)_{j,k} \in \mathbb{R}^{d\times d}$ as follows:

$$\big|\partial^2_{jk} f_i(\mathbf{x})\big| = |R^i_{jk}(\mathbf{x})| \le R^i_{jk} \text{ and} \tag{3}$$

$$\big|\partial^2_{jk} g^u_i(\mathbf{x})\big| = |R^{ui}_{jk}(\mathbf{x})| \le R^{ui}_{jk}, \tag{4}$$

for all $\mathbf{x} \in \mathcal{N}$, where $\mathcal{N}$ is a neighbourhood of the origin to be defined later. Finally we fix the constants $\mathcal{R}^i$ and $\mathcal{R}^{ui}$ as

$$\mathcal{R}^i := \|R^i\| \text{ and } \mathcal{R}^{ui} := \|R^{ui}\|. \tag{5}$$

The action of the generator (2) of the system (1) on some $V \in C(\mathcal{U}) \cap C^2(\mathcal{U} \setminus \{\mathbf{0}\})$ can be written as

$$\begin{aligned} LV(\mathbf{x}) &= \frac{1}{2} \sum_{i,j} m_{ij}(\mathbf{x}) \partial^2_{ij} V(\mathbf{x}) + \sum_i f_i(\mathbf{x}) \partial_i V(\mathbf{x}) \\ &= L_0 V(\mathbf{x}) + E(\mathbf{x}) \end{aligned}$$

where $L_0 V(\mathbf{x})$ is the generator of the linearized system defined below and $E(\mathbf{x})$ the rest (containing all the Taylor remainders). We will now work out the details, first notice that:

$$\begin{aligned} m_{ij}(\mathbf{x}) &= \sum_{u=1}^U g^u_i(\mathbf{x}) g^u_j(\mathbf{x}) \\ &= \sum_{k,l} x_k x_l \sum_{u=1}^U G^u_{ik} G^u_{jl} \\ &\quad + \frac{1}{2} \sum_{k,l,m} x_k x_l x_m \sum_{u=1}^U \Big( G^u_{ik} R^{uj}_{lm}(\mathbf{x}) + G^u_{jk} R^{ui}_{lm}(\mathbf{x}) \Big) \\ &\quad + \frac{1}{4} \sum_{k,l,m,n} x_k x_l x_m x_n \sum_{u=1}^U R^{ui}_{kl}(\mathbf{x}) R^{uj}_{mn}(\mathbf{x}). \end{aligned}$$

We define $L_0$ as the generator associated to the *linearization* of the system (1), i.e. the system

$$\mathrm{d}\mathbf{X}(t) = F\,X(t)\,\mathrm{d}t + \sum_{u=1}^{U} G^u\,\mathbf{X}(t)\,\mathrm{d}W_u(t) \qquad (6)$$

or equivalently

$$\mathrm{d}X_i(t) = \sum_j F_{ij}X_j(t)\,\mathrm{d}t + \sum_{u=1}^{U}\sum_j G^u_{ij}X_j(t)\,\mathrm{d}W_u(t)$$

for $i = 1,2,\dots,d$, which means that

$$L_0V(\mathbf{x}) = \qquad (7)$$

$$\sum_{i,j} F_{ij}x_j\partial_i V(\mathbf{x}) + \frac{1}{2}\sum_{i,j}\left(\sum_{k,l} x_kx_l\sum_{u=1}^{U} G^u_{ik}G^u_{jl}\right)\partial^2_{ij}V(\mathbf{x}).$$

We gather together the nonlinear parts of the full SDE generator into the expression for $E(\mathbf{x})$:

$$E(\mathbf{x}) = \underbrace{\sum_s E_s(\mathbf{x})\partial_s V(\mathbf{x})}_{E_F(\mathbf{x})} + \underbrace{\frac{1}{2}\sum_{r,s} E_{rs}(\mathbf{x})\partial^2_{rs}V(\mathbf{x})}_{E_G(\mathbf{x})},$$

where

$$E_s(\mathbf{x}) = \frac{1}{2}\sum_{j,k} x_jx_kR^s_{jk}(\mathbf{x}) \qquad \text{and}$$

$$E_{rs}(\mathbf{x}) = \frac{1}{2}\sum_{k,l,m} x_kx_lx_m\sum_{u=1}^{U}(G^u_{rk}R^{us}_{lm}(\mathbf{x}) + G^u_{sk}R^{ur}_{lm}(\mathbf{x}))$$
$$+\frac{1}{4}\sum_{k,l,m,n} x_kx_lx_mx_n\sum_{u=1}^{U} R^{ur}_{kl}(\mathbf{x})R^{us}_{mn}(\mathbf{x}).$$

The plan for the rest of this section is as follows: With $LV(\mathbf{x})$ broken up into a linear part $L_0V(\mathbf{x})$ and a nonlinear correction $E(\mathbf{x})$, we take the explicit function

$$V(\mathbf{x}) = \|\mathbf{x}\|_Q^p = \left(\mathbf{x}^\top Q\mathbf{x}\right)^{\frac{p}{2}} \qquad (8)$$

as the *ansatz* for the Lyapunov function candidate, where $Q \in \mathbb{R}^{d\times d}$ is a symmetric and positive definite matrix and $p > 0$. As argued in (Hafstein et al., 2018, §4) this is the expected form of a Lyapunov function for the linearized system (6) just as $\mathbf{x} \mapsto \mathbf{x}^\top P\mathbf{x}$ for a symmetric and positive definite $P$ is the usual form for a Lyapunov function for a linear deterministic system $\dot{\mathbf{x}} = A\mathbf{x}$. Note that typically $p < 2$ so $V$ is not differentiable at the origin. For this reason take $\mathbf{x} \neq \mathbf{0}$ in the calculations below. Assuming that we have fixed $Q$ and $p > 0$ such that $L_0V(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, we derive a neighbourhood of the origin such that $|L_0V(\mathbf{x})| > |E(\mathbf{x})|$, which implies $LV(\mathbf{x}) < 0$.

From (Hafstein et al., 2018, Lemma 4.1) we can state the following: for $V(\mathbf{x}) = \|\mathbf{x}\|_Q^p$ we have

$$L_0V(\mathbf{x}) = -\frac{1}{2}p\|\mathbf{x}\|_Q^{p-4}H(\mathbf{x}) \ \text{ for all } \mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\},$$

where

$$H(\mathbf{x}) = -\mathbf{x}^\top\left(F^\top Q + QF + \sum_{u=1}^{U}(G^u)^\top QG^u\right)\mathbf{x}\|\mathbf{x}\|_Q^2$$
$$+(2-p)\sum_{u=1}^{U}\left(\frac{1}{2}\mathbf{x}^\top(QG^u + (G^u)^\top Q)\mathbf{x}\right)^2.$$

This $V$ is a Lyapunov function for the linear system (6) if there is a constant $C > 0$ such that

$$H(\mathbf{x}) \geq C\|\mathbf{x}\|_Q^2\|\mathbf{x}\|^2 \ \text{ for all } \mathbf{x} \in \mathbb{R}^d,$$

because then

$$L_0V(\mathbf{x}) \leq -\frac{1}{2}pC\|\mathbf{x}\|_Q^{p-2}\|\mathbf{x}\|^2 \qquad (9)$$

for all $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$.

Before we state and prove our results we prove a simple but useful lemma:

**Lemma 3.1.** *Let $A = (A_{ij}), \widetilde{A} = (\widetilde{A}_{ij}) \in \mathbb{R}^{d\times d}$ be such that $|A_{ij}| \leq \widetilde{A}_{ij}$ for $i,j = 1,2,\dots,d$. Then*

$$\|A\| \leq \|\widetilde{A}\|. \qquad (10)$$

*In particular*

$$\left|\sum_{i,j} x_iA_{ij}y_j\right| \leq \|\widetilde{A}\|\|\mathbf{x}\|\|\mathbf{y}\| \qquad (11)$$

*and*

$$\left|\sum_{i,j,k} x_iQ_{ik}A_{kj}y_j\right| \leq \|\widetilde{A}\|\|Q^{\frac{1}{2}}\|\|\mathbf{x}\|_Q\|\mathbf{y}\| \qquad (12)$$
$$\leq \|\widetilde{A}\|\kappa(Q)^{\frac{1}{2}}\|\mathbf{x}\|_Q\|\mathbf{y}\|_Q.$$

*for every symmetric and positive definite $Q \in \mathbb{R}^{d\times d}$. If $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}}A$ we even have*

$$\left|\sum_{i,j,k} x_iQ_{ik}A_{kj}y_j\right| \leq \|\widetilde{A}\|\|\mathbf{x}\|_Q\|\mathbf{y}\|_Q. \qquad (13)$$

*Proof.* For $\mathbf{x} = (x_1,x_2,\dots,x_d)^\top$ set $\tilde{\mathbf{x}} = (|x_1|,|x_2|,\dots,|x_d|)^\top$. Clearly $\|\mathbf{x}\| = \|\tilde{\mathbf{x}}\|$. The estimate (10) follows from

$$\|A\mathbf{x}\|^2 = \mathbf{x}^\top A^\top A\mathbf{x} = \left|\sum_{i,j,k} x_iA_{ki}A_{kj}x_j\right|$$
$$\leq \sum_{i,j,k}|x_i|\cdot|A_{ki}|\cdot|A_{kj}|\cdot|x_j|$$
$$\leq \sum_{i,j,k}|x_i|\cdot\widetilde{A}_{ki}\widetilde{A}_{kj}|x_j| = \tilde{\mathbf{x}}^\top\widetilde{A}^\top\widetilde{A}\tilde{\mathbf{x}} = \|\widetilde{A}\tilde{\mathbf{x}}\|^2$$
$$\leq \|\widetilde{A}\|^2\|\tilde{\mathbf{x}}\|^2 = \|\widetilde{A}\|^2\|\mathbf{x}\|^2$$

and thus

$$\|A\| := \sup_{\mathbf{x}\neq\mathbf{0}}\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} \leq \|\widetilde{A}\|.$$

The inequality (12) follow from

$$
\begin{aligned}
|\sum_{i,j,k} x_i Q_{ik} A_{kj} y_j| &= \left|\sum_{i,j} x_i \left(\sum_k Q_{ik} A_{kj}\right) y_j\right| \\
&= |\mathbf{x}^\top Q A \mathbf{y}| = |(Q^{\frac{1}{2}}\mathbf{x})^\top Q^{\frac{1}{2}} A \mathbf{y}| \\
&\leq \|Q^{\frac{1}{2}}\mathbf{x}\| \|Q^{\frac{1}{2}} A \mathbf{y}\| = \|\mathbf{x}\|_Q \|Q^{\frac{1}{2}} A \mathbf{y}\| \\
&\leq \|\mathbf{x}\|_Q \|Q^{\frac{1}{2}}\| \|A\| \|\mathbf{y}\| \\
&\leq \|\widetilde{A}\| \|Q^{\frac{1}{2}}\| \|Q^{-\frac{1}{2}}\| \|\mathbf{x}\|_Q \|\mathbf{y}\|_Q
\end{aligned}
$$

and (11) follows form (12) with $Q$ as the identity matrix. To see (13) just note that if $AQ^{\frac{1}{2}} = Q^{\frac{1}{2}} A$ we have

$$\|Q^{\frac{1}{2}} A \mathbf{y}\| = \|A Q^{\frac{1}{2}} \mathbf{y}\| \leq \|A\| \|Q^{\frac{1}{2}} \mathbf{y}\| \leq \|\widetilde{A}\| \|\mathbf{y}\|_Q$$

which can be used to improve the estimate above. □

**Remark 3.2.** *If $A$ in* (12) *is symmetric we have*

$$\mathbf{x}^\top Q A \mathbf{y} = \sum_{i,j,k} x_i Q_{ik} A_{kj} y_j = \sum_{i,j,k} y_j A_{jk} Q_{ki} x_i = \mathbf{y}^\top A Q \mathbf{x}.$$

**Remark 3.3.** *For vectors $\mathbf{x}, \widetilde{\mathbf{x}} \in \mathbb{R}^d$, $|x_i| \leq \widetilde{x}_i$ for $i = 1, 2, \ldots, d$, we obviously have $\|\mathbf{x}\| \leq \|\widetilde{\mathbf{x}}\|$, but in general $\|\mathbf{x}\|_Q$ is not necessarily smaller than $\|\widetilde{\mathbf{x}}\|_Q$. Take for example $\mathbf{x} = (1,-1)^\top$, $\widetilde{\mathbf{x}} = (1,1)^\top$, and $Q = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Then $\|\mathbf{x}\|_Q = \sqrt{\mathbf{x}^\top Q \mathbf{x}} = \sqrt{6}$ but $\|\mathbf{y}\|_Q = \sqrt{2}$. For this reason one cannot expect $|A_{ij}| \leq \widetilde{A}_{ij}$ to imply $\|A\|_Q \leq \|\widetilde{A}\|_Q$ for matrices $A, \widetilde{A} \in \mathbb{R}^{d\times d}$.*

We now come to the main contribution of this paper:

**Theorem 3.4.** *Consider the system* (1)*, assume that $V$ as in* (8) *is a Lyapunov function for its linearization* (6)*, and let $C > 0$ be a constant as in* (9)*. Let $\rho^* > 0$ and assume the estimates* (3)*,* (4)*, and* (5) *hold true on $\mathcal{N} = \mathcal{D}^* := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_Q \leq \rho^*\}$. Define*

$$
\begin{aligned}
p^* &:= 1 + |p-2|, \\
\mathcal{R}^i &:= \|R^i\|, \\
\mathcal{R}^{ui} &:= \|R^{ui}\|, \\
\mathcal{R}_F &:= \|(\mathcal{R}^1, \mathcal{R}^2, \ldots, \mathcal{R}^d)\|, \\
\mathcal{R}_G^u &:= \|(\mathcal{R}^{u1}, \mathcal{R}^{u2}, \ldots, \mathcal{R}^{ud})\|, \\
\mathcal{R}_G &:= \|(\mathcal{R}_G^1, \mathcal{R}_G^2, \ldots, \mathcal{R}_G^U)\|^2, \\
\widetilde{E}_G &:= \|Q^{\frac{1}{2}}\| \left(\mathcal{R}_F + p^* \sum_{u=1}^{U} \mathcal{R}_G^u \|Q^{\frac{1}{2}} G^u Q^{-\frac{1}{2}}\|\right), \\
\widetilde{E}_G^* &:= \frac{1}{4} p^* \kappa(Q) \mathcal{R}_G.
\end{aligned}
$$

*Then*

$$LV(\mathbf{x}) = L_0 V(\mathbf{x}) + E(\mathbf{x})$$

*where $L_0 V$ is defined in* (7) *and*

$$|E(\mathbf{x})| \leq \frac{1}{2} p \|\mathbf{x}\|_Q^{p-2} \|\mathbf{x}\|^2 \cdot \|\mathbf{x}\|_Q \left(\widetilde{E}_G + \widetilde{E}_G^* \|\mathbf{x}\|_Q\right)$$

*for $\mathbf{x} \in \mathcal{D}^* := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_Q \leq \rho^*\}$. In particular, $V$ is a Lyapunov function for the nonlinear system* (1) *satisfying the condition of Definition 2.4 on*

$$\mathcal{U} = \mathcal{D} := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_Q \leq \rho\},$$

*with*

$$\rho < \min\left\{\rho^*, \frac{1}{2\widetilde{E}_G^*}\left(\sqrt{(\widetilde{E}_G)^2 + 4C\widetilde{E}_G^*} - \widetilde{E}_G\right)\right\}.$$

*Proof.* Let us first compute $\partial_s V(\mathbf{x})$ and $\partial_{rs}^2 V(\mathbf{x})$,

$$
\begin{aligned}
\partial_s V(\mathbf{x}) &= \left(\sum_j Q_{sj} x_j + \sum_i Q_{is} x_i\right) \frac{p}{2} \left(\sum_{i,j} Q_{ij} x_i x_j\right)^{\frac{p}{2}-1} \\
&= p \sum_i x_i Q_{is} \|\mathbf{x}\|_Q^{p-2} \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
\partial_{rs}^2 V(\mathbf{x}) &= p Q_{rs} \|\mathbf{x}\|_Q^{p-2} + p \left(\sum_j x_j Q_{js}\right) \left(\frac{p}{2} - 1\right) \\
&\quad \times 2 \left(\sum_i x_i Q_{ir}\right) \left(\sum_{i,j} Q_{ij} x_i x_j\right)^{\frac{p}{2}-2} \\
&= p \|\mathbf{x}\|_Q^{p-2} Q_{rs} + p(p-2) \sum_{i,j} x_i x_j Q_{ir} Q_{js} \|\mathbf{x}\|_Q^{p-4} \\
&= p \|\mathbf{x}\|_Q^{p-4} \left(Q_{rs} \|\mathbf{x}\|_Q^2 + (p-2) \sum_{i,j} x_i x_j Q_{ir} Q_{js}\right).
\end{aligned}
$$

Now set $\mathbf{z} = (z_1, z_2, \ldots, z_d)^\top$ with $z_s := \mathbf{x}^\top R^s(\mathbf{x}) \mathbf{x}$ and then $|z_s| \leq \mathcal{R}^s \|\mathbf{x}\|^2$ and $\|\mathbf{z}\| \leq \|\mathbf{x}\|^2 \mathcal{R}_F$ for $\mathbf{x} \in \mathcal{D}^*$. Then

$$
\begin{aligned}
|E_F(\mathbf{x})| &\leq \left|\sum_s E_s(\mathbf{x}) \partial_s V(\mathbf{x})\right| \\
&\leq \frac{p}{2} \|\mathbf{x}\|_Q^{p-2} \left|\sum_{s,i,j,k} x_j x_k R_{jk}^s(\mathbf{x}) x_i Q_{is}\right| \\
&= \frac{p}{2} \|\mathbf{x}\|_Q^{p-2} \left|\sum_{s,i} x_i Q_{is} \left(\sum_{j,k} x_j R_{jk}^s(\mathbf{x}) x_k\right)\right| \\
&= \frac{p}{2} \|\mathbf{x}\|_Q^{p-2} \left|\sum_{s,i} x_i Q_{is} \left(\mathbf{x}^\top R^s(\mathbf{x}) \mathbf{x}\right)\right| \\
&= \frac{p}{2} \|\mathbf{x}\|_Q^{p-2} \left|\sum_{s,i} x_i Q_{is} z_s\right| \\
&= \frac{p}{2} \|\mathbf{x}\|_Q^{p-2} \left|\mathbf{x}^\top Q \mathbf{z}\right| \\
&= \frac{p}{2} \|\mathbf{x}\|_Q^{p-2} \|Q^{\frac{1}{2}} \mathbf{x}\| \|Q^{\frac{1}{2}} \mathbf{z}\| \\
&\leq \frac{p}{2} \|\mathbf{x}\|_Q^{p-2} \|\mathbf{x}\|_Q \|Q^{\frac{1}{2}}\| \|\mathbf{z}\| \\
&\leq \frac{p}{2} \|\mathbf{x}\|_Q^{p-1} \|\mathbf{x}\|^2 \|Q^{\frac{1}{2}}\| \mathcal{R}_F.
\end{aligned}
$$

Since $E_G(\mathbf{x}) = \frac{1}{2}\sum_{r,s} E_{rs}(\mathbf{x})\partial^2_{rs}V(\mathbf{x})$ and by using our expressions for $E_{rs}$ and $\partial^2_{rs}V(\mathbf{x})$ we obtain:

$$\begin{aligned}|E_G(\mathbf{x})| \le \frac{1}{4}p\|\mathbf{x}\|_Q^{p-4}\Bigg| &\sum_{r,s,k,l,m} x_k x_l x_m \\ &\times \sum_{u=1}^{U}\left(G^u_{rk}R^{us}_{lm}(\mathbf{x}) + G^u_{sk}R^{ur}_{lm}(\mathbf{x})\right) \\ &\times \left(Q_{rs}\|\mathbf{x}\|_Q^2 + (p-2)\sum_{i,j}x_i x_j Q_{ir}Q_{js}\right)\Bigg| \\ + \frac{1}{8}p\|\mathbf{x}\|_Q^{p-4}\Bigg| &\sum_{r,s,k,l,m,n} x_k x_l x_m x_n \\ &\times \left(Q_{rs}\|\mathbf{x}\|_Q^2 + (p-2)\sum_{i,j}x_i x_j Q_{ir}Q_{js}\right) \\ &\times \sum_{u=1}^{U} R^{ur}_{kl}(\mathbf{x})R^{us}_{mn}(\mathbf{x})\Bigg|.\end{aligned}$$

We now estimate the expression on the right-hand side term by term: Set $\mathbf{z}^u = (z^u_1, z^u_2, \ldots, z^u_d)^\top$, where $z^u_i := \mathbf{x}^\top R^{ui}(\mathbf{x})\mathbf{x}$, and then $|z^u_i| \le \mathcal{R}^{ui}\|\mathbf{x}\|^2$ and $\|\mathbf{z}^u\| \le \|\mathbf{x}\|^2\mathcal{R}^u_G$ for $\mathbf{x} \in \mathcal{D}^*$. Then

$$\begin{aligned}&\sum_{r,s,k,l,m} x_k x_l x_m \sum_{u=1}^{U} G^u_{rk}R^{us}_{lm}(\mathbf{x})Q_{rs}\|\mathbf{x}\|_Q^2 \\ &= \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U}\sum_{r,s,k} x_k \left(\sum_{l,m} x_l R^{us}_{lm} x_m\right) Q_{sr}G^u_{rk} \\ &= \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U}\sum_{r,s,k} x_k \left(\mathbf{x}^\top R^{us}\mathbf{x}\right) Q_{sr}G^u_{rk} \\ &= \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U}\sum_{r,s,k} z^u_s Q_{sr}G^u_{rk}x_k \\ &= \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U} (\mathbf{z}^u)^\top Q G^u \mathbf{x} \\ &= \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U} (\mathbf{z}^u)^\top Q G^u Q^{-\frac{1}{2}}Q^{\frac{1}{2}} \mathbf{x} \\ &\le \|\mathbf{x}\|_Q^3\|\mathbf{x}\|^2 \sum_{u=1}^{U} \|QG^uQ^{-\frac{1}{2}}\|\mathcal{R}^u_G \\ &\le \|\mathbf{x}\|_Q^3\|\mathbf{x}\|^2\|Q^{\frac{1}{2}}\| \sum_{u=1}^{U} \|Q^{\frac{1}{2}}G^uQ^{-\frac{1}{2}}\|\mathcal{R}^u_G\end{aligned}$$

and similarly

$$\begin{aligned}&\sum_{r,s,k,l,m} x_k x_l x_m \sum_{u=1}^{U} G^u_{sk}R^{ur}_{lm}(\mathbf{x})Q_{rs}\|\mathbf{x}\|_Q^2 \\ &= \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U}\sum_{r,s,k} \left(\mathbf{x}^\top R^{ur}\mathbf{x}\right) Q_{rs}G^u_{sk}x_k \\ &= \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U}\sum_{r,s,k} z^u_r Q_{rs}G^u_{sk}x_k \\ &= \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U} (\mathbf{z}^u)^\top Q G^u \mathbf{x} \\ &\le \|\mathbf{x}\|_Q^3\|\mathbf{x}\|^2 \sum_{u=1}^{U} \|QG^uQ^{-\frac{1}{2}}\|\mathcal{R}^u_G \\ &\le \|\mathbf{x}\|_Q^3\|\mathbf{x}\|^2\|Q^{\frac{1}{2}}\| \sum_{u=1}^{U} \|Q^{\frac{1}{2}}G^uQ^{-\frac{1}{2}}\|\mathcal{R}^u_G.\end{aligned}$$

Further

$$\begin{aligned}&\sum_{r,s,k,l,m} x_k x_l x_m \sum_{u=1}^{U} G^u_{rk}R^{us}_{lm}(\mathbf{x})(p-2)\sum_{i,j}x_i x_j Q_{ir}Q_{js} \\ &= (p-2)\sum_{u=1}^{U}\sum_{j,s}\left(\sum_{i,k,r} x_i Q_{ir}G^u_{rk}x_k\right) \\ &\quad\times \left(\sum_{l,m} x_l R^{us}_{lm}(\mathbf{x})x_m\right) Q_{sj}x_j \\ &= (p-2)\sum_{u=1}^{U}\sum_{j,s}\left(\mathbf{x}^\top QG^u\mathbf{x}\right)\left(\mathbf{x}^\top R^{us}\mathbf{x}\right) Q_{sj}x_j \\ &= (p-2)\sum_{u=1}^{U}\sum_{j,s}\left(\mathbf{x}^\top QG^uQ^{-\frac{1}{2}}Q^{\frac{1}{2}}\mathbf{x}\right)\left(\mathbf{x}^\top R^{us}\mathbf{x}\right) Q_{sj}x_j \\ &\le |p-2|\sum_{u=1}^{U}\|\mathbf{x}\|_Q^2\|Q^{\frac{1}{2}}G^uQ^{-\frac{1}{2}}\|\left|\sum_{j,s} z^u_s Q_{sj}x_j\right| \\ &\le |p-2|\|\mathbf{x}\|_Q^2\sum_{u=1}^{U}\|Q^{\frac{1}{2}}G^uQ^{-\frac{1}{2}}\|\left|(\mathbf{z}^u)^\top Q\mathbf{x}\right| \\ &\le |p-2|\|\mathbf{x}\|_Q^2\sum_{u=1}^{U}\|Q^{\frac{1}{2}}G^uQ^{-\frac{1}{2}}\|\|\mathbf{z}^u\|\|Q^{\frac{1}{2}}\|\|\mathbf{x}\|_Q \\ &\le |p-2|\|\mathbf{x}\|_Q^3\|\mathbf{x}\|^2\|Q^{\frac{1}{2}}\|\sum_{u=1}^{U}\|Q^{\frac{1}{2}}G^uQ^{-\frac{1}{2}}\|\mathcal{R}^u_G\end{aligned}$$

and similarly

$$
\begin{aligned}
&\sum_{r,s,k,l,m} x_k x_l x_m \sum_{u=1}^{U} G^u_{sk} R^{ur}_{lm}(\mathbf{x})(p-2)\sum_{i,j} x_i x_j Q_{ir} Q_{js}\\
&= (p-2)\sum_{u=1}^{U}\sum_{i,r}\left(\sum_{j,k,s} x_i Q_{js} G^u_{sk} x_k\right)\\
&\quad \times \left(\sum_{l,m} x_l R^{ur}_{lm}(\mathbf{x}) x_m\right) Q_{ri} x_i\\
&= (p-2)\sum_{u=1}^{U}\sum_{i,r}\left(\mathbf{x}^\top Q G^u \mathbf{x}\right)\left(\mathbf{x}^\top R^{ur}\mathbf{x}\right) Q_{ri} x_i\\
&\le |p-2|\sum_{u=1}^{U}\|\mathbf{x}\|_Q^2 \|Q^{\frac{1}{2}} G^u Q^{-\frac{1}{2}}\| \left|\sum_{i,r} z^u_r Q_{ri} x_i\right|\\
&\le |p-2|\|\mathbf{x}\|_Q^3 \|\mathbf{x}\|^2 \|Q^{\frac{1}{2}}\| \sum_{u=1}^{U}\|Q^{\frac{1}{2}} G^u Q^{-\frac{1}{2}}\| \mathcal{R}^u_G.
\end{aligned}
$$

Further

$$
\begin{aligned}
&\sum_{r,s,k,l,m,n} x_k x_l x_m x_n Q_{rs}\|\mathbf{x}\|_Q^2 \sum_{u=1}^{U} R^{ur}_{kl}(\mathbf{x}) R^{us}_{mn}(\mathbf{x})\\
&= \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U}\sum_{r,s}\left(\sum_{k,l} x_k R^{ur}_{kl}(\mathbf{x}) x_l\right) Q_{rs}\\
&\quad \times \left(\sum_{m,n} x_m R^{us}_{mn}(\mathbf{x}) x_n\right)\\
&= \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U}\sum_{r,s} z^u_r Q_{rs} z^u_s\\
&= \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U}\left((\mathbf{z}^u)^\top Q \mathbf{z}^u\right)\\
&\le \|\mathbf{x}\|_Q^2 \sum_{u=1}^{U}\|Q\|\|\mathbf{z}^u\|^2\\
&\le \|\mathbf{x}\|_Q^2 \|\mathbf{x}\|^4 \|Q\| \sum_{u=1}^{U}(\mathcal{R}^u_G)^2\\
&= \|\mathbf{x}\|_Q^2 \|\mathbf{x}\|^4 \|Q\| \mathcal{R}_G\\
&\le \|\mathbf{x}\|_Q^4 \|\mathbf{x}\|^2 \|Q^{-1}\|\|Q\| \mathcal{R}_G\\
&= \|\mathbf{x}\|_Q^4 \|\mathbf{x}\|^2 \kappa(Q) \mathcal{R}_G.
\end{aligned}
$$

Finally

$$
\begin{aligned}
&\sum_{r,s,k,l,m,n} x_k x_l x_m x_n (p-2)\sum_{i,j} x_i x_j Q_{ir} Q_{js}\sum_{u=1}^{U} R^{ur}_{kl}(\mathbf{x}) R^{us}_{mn}(\mathbf{x})\\
&= (p-2)\sum_{u=1}^{U}\sum_{i,j,r,s} x_i Q_{ir}\left(\sum_{k,l} x_k R^{ur}_{kl}(\mathbf{x}) x_l\right) x_j Q_{js}\\
&\quad \times \left(\sum_{m,n} x_m R^{us}_{mn}(\mathbf{x}) x_n\right)\\
&= (p-2)\sum_{u=1}^{U}\sum_{i,j,r,s} x_i Q_{ir} z^u_r x_j Q_{js} z^u_s\\
&= (p-2)\sum_{u=1}^{U}\left(\sum_{i,r} x_i Q_{ir} z^u_r\right)\left(\sum_{i,r} x_j Q_{js} z^u_s\right)\\
&= (p-2)\sum_{u=1}^{U}\left(\mathbf{x}^\top Q \mathbf{z}^u\right)^2\\
&\le |p-2|\sum_{u=1}^{U}\|\mathbf{x}\|_Q^2 \|Q^{\frac{1}{2}}\|^2 \|\mathbf{z}^u\|^2\\
&\le |p-2|\|\mathbf{x}\|_Q^2 \|\mathbf{x}\|^4 \|Q\| \mathcal{R}_G\\
&\le |p-2|\|\mathbf{x}\|_Q^4 \|\mathbf{x}\|^2 \|Q^{-1}\|\|Q\| \mathcal{R}_G\\
&= |p-2|\|\mathbf{x}\|_Q^4 \|\mathbf{x}\|^2 \kappa(Q) \mathcal{R}_G.
\end{aligned}
$$

By combining the results from these estimates we get

$$
\begin{aligned}
&|E_G(\mathbf{x})| \le\\
&\frac{1}{4}p\|\mathbf{x}\|_Q^{p-4}\Bigg|2\|\mathbf{x}\|_Q^3\|\mathbf{x}\|^2\|Q^{\frac{1}{2}}\|\sum_{u=1}^{U}\|Q^{\frac{1}{2}} G^u Q^{-\frac{1}{2}}\|\mathcal{R}^u_G\\
&\qquad + 2|p-2|\|\mathbf{x}\|_Q^3\|\mathbf{x}\|^2\|Q^{\frac{1}{2}}\|\sum_{u=1}^{U}\|Q^{\frac{1}{2}} G^u Q^{-\frac{1}{2}}\|\mathcal{R}^u_G\Bigg|\\
&+\frac{1}{8}p\|\mathbf{x}\|_Q^{p-4}\Bigg|\|\mathbf{x}\|_Q^4\|\mathbf{x}\|^2\kappa(Q)\mathcal{R}_G\\
&\qquad + |p-2|\|\mathbf{x}\|_Q^4\|\mathbf{x}\|^2\kappa(Q)\mathcal{R}_G\Bigg|\\
&= \frac{1}{2}p\|\mathbf{x}\|_Q^{p-1}\|\mathbf{x}\|^2(1+|p-2|)\\
&\times\left(\|Q^{\frac{1}{2}}\|\sum_{u=1}^{U}\mathcal{R}^u_G\|Q^{\frac{1}{2}} G^u Q^{-\frac{1}{2}}\| + \frac{1}{4}\kappa(Q)\mathcal{R}_G\|\mathbf{x}\|_Q\right)
\end{aligned}
$$

and we can estimate

$$
\begin{aligned}
|E(\mathbf{x})| &\le |E_F(\mathbf{x})| + |E_G(\mathbf{x})|\\
&\le \frac{1}{2}p\|\mathbf{x}\|_Q^{p-2}\|\mathbf{x}\|^2\cdot\|\mathbf{x}\|_Q\left(\widetilde{E}_G + \widetilde{E}^*_G\|\mathbf{x}\|_Q\right),
\end{aligned}
$$

which proves the first stated inequality.

Since

$$\begin{aligned} LV(\mathbf{x}) &= L_0 V(\mathbf{x}) + E(\mathbf{x}) \\ &\le -\frac{1}{2} pC \|\mathbf{x}\|_Q^{p-2} \|\mathbf{x}\|^2 + E(\mathbf{x}) \\ &\le -\frac{1}{2} p \|\mathbf{x}\|_Q^{p-2} \|\mathbf{x}\|^2 \left[ C - \|\mathbf{x}\|_Q \left( \widetilde{E}_G + \widetilde{E}_G^* \|\mathbf{x}\|_Q \right) \right] \end{aligned}$$

we have $LV(\mathbf{x}) < 0$ if

$$\|\mathbf{x}\|_Q \left( \widetilde{E}_G + \widetilde{E}_G^* \|\mathbf{x}\|_Q \right) < C,$$

i.e.

$$\|\mathbf{x}\|_Q < \frac{-\widetilde{E}_G + \sqrt{(\widetilde{E}_G)^2 + 4C\widetilde{E}_G^*}}{2\widetilde{E}_G^*}.$$

Thus for

$$\mathbf{x} \in \mathcal{D} = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_Q \le \rho\}$$

with

$$\rho < \min\left\{ \rho^*, \frac{1}{2\widetilde{E}_G^*} \left( \sqrt{(\widetilde{E}_G)^2 + 4C\widetilde{E}_G^*} - \widetilde{E}_G \right) \right\},$$

we have $LV(\mathbf{x}) < 0$, which concludes the proof. □

## 4 CONCLUSIONS

We derived rigid bounds on a domain, on which a Lyapunov function for a linearized stochastic differential equation is also a Lyapunov function for the original nonlinear system. This allows for the derivation of a lower bound on the equilibrium's $\gamma$-basin of attraction, i.e. the area in which all started solutions converge to the equilibrium with probability no less than $\gamma$. Another application is the facilitation of a numerical method to compute Lyapunov functions for nonlinear stochastic differential equations on a larger domain as discussed in (Gudmundsson and Hafstein, 2018), because one first needs a local Lyapunov function at the equilibrium.

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