Reliability-based Topology Optimization of Continuum Structure

Zhaokun Li [1a], Jinying Chen [1b], Lijuan Shi [1c] and Huamei Bian [1d]
[1]*Engineering of mechanical and electrical department, Beijing polytechnic college, Beijing, 100042, China*

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Abstract: The reliability-based topology optimization method of continuous structures is investigated considering structural applied loads and the geometry description. Firstly, based on the solid isotropic material with penalization approach, the deterministic mathematical model of topology optimization is developed, in which minimization of the compliance is taken as objective function and the volume is taken as constraint function; Secondly, using the relation between failure probability and reliability index, the mathematical model of reliability topology optimization based on the reliability index constraint is established. Reliability index probabilistic constraint problem considering uncertainties is solved using first order reliability method, in which the reliability index constraints are transformed into random variables, and then the modified random variables are used as deterministic variables to carry on the deterministic topology optimization; Finally, several numerical examples are simulated to show that reliability-based topology optimization yields structures that are more reliable than those produced by deterministic topology optimization and also certificate the validity of the proposed method..

1 INTRODUCTION

In recent years, the research on the topology optimization has been an attractive research area and made great progress. Topological optimization is a design method based on structural optimization, which is under the action of some external forces and constraints, seeking the optimal structure arrangement. At present, the depth and breadth of topological optimization research of continuous structure has been extended from single object to multi-objective function, from single physical field to multi-physical field design, from material and geometric linear problem to nonlinear problem, from static topology to dynamic topology [1-3].

But so far, papers on the topology optimization of continuum structure presented mostly deal with the solution without taking into account the effects of uncertainties. Actually, because continuum structures may be subject to inherent uncertainties such as external loading, material properties, and manufacturing quality, the prototypes or manufactured products may not satisfy the necessary performance requirements.

Therefore, in order to reduce the mechanism performance degradation caused by the uncertainty in manufacturing process, these uncertainties must be considered in topology optimization. The reliability optimization design accounting for uncertainties has become a research hot topic in the field of structural design. At present, structural reliability optimization is widely used in the field of dimension and shape optimization [5]. However, in the field of topology optimization of continuum structures, there are few references to the application of reliability optimization methods. [5-8] the research on reliability-based topology optimization for continuum structure is currently processed in the initial stage and many problems need further investigate.

In this paper, a new reliability-based topology optimization methodology for continuum structure is presented. Firstly, Based on the solid isotropic material with penalization approach, the deterministic mathematical model of topology optimization is developed, in which minimization of the compliance is taken as objective function and the volume is taken as constraint function; Secondly, using the relation between failure probability and reliability index, the mathematical model of reliability topology optimization based on the reliability index constraint is established. Reliability index probabilistic constraint problem considering uncertainties is solved using first order reliability method, in which the reliability index constraints are transformed into random variables, and then the modified random variables are used as deterministic variables to carry on the deterministic topology optimization; Finally, several numerical examples are simulated to certificate the validity of the proposed method.

2 THE DETERMINISTIC TOPOLOGY OPTIMIZATION OF CONTINUUM STRUCTURE

The need for the continuum structure to be stiff enough to withstand the external load is captured as the stiffness requirement. Maximizing the stiffness requirement is determined by minimizing Strain Energy (SE) which is equivalent to minimizing the mean compliance $(\boldsymbol{P})$ of the structures and the formulation is defined as:

$$\min SE \Leftrightarrow \min \boldsymbol{P} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{U} \tag{1}$$

Where $\boldsymbol{U}$ is the displacement vector and $\boldsymbol{F}$ is the sum of all the external force vectors.

Using the SIMP approach, the relative density x_e of material in each element is a design variable. The N -vector containing the design variables is denoted $\boldsymbol{x}$. The overall topology optimization solving the problem of distributing a limited amount of material in the design domain such that the objective function is minimized and the volume and input displacement is constrained can be expressed as:

$$\begin{cases} \min\limits_{x_e} & f(x) = \boldsymbol{U}^T\boldsymbol{K}\boldsymbol{U} = \sum\limits_{e=1}^{N}(x_e)^p u_e^T k_0 u_e \\ \text{subject to} & \boldsymbol{F} = \boldsymbol{K}\boldsymbol{U} \\ & \mathrm{V} = f \cdot \boldsymbol{V}_0 = \sum\limits_{e=1}^{N} x_e v_e \le \boldsymbol{V}^* \\ & 0 < x_{\min} \le x_e \le 1 \end{cases} \tag{2}$$

where $\boldsymbol{V}$ is N -vector containing the element volume, $\boldsymbol{V}^*$ is the upper bound on material volume and $x_{\min}$ is an N -vector with the minimum values of the densities, ${}^0\boldsymbol{V}$ is the volume. K is the tangent stiffness matrix.

The constitutive tensor for element e with intermediate densities $\boldsymbol{C}_{ijkl}^{\mathrm{e}}$ can be expressed as:

$$\boldsymbol{C}_{ijkl}^{\mathrm{e}} = (x_{\mathrm{e}})^P \boldsymbol{C}_{ijkl}^0 \tag{3}$$

The second Piola-Kirchhoff stresses are calculated:

$$\boldsymbol{s}_{ij} = (x_{\mathrm{e}})^P \boldsymbol{C}_{ijkl}^0 \boldsymbol{\varepsilon}_{kl} \tag{4}$$

Where $\boldsymbol{s}_{ij}$ is stress tensor, $\boldsymbol{\varepsilon}_{\mathrm{k1}}$ is Green-Lagrange strain tensor, p is the penalization factor (typically $p = 3$), $\boldsymbol{C}_{ijkl}^0$ is the constitutive tensor for solid isotropic material.

3 RELIABILITY-BASED TOPOLOGY OPTIMIZATION

3.1 Reliability Analysis

In reliability-based topology optimization, three kinds of variables will be distinguished [18]: the design variables $\boldsymbol{x}$, the random variables $\boldsymbol{y}$, and the normalized variables $\boldsymbol{s}$. In contrast to the deterministic optimization, probabilistic mechanism design optimization can be characterized by the probabilistic constraints. The random variables as well as the design variables are involved in defining the problem of probabilistic optimization [19]:

$$\begin{aligned} &\text{min:} \quad f(\mathbf{x}) \\ &subject\ \ to:\ \ P_r[G_i(\mathbf{x},\mathbf{y}) \le 0] \le P_i \quad (i = 1,2,..m) \end{aligned} \tag{5}$$

Where $P_r[G_i(\boldsymbol{x},\boldsymbol{y}) \le 0] \le P_i$ and P_i are the probability of constraint violation and the allowable probability violation ,respectively, and $G(\boldsymbol{x},\boldsymbol{y})$ is defined as a limit state function . Safety is the state in which the structure is able to fulfill all the functioning requirements. The safety of components depends on external loading S and resisting force R , and active states according to a limit state function $G(\boldsymbol{x},\boldsymbol{y})$ can be expressed as [18]:

$$\begin{cases} G(\boldsymbol{x},\boldsymbol{y}) = R(\boldsymbol{x},\boldsymbol{y}) - S(\boldsymbol{x},\boldsymbol{y}) < 0 & \text{failure state} \\ G(\boldsymbol{x},\boldsymbol{y}) = R(\boldsymbol{x},\boldsymbol{y}) - S(\boldsymbol{x},\boldsymbol{y}) > 0 & \text{safety state} \\ G(\boldsymbol{x},\boldsymbol{y}) = R(\boldsymbol{x},\boldsymbol{y}) - S(\boldsymbol{x},\boldsymbol{y}) = 0 & \text{limit state} \end{cases} \tag{6}$$

The uncertainties of S and R which is mostly not statistical information are modeled by a vector of stochastic physical variables. No matter what regularities of distribution of S and R, distribution characteristics generally can be described using mean values and standard deviation. Defining mean values and standard deviation of S and R are μ_R 、 μ_S and σ_R^2 、 σ_S^2, supposed that S and R are uncorrelated independent random variables then mean value and standard deviation of a limit state function :

$$\mu_Z = \mu_R - \mu_S \tag{7}$$

$$\sigma_Z^2 = \sigma_R^2 + \sigma_S^2 \tag{8}$$

The failure probability P_f is then calculated by

$$P_f = P[G(\boldsymbol{x},\boldsymbol{y}) \le 0] = \Phi((\mu_S - \mu_R) \big/ \sqrt{\sigma_R^2 + \sigma_S^2}) \tag{9}$$

Reliability index β is defined as:

$$\beta = \mu_Z / \sigma_Z = (\mu_R - \mu_S) \big/ \sqrt{\sigma_R^2 + \sigma_S^2} \tag{10}$$

The relation between the probability of failure and reliability index is expressed as:

$$P_f = \Phi(-\beta) = 1 - \Phi(\beta) \Leftrightarrow \beta = -\Phi^{-1}(P_f) \tag{11}$$

Where $\Phi(\beta)$ is the standard normal cumulative distribution function.

That indicates from the formula (11) that the failure rate and reliability index are corresponding. The permissible value of the probability of failure may thus be expressed as:

$$P_f = \Phi(-\beta) \tag{12}$$

where $\Phi(\beta)$ is increasing with the enlargement of β, and β is corresponding to P_f. Since reliability index β is corresponding to the probability of failure P_f, we may solve the reliability level by introducing reliability index β .according to the formula (11) (12) and (5), the reliability constraint can be transformed into:

$$\Phi(-\beta_{\mathrm{t}}) - \Phi(-\beta) \ge 0 \Rightarrow \beta_{\mathrm{t}} - \beta \le 0 \tag{13}$$

Therefore, we can solve the reliability index and then solve the reliability. First order reliability method is used.

3.2 First Order Reliability Method

The first order reliability method is developed in which the probabilistic constraints are stated in terms of the reliability index as a measure of the probabilistic safety. The reliability index β was introduced by Hasofer and Lind (1974), who proposed working in the space of standard independent Gaussian variables instead of the space of physical variables. The transformation from the random variable $\boldsymbol{y}$ to standard normal $\boldsymbol{s}$ is given by

$$\boldsymbol{s} = T(\boldsymbol{y}), \text{ or } \boldsymbol{y} = T^{-1}(\boldsymbol{s}) \tag{14}$$

where $T(.)$ is generally a non-linear mapping that depends on the type of random distribution of $\boldsymbol{y}$. In the case of a normal distribution, normal random variables $\boldsymbol{y}_j$ can be transformed into a standard normal random variable $\boldsymbol{s}_j$ by

$$\boldsymbol{s}_j = (\boldsymbol{y}_j - \boldsymbol{\mu}_j) \big/ \boldsymbol{\sigma}_j \tag{15}$$

where $\boldsymbol{y}_j$ is the j -th random variable, with mean value $\boldsymbol{\mu}_j$ and standard-deviation $\boldsymbol{\sigma}_j$ and j is the number of selected random variables. The reliability index β is defined as the minimum distance from the origin in the standard normal space to the limit state surface and the calculation of the reliability index can be realized by the following form [21]:

$$\beta = \min d(\boldsymbol{s}) = (\sqrt{\boldsymbol{s}^T \boldsymbol{s}})$$
$$\text{subject} \quad \text{to}: \quad H(\boldsymbol{x}, \boldsymbol{s}) = 0 \tag{16}$$

where $H(\boldsymbol{x}, \boldsymbol{s}) = 0$ is the limit state function in the standard space. According to the structural reliability index, the normal random variables are solved as follows:

$$\begin{aligned} \min \quad & \beta(\boldsymbol{s}) = \min d(\boldsymbol{s}) = (\sqrt{\boldsymbol{s}^T \boldsymbol{s}}) \\ \textit{subject} \quad \textit{to} \quad & \beta(\boldsymbol{s}) \geq \beta_t \end{aligned} \tag{17}$$

The formula (17) directly reflects the meaning of the structural reliability index and transforms the reliability constraint into a correction of random variables.

3.3 Formulation

When probabilistic constraints accounting for the randomness of the applied loads and the description of the geometry are estimated in terms of the reliability index, the reliability based topology optimization may be expressed as:

$$\begin{cases} \min\limits_{x_e} & f(\mathbf{x}) \\ \textit{subject to} & \beta(\mathbf{s}) \geq \beta_t \\ & R(\mathbf{x}, \mathbf{y}, \mathbf{s}) = 0 \\ & V(\mathbf{x}, \mathbf{y}, \mathbf{s}) / V_0 \leq f \\ & 0 < x_{\min} \leq x_e \leq 1 \end{cases} \tag{18}$$

where β and β_t are the reliability index of the system and the target reliability index, respectively, and s is the normalized variable.

In the evaluation of the reliability index, the derivative of β with respect to normalized variables s can be written as [18]:

$$\frac{\partial \beta}{\partial \boldsymbol{s}_j} = \frac{1}{2} (\sum \boldsymbol{s}_j^2)^{1/2} 2\boldsymbol{s}_j = \frac{\boldsymbol{s}_j}{\beta} \tag{19}$$

The resulting $\boldsymbol{s}$ of problem in (19) will be used to evaluate the random variable $\boldsymbol{y}: \boldsymbol{y}_j = \sigma_j \mathbf{s}_j + \mu_j$ .using (15) with the standard deviations given by $\boldsymbol{\sigma}_j = 0.1\boldsymbol{\mu}_j$.

Reliability analysis design proceed:

First, determine design variables and random variables. The design variables $\boldsymbol{x}$ based on variable density method are relative density x_e in finite element, considering the uncertainty of geometric size and action load, random variables $\boldsymbol{y}$ are action load F, discrete units nelx and nely in the horizontal and vertical direction and volumetric ratio f .

Secondly, using the above random variable $\boldsymbol{y}$ as the initial value and constructing the mean vector $\boldsymbol{\mu}$, the influence of the mean on the objective function is positive and negative. The least squares method is used to analyze the sensitivity of the objective function, considering the uncertainty of the load and geometric dimensions:

$$\partial f / \partial \mu_j = \Delta f / \Delta \mu_j = \left(f(\mu_j + \Delta \mu_j) - f(\mu_j) \right) / \Delta \mu_j \tag{20}$$

where $\Delta \mu_j / \mu_j = 0.01$.

Thirdly, using the formula (17) to calculate the standardized variable s under the constraint of the reliability index, the standardized variable s is used to modify the random variable y by using the formula (15). The revised random variable y is the known quantity;

Finally, the deterministic topology optimization module is called.

Thus the reliability analysis and topology optimization are composed of two independent modules,

Namely, the correction of random variables and the deterministic topology optimization. The result depends on the given reliability index. In this method, the influence of reliability constraint on the optimization of the mechanism is transformed into the range of random variable due to reliability constraint, avoiding the cumbersome reliability analysis in the process of topology optimization. The optimization problem is solved using the MMA method proposed by Svanberg [10]. Mesh-independency scheme is used to circumvent the problem of checkerboard patterns and mesh-dependencies proposed by Diaz and Sigmund [9-10].

4 NUMERICAL EXAMPLES

4.1 Example 1

The first example considers a beam structure. Figure 1 shows the half symmetric design domain. The dimension of the design domain, the material properties, and the input parameters for the optimization program are shown in Table 1.

The resulting optimal topology principally depends on the reliability index value. In the case of the target reliability index $\beta = 4$ modified parameters are shown in Table 2.From intermediate results during reliability analysis in Table 2, it is noted that volume ratio is modified from 0.5to 0.4 which reduces manufacture cost and the applied load is changed into 1.2KN more than the initial value 1KNwhich indicates that reliability based topology mechanism may bear greater external force than the deterministic topology mechanism. Afterwards, topology optimization is implemented to demonstrate the global system performances as below.

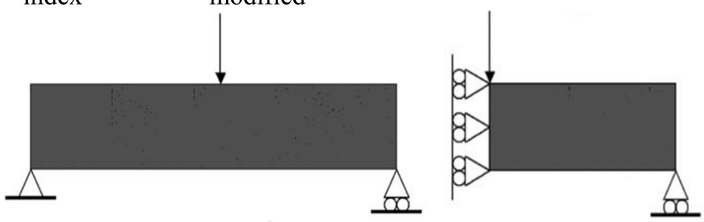


Fig.1 beam structure. (Left) design domain and boundary conditions. (Right) Equivalent model.

Table 1. Input parameters for topology optimization.

Variable name	Setting value
Design domain size S/μm	60×20
Input force(KN)	1
Poisson ratio ν	0.3
Young modulu E/Gpa	1
V/V_0	*0.5*

Table 2. Random variable parameters.

Type	Random variables	Mean value $\boldsymbol{\mu}$	normalized value $\boldsymbol{s}$	Modified value $\boldsymbol{y}$
Geometry dimension	Horizon size(μm)	60	2	72
	Vertical size (μm)	20	-2	16
	Volume ratio	0.5	-2.003	0.4
applied load	load (KN)	1	2.003	1.2

The layouts are obtained from the topology optimization neglecting all uncertainties and considering uncertainties respectively shown in Table5.From topology results, it can be seen that the main difference between considering uncertainties and neglecting uncertainties in this example is to more properly redistribute the arms, which Considering uncertainties has additional arms, which obviously improve the reliabilities of the mechanisms.

4.2 Example 2

Fig.2 design domain and boundary conditions of cantilevered beam.

The second example is designing a cantilevered beam. The design domain is sketched in Figure2. The dimension of the design domain, the material properties, and the other initial values are all list in Table 3.

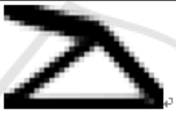
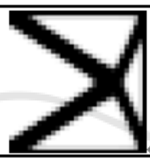
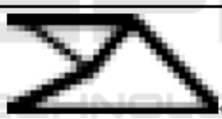
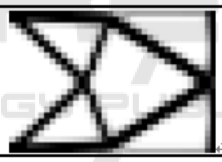
Table 3. Input parameters for topology optimization.

Variable name	Setting value
Design domain size S/μm	32×20
Input force(KN)	1
Poisson ratio v	0.3
Young modulu E/Gpa	1
V/V0	0.4

Table 4. Random variable parameters.

Type	Random variables	Mean value $\boldsymbol{\mu}$	normalized value $\boldsymbol{s}$	Modified value $\boldsymbol{y}$
Geometry dimension	Horizon size(μm)	32	1.56	37
	Vertical size (μm)	20	-1.5	17
	Volume ratio	0.4	-1.48	0.34
applied load	load (KN)	1	1.48	1.15

Table5. Topological diagrams.

	Example 1	**Example 2**	**Example 3**	**Example4**
deterministic Topological				
$\boldsymbol{\beta}$	$\boldsymbol{\beta}$ =0			
reliability Topological				
$\boldsymbol{\beta}$	$\boldsymbol{\beta}$ =4	$\boldsymbol{\beta}$ =3	$\boldsymbol{\beta}$ =4	$\boldsymbol{\beta}$ =3

The resulting optimal topology principally depends on the reliability index value. In the case of the target reliability index $\beta = 3$ modified parameters are shown in Table3.From intermediate results during reliability analysis in Table 4, it is noted that volume ratio is modified from 0.4 to 0.34 which reduces manufacture cost and the applied load is changed into 1KN more than the initial value 1.15KNwhich indicates that reliability based topology mechanism may bear greater external force than the deterministic topology mechanism. Afterwards, Topology optimization is implemented to demonstrate the global system performances as below. From topology results in Table 5 , it can be seen that the main difference between considering uncertainties and neglecting uncertainties in this example is to more properly redistribute the arms, which Considering uncertainties has additional arms, which obviously improve the reliabilities of the mechanisms.

5 CONCLUSIONS

（1） The reliability-based topology optimization method is investigated considering structural applied loads and the geometry description. Using the relation between failure probability and reliability index, the reliability mathematical model based on the reliability index constraint is established;

（2） Reliability index probabilistic constraint problem is solved using first order reliability method, in which the reliability index constraints are transformed into random variables, and then the modified random variables are used as deterministic variables to carry on the deterministic topology optimization；

（3）Several numerical examples are simulated to show that reliability-based topology optimization yields structures that are more reliable than those produced by deterministic topology optimization. That is, the use of reliability-based topology can improve the performance. Meanwhile, numerical examples also certificate the validity of the proposed method.

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