Duality for a Class of Multiobjective Semi-infinite Programming Problems

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Keywords: Semi-infinite programming, multiobjective optimization, sufficient optimality condition, duality, generalized convexity.

Abstract: In this paper, a class of multiobjective semi-infinite programming problems is considered. Sufficient optimality condition is established for an efficient solution firstly. Furthermore, we formulate Mond-Weir type dual for multiobjective semi-infinite programming problems and establish weak, strong and converse duality theorems relating the problem and the dual problems under G-invex assumptions.

1 INTRODUCTION AND PRELIMINARIES

Generalized convexity has been playing a vital role in mathematical programming and optimization theory. A class of generalized convex functions called *G*-invex functions was defined by Antczak (2007) for scalar differentiable functions. Then, the definition of a real-valued *G*-invex function introduced by Antczak was generalized to the vectorial case in (2009). They used vector *G*-invexity to develop optimality and duality for differentiable multiobjective programming problems with both inequality and equality constraints.

A semi-infinite programming problem is an optimization problem on a feasible set described by infinite number of inequality constraints. Recently, semi-infinite optimization became an active field of research. Many scholars have been interested in semi-infinite programming problem, especially their optimality conditions and duality results(see (Heettich R., 1993; Jeyakumar V., 2008; Lopez M.2007; Shapiro A., 2009; Kanzi N., 2010) and the references therein). S.K.Mishra et al. studied the duality results of this nonsmooth semi-infinite programming problem.

Motivated by the works of (T. Antczak., 2009), (T. Antczak., 2009), and (Mishra S.K.), in this paper, we study a class of multiobjective semi-infinite optimization problems. We formulate Mond-Weir type dual for multiobjective semi-infinite programming problems. Furthermore, by using *G*-invex assumption, related duality theorems are established.

Next, we first introduce some basic concepts and results which will be used in the sequel. The following convention for equalities and inequalities will be used throughout the paper.

We define:

$$\forall x=\left(x_1,x_2,\cdots,x_n\right)^{\mathbf{T}},\ y=\left(y_1,y_2,\cdots,y_n\right)^{\mathbf{T}}$$
$$x=y\Leftrightarrow x_i=y_i,i=1,2,\cdots,n;$$
$$x<y\Leftrightarrow x_i<y_i,i=1,2,\cdots,n;$$
$$x\leqq y\Leftrightarrow x_i\le y_i,i=1,2,\cdots,n;$$
$$x\le y\Leftrightarrow x_i\leqq y_i,x\ne y,n>1.$$

Throughout the paper, we will use the same notation for row and column vectors when the interpretation is obvious.

We say that a vector $z\in R^n$ is negative if $z\leqq 0$ and strictly negative if $z<0$.

Definition 1.1 A function $f:R\to R$ is said to be strictly increasing if and only if

$$\forall x,y\in R,x<y\Rightarrow f\left(x\right)<f\left(y\right).$$

Let $f=\left(f_1,f_2,\cdots,f_k\right):X\to R^k$ be a vector-valued differentiable function defined on a nonempty open set $X\subset R^n$, and $I_{f_i}\left(X\right),i=1,2,\cdots,k$ be the range of f_i, that is, the image of X under f_i.

Zhao, J. and Yan, X.
Duality for a Class of Multiobjective Semi-infinite Programming Problems.
In *3rd International Conference on Electromechanical Control Technology and Transportation (ICECTT 2018)*, pages 314-317
ISBN: 978-989-758-312-4

Definition 1.2[2] Let $f: X \to R^k$ be a vector-valued differentiable function defined on a nonempty open set $X \subset R^n$ and $u \in X$. If there exists a differentiable vector-valued function $G_f = (G_{f_1}, \cdots, G_{f_k}): R \to R^k$ such that any its component $G_{f_i}: I_{f_i}(X) \to R$ is a strictly increasing function on its domain. And we assume that there exists a vector-valued function $\eta: X \times X \to R^n$ such that, for any $i = 1, 2, \cdots, k$, and all $x \in X\ (x \neq u)$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) - G'_{f_i}(f_i(u)) \nabla f_i(u) \eta(x, u) \geqq 0 \quad (>)$$

Then f is said to be a (strictly) vector G_f-invex function at u on X with respect to η. If (1) is satisfied for each $u \in X$, then f is vector G_f-invex function on X with respect to η.

Remark 1.1 In order to define an analogous class of (strictly) vector G_f-incave functions with respect to η, the direction of the inequality in the definition of these functions should be changed to the opposite one.

Remark 1.2 In the case, When $G_{f_i}(a) = a,\ i = 1, 2, \cdots, k$, for any $a \in I_{f_i}(X)$, we obtain a definition of a vector-valued invex function.

Definition 1.3 A point $\bar{x} \in X$ is said to be an efficient solution of the problem if there is no $x \in X$ such that $f(x) \leq f(\bar{x})$.

In this paper, we consider the following multiobjective semi-infinite programming problem (SIMP).

$$\min \quad f(x) = (f_1(x), f_2(x), \cdots, f_k(x))$$

$$s.t. \quad g_j(x) \leq 0, j \in J.$$

where J is an (possibly infinite) index set, $f_i: X \to R, i \in I = \{1, 2, \cdots, k\}$ are vector differentiable functions, $g_j: X \to R, j \in J$,are strictly vector differentiable functions. Let $D = \{x \in X : g_j(x) \leq 0, j \in J\}$ be the set of all feasible solutions for problem (SIMP). Further, We assume that, if $\bar{x} \in D$ is an efficient point, then there exits $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \cdots \bar{\lambda}_k), \bar{\mu}_j \geq 0, \forall j \in J$, $\bar{\mu}_j \neq 0$ for finite $j \in J$, such that

$$\sum_{i \in I} \bar{\lambda}_i G'_{f_i} f_i(\bar{x}) \nabla f_i(\bar{x}) + \sum_{j \in J} \bar{\mu}_j G'_{g_j}(g_j(\bar{x})) \nabla g_j(\bar{x}) = 0 \qquad (1)$$

$$\bar{\mu}_j G_{g_j}(g_j(\bar{x})) = 0, j \in J \qquad (2)$$

$$\bar{\lambda} \geq 0, \bar{\mu} \geqq 0, \bar{\mu}_j \neq 0 \text{ for finitely many } j \in J \qquad (3)$$

2 OPTIMALITY CONDITION

In this section, we give Karush-Kuhn-Tucker suffcient optimality condition of efficient solution for the problem (SIMP).

Theorem 2.1 Suppose that $\bar{x} \in D$ is a feasible solution of the problem (SIMP), and $f_i, i \in I$ are G_{f_i} -invex with respect to η, $g_j: X \to R, j \in J$ are strictly G_{g_j}- invex with respect to η, Then $\bar{x}$ is the efficient solution of the problem (SIMP).

Proof Contrary to the result of theorem. Suppose that there exist $\hat{x} \in D$ such that $f(\hat{x}) \leq f(\bar{x})$. Since $\hat{x} \in D$, G_{g_j} are strictly increasing functions and (2)-(3), we get

$$G_{g_j}(g_j(\hat{x})) < G_{g_j}(g_j(\bar{x})) = 0.$$

By assumption $g_j,\ j \in J$ are strictly G_{g_j} -invex, then

$$G_{g_j}(g_j(\hat{x})) - G_{g_j}(g_j(\bar{x})) - G'_{g_j}(g_j(\bar{x})) \nabla g_j(\bar{x}) \eta(\hat{x}, \bar{x}) > 0, j \in J.$$

From (1) and (3) it follows that

$$G'_{g_j}(g_j(\bar{x})) \nabla g_j(\bar{x}) \eta(\hat{x}, \bar{x}) < 0, j \in J.$$

$$G'_{f_i}(f_i(\bar{x})) \nabla f_i(\bar{x}) \eta(\hat{x}, \bar{x}) > 0.$$

Since $f_i, i \in I$ are G_{f_i} -invex functions, we have

$$G_{f_i}(f_i(\hat{x})) - G_{f_i}(f_i(\bar{x})) - G'_{f_i}(f_i(\bar{x})) \nabla f_i(\bar{x}) \eta(\hat{x}, \bar{x}) \geq 0, i \in I.$$

Moreover, for $G_{f_i},\ i \in I$ are strictly increasing functions, then

$$f_i(\hat{x}) > f_i(\overline{x})$$

which is a contradiction to the assumption. The proof is completed.□

3 VECTOR DUALITY

Now, we consider the following Mond-Weir type dual (SMWD) for the problem (SIMP).

$$\max \quad f(y) = \left(f_1(y), \cdots, f_k(y)\right)$$

$$s.t. \quad \sum_{i\in I} \lambda_i G'_{f_i} f_i(y) \nabla f_i(y)$$

$$+\sum_{j\in J} \mu_j G'_{g_j}\left(g_j(y)\right)\nabla g_j(y) = 0. \tag{4}$$

$$\sum_{j\in J} \mu_j G_{g_j}\left(g_j(y)\right) \geq 0. \tag{5}$$

$$\lambda \in R_+^k, \lambda \geq 0, \lambda^T e = 1, e = (1,1,\cdots,1) \in R^k. \tag{6}$$

$$\mu_j \geq 0, \forall j \in J \quad and \quad \mu_j \neq 0$$

for finitely many $j \in J$. (7)

Theorem 3.1 (weak duality). Let x be feasible for (SIMP) and (y,λ,μ) where $\lambda \in (\lambda_i), i \in I$, be feasible for (SMWD). Let $f_i, i \in I$ be G_{f_i}- invex functions with respect to η, $g_j : X \to R, j \in J$ be strictly G_{g_j}- invex functions with respect to η, $G_{g_j}(0) = 0, j \in J$. Then $f(x) \not< f(y)$

Proof We proceed by contradiction. Suppose that

$$f(x) < f(y).$$

For $f_i, i \in I$ be G_{f_i}- invex functions with respect to η, G_{f_i} are strictly increasing functions, we can obtain that

$$G'_{f_i}\left(f_i(y)\right)\nabla f_i(y)\eta(x,y) < 0, i \in I.$$

From the (4), (6-7), it follows that

$$G'_{g_j}\left(g_j(y)\right)\nabla g_j(y)\eta(x,y) > 0, j \in J.$$

By strict G_{g_j}- invexity of $g_j, j \in J$, we have

$$G_{g_j}\left(g_j(y)\right) < G_{g_j}\left(g_j(x)\right) \leq 0.$$

Again from (7), we get

$$\sum_{j\in J} \mu_j G_{g_j}\left(g_j(y)\right) < 0.$$

which is a contradiction to (5). □

Theorem 3.2 (strong duality). Let $f_i, i \in I$ be G_{f_i}- invex functions with respect to η, $g_j, j \in J$ be strictly G_{g_j}- invex functions with respect to η. If $\overline{x}$ is efficient solution for (SIMP), then $\exists \overline{\lambda} \in R_+^k, \overline{\lambda} \geq 0, \overline{\lambda}^T e = 1, e = (1,1,\cdots,1) \in R^k$, $\overline{\mu} = (\overline{\mu}_j) \geqq 0, \forall j \in J, \overline{\mu}_j \neq 0$. *for finitely many* $j \in J(\overline{x})$ such that $(\overline{x},\overline{\lambda},\overline{\mu})$ is an efficient solution of (SMWD), and the respective objective values are equal.

Proof As $\overline{x}$ is efficient solution for (SIMP) and the suitable constraint qualification is satisfied, that is, $\exists \overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2, \cdots \overline{\lambda}_k), \overline{\mu}_j \geq 0, \forall j \in J, \overline{\mu}_j \neq 0$ for finite $j \in J$, such that (1-2) are satisfied.

Since $\overline{\lambda}^T e = 1, e = (1,1,\cdots,1) \in R^k$, then $(\overline{x},\overline{\lambda},\overline{\mu})$ is a feasible solution of (SMWD).

On the other hand by weak theorem, we have

$$f(\overline{x}) \geq f(y)$$

for any efficient solution (y,λ,μ). Hence we get that $(\overline{x},\overline{\lambda},\overline{\mu})$ is a feasible solution of (SMWD) and the respective objective values are equal.□

Theorem 3.3 (converse duality). Let $(\overline{y},\overline{\lambda},\overline{\mu})$ be efficient solution of (SMWD) and $\overline{y} \in D$. Assume $f_i, i \in I$ be G_{f_i}-invex functions with respect to $\eta, g_j, j \in J$ be strictly G_{g_j}- invex functions with respect to η, $G_{g_j}(0) = 0, j \in J$, then $\overline{y}$ be efficient solution of (SIMP).

Proof Contrary to the result of theorem. Assume $\exists \overline{x} \in D$ such that $f(\overline{x}) \leq f(\overline{y})$.

As $(\overline{y},\overline{\lambda},\overline{\mu})$ be efficient solution of (SMWD), then

$$\sum_{i\in I} \overline{\lambda}_i G'_{f_i} f_i(\overline{y}) \nabla f_i(\overline{y})$$

$$+\sum_{j\in J} \overline{\mu}_j G'_{g_j}\left(g_j(\overline{y})\right)\nabla g_j(\overline{y}) = 0 \tag{8}$$

$$\sum_{j \in J} \bar{\mu}_j G_{g_j}\left(g_j\left(\bar{y}\right)\right) \geq 0. \tag{9}$$

From $\bar{y} \in D$, it follows that

$$g_j\left(\bar{y}\right) \leq 0, j \in J. \tag{10}$$

Combining (9)-(10) and $\bar{\mu}_j \geq 0, j \in J$, we obtain

$$\sum_{j \in J} \bar{\mu}_j G_{g_j}\left(g_j\left(\bar{y}\right)\right) = 0. \tag{11}$$

From strict G_{g_j} - invexity of $g_j, j \in J$, G_{f_i} invexity of $f_i, i \in I$ and $\bar{\lambda}_i \geq 0, i \in I, \bar{\mu}_j \geq 0$, $j \in J$, we have

$$\sum_{i \in I} \bar{\lambda}_i G_{f_i}\left(f_i(\bar{x})\right) - \sum_{i \in I} \bar{\lambda}_i G_{f_i}\left(f_i(\bar{y})\right) \geq \sum_{i \in I} \bar{\lambda}_i G'_{f_i}\left(f_i(\bar{y})\right) \nabla f_i\left(\bar{y}\right) \eta\left(\bar{x}, \bar{y}\right), i \in I. \tag{12}$$

$$\sum_{j \in J} \bar{\mu}_j G_{g_j}\left(g_j(\bar{x})\right) - \sum_{j \in J} \bar{\mu}_j G_{g_j}\left(g_j(\bar{y})\right) > \sum_{j \in J} \bar{\mu}_j G'_{g_j}\left(g_j(\bar{y})\right) \nabla g_j(\bar{y}) \eta\left(\bar{x}, \bar{y}\right), j \in J. \tag{13}$$

Adding both side of (12-13), using (8), we get

$$\sum_{i \in I} \bar{\lambda}_i G_{f_i}\left(f_i(\bar{x})\right) - \sum_{i \in I} \bar{\lambda}_i G_{f_i}\left(f_i(\bar{y})\right) + \sum_{j \in J} \bar{\mu}_j G_{g_j}\left(g_j(\bar{x})\right) - \sum_{j \in J} \bar{\mu}_j G_{g_j}\left(g_j(\bar{y})\right) > 0. \tag{14}$$

From $\bar{x} \in D$, $G_{g_j}\left(0\right) = 0, j \in J$ are strictly increasing functions, we have

$$G_{g_j}\left(g_j(\bar{x})\right) < 0. \tag{15}$$

Combining (11), (14-15), and $\bar{\lambda}_i \geq 0, i \in I, \bar{\mu}_j \geq 0, j \in J$, it is obvious that

$$G_{f_i}\left(f_i(\bar{x})\right) > G_{f_i}\left(f_i(\bar{y})\right).$$

Furthermore, from $G_{f_i}, i \in I$ are strictly increasing functions, it follows that

$$f(\bar{x}) > f(\bar{y})$$

which is a contradiction to the suppose.

4 CONCLUSIONS

Sufficient optimality condition is established for an efficient solution of a multiobjective semi-infinite programming problem called (SIMP). Mond-Weir type dual for (SIMP) is formulated. And we establish weak, strong and converse duality theorems under G-invex assumptions.

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