Two Modified Three-Step Iterative Methods for Solving Nonlinear Equations

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Abstract: With the rapid development of information and engineering technology and wide application of science and technology, nonlinear problems become an important direction of research in the field of numerical calculation and analysis. In this paper, we mainly study modified iterative methods for solving nonlinear equations. We present and analyze a sixth-order convergent modified three-step Newton-type method for solving nonlinear equations. Then we give a seventh-order convergence algorithm. The convergence analysis of the presented algorithms are given. Both of the given methods are free from second derivatives. The efficiency indices of the presented methods are 1.431 and 1.476, respectively, which are better than that of the classical Newton's method 1.414. Some numerical experiments illustrate the efficiency and performance of the proposed two methods.

1 INTRODUCTION

Nonlinear problems are an important direction of research in the field of numerical calculation and analysis. The solution of nonlinear equations is one of the most investigated topics in applied mathematics, numerical analysis, and the problem of solving nonlinear equations by numerical methods has gained more importance than before, since many practical problems in the applied information technology, as well as in intelligent materials and mechanical engineering, can build a suitable mathematical model, and then be transformed into nonlinear equations to solve.

In this paper, in order to improve the efficiency, we consider iterative methods to find a simple root of a nonlinear equation

$$f(x)=0, \tag{1}$$

where $f:\Omega\subseteq R\rightarrow R$ for an open interval Ω is a scalar function and it is sufficiently differentiable in a neighborhood of α.

It is well known that the classical Newton's method (NM) is an important and basic method for solving nonlinear equation [1] by the iterative scheme

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \tag{2}$$

which is quadratically convergent in the neighborhood of α.

In recent years, much attention has been given to develop iterative methods for solving nonlinear equations and a vast literature has been produced [2-12].

Motivated and inspired by the on-going activities in this direction, in this paper, we present a sixth-order convergent three-step iterative method and a seventh-order convergent method. Both of the three-step methods are free from second derivatives. Several numerical results are given to illustrate the efficiency and advantage of the algorithms.

2 TWO MODIFIED THREE-STEP METHODS AND THEIR CONVERGENCE ANALYSIS

Let us consider the following two three-step iterative methods.

Algorithm 1. For given x_0, we consider the following iteration scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{3}$$

$$z_n = y_n - \frac{2f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{f'(x_n)^2}, \tag{4}$$

$$x_{n+1} = z_n - \frac{5f'(x_n)^2 + 3f'(y_n)^2}{f'(x_n)^2 + 7f'(y_n)^2}\frac{f(z_n)}{f'(x_n)}. \tag{5}$$

For Algorithm 1, we have the following convergence result.

THEOREM 1. Assume that the function $f:\Omega \subseteq R \rightarrow R$ has a single root $\alpha \in \Omega$, where Ω is an open interval. If $f(x)$ has first, second and third derivatives in the interval Ω, then Algorithm 1 defined by (3)-(5) is sixth-order convergent in a neighborhood of α and it satisfies error equation

$$e_{n+1} = 5c_2^5 e_n^6 + O(e_n^7) \tag{6}$$

where

$$e_n = x_n - \alpha, \tag{7}$$

$$c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}, k = 1, 2, \cdots. \tag{8}$$

Proof. Let α be the simple root of $f(x)$, and

$$c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}, k = 1, 2, \cdots$$

$$e_n = x_n - \alpha.$$

Consider the iteration function $F(x)$ defined by

$$F(x) = z(x) - \frac{5f'(x)^2 + 3f'(y(x))^2}{f'(x)^2 + 7f'(y(x))^2}\frac{f(z(x))}{f'(x)} \tag{9}$$

where

$$z(x) = y(x) - \frac{2f(y(x))}{f'(x)} + \frac{f(y(x))f'(y(x))}{f'(x)^2},$$

$$y(x) = x - \frac{f(x)}{f'(x)}.$$

By some computations using Maple we can obtain

$$F(\alpha) = \alpha, F^{(i)}(\alpha) = 0, i = 1, 2, 3, 4, 5, \tag{10}$$

$$F^{(6)}(\alpha) = \frac{225 f^{(2)}(\alpha)^5}{2f'(\alpha)^5}. \tag{11}$$

Further more, from the Taylor expansion of $F(x_n)$ around α, we get

$$\begin{aligned} x_{n+1} &= F(x_n) = F(\alpha) + F'(\alpha)(x_n - \alpha) \\ &+ \frac{F^{(2)}(\alpha)}{2!}(x_n - \alpha)^2 + \frac{F^{(3)}(\alpha)}{3!}(x_n - \alpha)^3 \\ &+ \frac{F^{(4)}(\alpha)}{4!}(x_n - \alpha)^4 + \frac{F^{(5)}(\alpha)}{5!}(x_n - \alpha)^5 \\ &+ \frac{F^{(6)}(\alpha)}{6!}(x_n - \alpha)^6 + O((x_n - \alpha)^7). \end{aligned} \tag{12}$$

Substituting (11) into (12) yields

$$x_{n+1} = \alpha + e_{n+1} = \alpha + 5c_2^5 e_n^6 + O(e_n^7).$$

Therefore, we have

$$e_{n+1} = 5c_2^5 e_n^6 + O(e_n^7)$$

which shows that Algorithm 1 defined by (3)-(5) is sixth-order convergent.

Algorithm 2. For given x_0, we consider the three-step Newton-type iteration scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{13}$$

$$z_n = y_n - \frac{2f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{f'(x_n)^2}, \tag{14}$$

$$x_{n+1} = z_n - \frac{f'(x_n) + f'(y_n)}{-f'(x_n) + 3f'(y_n)}\frac{f(z_n)}{f'(x_n)}. \tag{15}$$

THEOREM 2. Assume that the function $f:\Omega \subseteq R \rightarrow R$ has a single root $\alpha \in \Omega$, where Ω is an open interval. If $f(x)$ has first, second and third derivatives in the interval Ω, then Algorithm 2 is seventh-order convergent in a neighborhood of α and it satisfies error equation

$$e_{n+1} = 10c_2^4(c_2^2 - c_3)e_n^7 + O(e_n^8),$$

where

$$e_n = x_n - \alpha,$$

$$c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}, k = 1, 2, \cdots.$$

Proof. Let α be the simple root of $f(x)$, and

$$c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}, k = 1, 2, \cdots, \quad e_n = x_n - \alpha.$$

Consider the iteration function $F(x)$ defined by

$$z(x) = y(x) - \frac{2f(y(x))}{f'(x)} + \frac{f(y(x))f'(y(x))}{f'(x)^2},$$

$$y(x) = x - \frac{f(x)}{f'(x)}.$$

$$F(x) = z(x) - \frac{f'(x) + f'(y(x))}{-f'(x) + 3f'(y(x))}\frac{f(z(x))}{f'(x)} \quad (16)$$

Where

By some computations using Maple we can obtain

$$F(\alpha) = \alpha, F^{(i)}(\alpha) = 0, i = 1, 2, 3, 4, 5, 6 \quad (17)$$

$$F^{(7)}(\alpha) = -\frac{525 f^{(2)}(\alpha)^4[-3f^{(2)}(\alpha)^2 + 2f'(\alpha)f^{(3)}(\alpha)]}{2f'(\alpha)^6}. \quad (18)$$

$$x_{n+1} = F(x_n) = F(\alpha) + \sum_{i=1}^{7}\frac{F^{(i)}(\alpha)}{i!}(x_n - \alpha)^i + O((x_n - \alpha)^8)$$

Furthermore, from the Taylor expansion of $F(x_n)$ around α, we get

Substituting (18) into (19) yields

$$x_{n+1} = \alpha + e_{n+1} = \alpha + 10c_2^4(c_2^2 - c_3)e_n^7 + O(e_n^8).$$

Therefore, we have

$$e_{n+1} = 10c_2^4(c_2^2 - c_3)e_n^7 + O(e_n^8)$$

which shows the seventh order of convergence.

To obtain an assessment of the efficiency of the proposed method, we shall make use of efficiency index, according to which the efficiency of an iterative method is given by $p^{1/\omega}$, where p is the order of the method and ω is the number of function evaluations per iteration required by the method. It is not hard to see that the efficiency indices of the Algorithm 1 and Algorithm 2 are 1.431 and 1.476 respectively, which are better than that of the classical Newton's method 1.414.

3 NUMERICAL RESULTS

Now, we employ Algorithm 1 and Algorithm 2 to solve some nonlinear equations and compare them with NM and the iterative method (PPM for short) Potra and Pták presented in [7]

$$x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)} \quad (19)$$

which is cubically convergent with efficiency index 1.442, where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$.

Displayed in Table 1 are the number of iterations (ITs) required such that $|f(x_n)| < 1.E-14$.

In table 1, we use the following functions.

$f_1(x) = x^3 + 4x^2 - 10,\ \alpha = 1.36523001341410.$

$f_2(x) = \cos x - x,\ \alpha = 0.73908513321516.$

$f_3(x) = (x-1)^3 - 1,\ \alpha = 2.$

$f_4(x) = e^{x^2+7x-30} - 1,\ \alpha = 3.$

$f_5(x) = \sin^2(x) - x^2 + 1, \alpha = 1.40449164885154.$

$f_6(x) = (x+2)e^x - 1, \alpha = -0.44285440096708.$

The computational results in Table 1 show that Algorithm 1 requires less ITs than NM. Therefore, Algorithm is of practical interest and can compete with NM.

Table 1: Comparison of Algorithm 1, Algorithm 2, PPM and NM

Func-tions	x_0	NM	PPM	Algorithm 1	Algorithm 2
f_1	1	5	3	3	3
	1.46	4	3	3	3
f_2	0.5	4	3	3	3
	3.12	7	4	3	3
f_3	2.5	6	3	3	3
	1.45	7	5	4	4
f_4	4.2	16	8	6	6
	3.4	12	5	4	4
f_5	1.8	5	4	3	3
	2.1	5	4	4	3
f_6	7.6	12	9	5	5
	-0.5	4	4	3	3

4 CONCLUSIONS

In order to obtain efficient iterative methods for the nonlinear equations which come from the practical problems in the materials science and manufacturing technology field, in this paper, we present and analyze two modified Newton-type iterative methods for solving nonlinear equations. Both of the algorithms are free from second derivatives. Several numerical results illustrate the convergence behavior and computational efficiency of the method proposed in this paper. Computational results demonstrate that they are more efficient and performs better than the classical NM.

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