A Class of Three-step Root-solvers with Order of Convergence Five for Nonlinear Equations

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Abstract: The root-finding problem of a univariate nonlinear equation is a fundamental and long-studied problem, and it has wide applications in mathematics and engineering computation. In this paper, a class of modified Newton-type methods for solving nonlinear equations is brought forward. Analytical discussions are reported and the theoretical efficiency of the method is studied. The proposed algorithm requires two evaluations of the functions and two evaluations of derivatives at each iteration. Therefore the efficiency indices of it is 1.4953. Hence, the index of the proposed algorithm is better than that of classical Newton's method 1.4142. The proposed algorithm in this paper is free from second derivatives. Some numerical results are finally provided to support the theoretical discussions of the proposed method.

1 INTRODUCTION

One of the most important questions in mathematics is how to find a solution of the nonlinear equation

$$f(x)=0,$$

where $f: D \subseteq R \to R$ for an open interval D has sufficient number of continuous derivatives in a neighborhood of the root α. Solving nonlinear equations is a classical problem which has interesting applications in various branches of science, and engineering computation. The nonlinear equations play an important role in many fields of science, and many numerical methods are developed. In this paper, we apply iterative method to find a simple root α of the above problem.

It is well known that Newton's method (NM for simplicity) is one of the most important and famous methods for computing approximations α by the following iterative scheme [5]

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

The Newton's method converges quadratically in some neighborhood of α for some appropriate start value x_0. The main advantage of this method is that the computation of the second derivative not required.

In the past decades, much attention has been paid to develop iterative methods for solving nonlinear equations, and a large number of researchers try to improve Newton's method in order to get a method with a higher order of convergence and more accuracy in open literatures, see for example [1-18] and the references therein for more details.

For example, the algoithm defined by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$

$$x_{n+1} = y_n - \frac{3f'(x_n)^2 + f'(y_n)^2}{2f'(x_n)f'(y_n) + 2f'(y_n)^2}\frac{f(y_n)}{f'(x_n)} \tag{3}$$

is fifth-order convergent, and it satisfies the following error equation

$$e_{n+1} = \frac{1}{2}c_2^2(6c_2^2 - c_3)e_n^5 + O(e_n^6).$$

The following fifth-order convergent iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{4}$$

$$x_{n+1} = y_n - \frac{3f'(x_n)+f'(y_n)}{-f'(x_n)+5f'(y_n)} \frac{f(y_n)}{f'(x_n)}, \tag{5}$$

satisfies error equation

$$e_{n+1} = \frac{1}{2} c_2^2 (3c_2^2 - 2c_3) e_n^5 + O(e_n^6).$$

Motivated and inspired by the ongoing activities in the direction, in this paper, to improve the local order of convergence properties, we present a class of modified Newton-type iterative method for solving nonlinear equations.

Based on above two efficient five-order convergent methods, in this paper, we construct a new iteration formula by introducing a real parameter $\lambda\,(0 \le \lambda \le 1)$. The order of convergence of the proposed method is five, and it does not depend on the parameter λ.

The proposed algorithm in this paper is free from second derivatives. At each iteration, it requires two evaluations of the functions and two evaluations of derivatives. Some numerical results are given to illustrate the advantage and effectiveness of the methods.

The rest of the paper is organized as follows: in Section 2 we describe a class of modified Newton-type iterative methods and analyze its convergence. Different numerical test confirm the theoretical results and allow us to compare our new method with some other known methods in Section 3. Finally, the conclusions are given in Section 4.

2 A CLASS OF MODIFIED NEWTON-TYPE ITERATIVE METHODS AND CONVERGENCE ANALYSIS

Now, we consider the following iterative scheme.

Algorithm 1. For given x_0, we consider the following iteration method for solving nonlinear equation

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{6}$$

$$x_{n+1} = y_n - [\lambda \cdot \frac{3f'(x_n)^2 + f'(y_n)^2}{2f'(x_n)f'(y_n) + 2f'(y_n)^2} + (1-\lambda) \cdot \frac{3f'(x_n)+f'(y_n)}{-f'(x_n)+5f'(y_n)}] \frac{f(y_n)}{f'(x_n)}, \tag{7}$$

where $\lambda\,(0 \le \lambda \le 1)$ is a real parameter.

For Algorithm 1, we have the following convergence result.

THEOREM 1. Assume that the function $f : D \subseteq R \to R$ has a single root $\alpha \in D$, where D is an open interval. If $f(x)$ has first, second and third derivatives in the interval D, then Algorithm 1 defined by (6)-(7) is fifth-order convergent in a neighborhood of α and it satisfies the following error equation

$$e_{n+1} = \frac{1}{2} c_2^2 \left[3(\lambda+1)c_2^2 - 2c_3 \right] e_n^5 + O(e_n^6), \tag{8}$$

where

$$e_n = x_n - \alpha, \tag{9}$$

$$c_k = \frac{f^{(k)}(\alpha)}{k!\,f'(\alpha)}, k = 1, 2, \cdots \tag{10}$$

Proof. Let α be the simple root of $f(x)$, and $e_n = x_n - \alpha$,

$$c_k = \frac{f^{(k)}(\alpha)}{k!\,f'(\alpha)}, k = 1, 2, \cdots.$$

Consider the iteration function $F(x)$ defined by

$$F(x) = y(x) - [\lambda \cdot \frac{3f'(x)^2 + f'(y(x))^2}{2f'(x)f'(y(x)) + 2f'(y(x))^2} + (1-\lambda) \cdot \frac{3f'(x)+f'(y(x))}{-f'(x)+5f'(y(x))}] \frac{f(y(x))}{f'(x)} \tag{11}$$

where

$$y(x) = x - \frac{f(x)}{f'(x)}. \tag{12}$$

By some computations using Maple we can obtain

$$F(\alpha) = \alpha, \tag{13}$$

$$F^{(i)}(\alpha) = 0, i = 1, 2, 3, 4,$$

Furthermore, from the Taylor expansion of $F(x_n)$ at α, we have

$$\begin{aligned} x_{n+1} &= F(x_n) \\ &= F(\alpha) + \sum_{k=1}^{5} \frac{F^{(k)}(\alpha)}{k!}(x_n - \alpha)^k + O((x_n - \alpha)^6). \end{aligned} \tag{14}$$

Substituting (13) into (14) yields

$$\begin{aligned} x_{n+1} &= \alpha + e_{n+1} \\ &= \alpha + \frac{1}{2}c_2^2\left[3(\lambda+1)c_2^2 - 2c_3\right]e_n^5 + O(e_n^6). \end{aligned}$$

Therefore, we have the error expression of the algorithm

$$e_{n+1} = \frac{1}{2}c_2^2\left[3(\lambda+1)c_2^2 - 2c_3\right]e_n^5 + O(e_n^6),$$

which means the order of convergence of the Algorithm 1 is five. The proof is completed.

3 NUMERICAL RESULTS

This section is devoted to checking the effectiveness and efficiency of our proposed method Algorithm 1 with NM, and PPM method defined by

$$x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)} \tag{15}$$

which is third-order convergent with efficiency index 1.4422.

Table 1 shows the number of iterations (ITs) required to satisfy the stopping criterion. In the numerical experiment, we take parameter $\lambda = 0.5$. For orher parameter λ we can obain similar results. All computations were done by using MATLAB 7.0 and using 64 digit floating point arithmetics. In table 1, we use the following functions.

$$\begin{aligned} &f_1(x) = \sin^2(x) - x^2 + 1, \ \alpha \approx 1.404492. \\ &f_2(x) = x^3 + 4x^2 - 10, \ \alpha \approx 1.365230. \\ &f_3(x) = (x-1)^3 - 1, \ \alpha = 2. \\ &f_4(x) = x^2 - e^x - 3x + 1, \ \alpha \approx 1.404492. \\ &f_5(x) = \cos x - xe^x + x^2, \ \alpha \approx 0.639154. \\ &f_6(x) = e^x - \arctan(x) - 1.5, \ \alpha \approx 0.767653. \end{aligned}$$

The computational results in Table 1 show that Algorithm 1 requires less ITs than NM and PPM. Therefore, the proposed method is of practical interest and can compete with NM and PPM.

Table 1: Comparison of Algorithm 1, PPM and NM.

Functions	x_0	NM	PPM	Algorithm 1
f_1	1.45	6	5	3
	2.1	8	6	5
f_2	1.12	6	5	4
	1.43	5	4	4
f_3	2.3	8	6	3
	1.5	9	7	6
f_4	0.8	8	5	4
	1.25	6	4	4
f_5	1.3	10	7	5
	0.8	7	6	4
f_6	1.4	12	9	8
	0.82	10	8	5

4 CONCLUSIONS

This paper presented and analyzed a class of modified three-step Newton-type iterative methods for solving nonlinear equations. The method is free from second derivatives, and it requires two evaluations of the functions and two evaluations of derivatives at each step. Several numerical tests demonstrate that the method proposed in the paper is more efficient and perform better than Newton's method, and PPM.

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REFERENCES

1. S. Huang, and Z. WAN, A new nonmonotone spectral residual method for nonsmooth nonlinear equations, Journal of Computational and Applied Mathematics 313 (2017), pp. 82-101.
2. Z. Liu, Q. Zheng, and C-E Huang, Third- and fifth-order Newton-Gauss methods for solving nonlinear equations with n variables, Applied Mathematics and Computation, 290(2016), pp. 250-257.
3. X-D. Chen, Y. Zhang, J. Shi, and Y. Wang, An efficient method based on progressive interpolation for solving nonlinear equations, Applied Mathematics Letters, 61(2016), pp. 67-72.
4. S. Amat, C. Bermúdez, M.A. Hernández-Verón, and E. Martínez, On an efficient image k-step iterative method for nonlinear equations, Journal of Computational and Applied Mathematics, 302(2016), pp. 258-271.
5. X-W FANG, Q. NI, and M-L ZENG, A modified quasi-Newton method for nonlinear equations, Journal of Computational and Applied Mathematics 328 (2018) , pp.44-58.
6. F. Ahmad, F. Soleymani, F. Khaksar Haghani, and S. Serra-Capizzano, Higher order derivative-free iterative methods with and without memory for systems of nonlinear equations, Applied Mathematics and Computation 314 (2017), pp. 199-211.
7. A. Khare, and A. Saxena, Novel PT-invariant solutions for a large number of real nonlinear equations, Physics Letters A 380 (2016), pp. 856-862.
8. Y.-C, Kima, and K.-A Lee, The Evans-Krylov theorem for nonlocal parabolic fully nonlinear equations, Nonlinear Analysis 160 (2017), pp.79-107.
9. M. Alquran, H. M. Jaradat Muhammed, and I. Syam, A modified approach for a reliable study of new nonlinear equation: two-mode Korteweg-de Vries-Burgers equation, Nonlinear Dynamics, 91(3), 2018, pp 1619-1626.
10. Z. Wan, and W. Liu, A modifed spectral conjugate gradient projection method for solving nonlinear monotone symmetric equations, Pac. J. Optim. 12(2016), pp. 603-622.
11. M. Grau-Sánchez, M. Noguera, and J.L. Diaz-Barrero, Note on the efficiency of some iterative methods for solving nonlinear equations, SeMA J. 71 (2015), pp.15-22.
12. M.S. Petkovic, and J.R. Sharma, On some efficient derivative-free iterative methods with memory for solving systems of nonlinear equations, Numer. Algorithm 71 (2015), pp. 1017-1398.
13. J.R. Sharma, and H. Arora, Efficient derivative-free numerical methods for solving systems of nonlinear equations, Comput. Appl. Math. 35 (2016), pp. 269-284.
14. X. Wang, T. Zhang, W. Qian, and M. Teng, Seventh-order derivative-free iterative method for solving nonlinear systems, Numer. Algorithm 70 (2015), pp. 545-558.
15. W.P. Li, F. Zhao, and T.Z. Wang, Small prime solutions of an nonlinear equation, Acta Math. Sinica (Chin. Ser.) 58 (2015), pp. 739-764 (in Chinese).
16. H. Chang-Lara, and D. Kriventsov, Further time regularity for nonlocal, fully nonlinear parabolic equations, Comm. Pure Appl. Math. 70 (2017), pp. 950-977.
17. S. Deng, Z. Wan, A three-term conjugate gradient algorithm for large-scale unconstrained optimization problems, Appl. Numer. Math. 92 (2015), pp. 70-81.
18. Y.H. Dai, M. Al-Baali, X.Q. Yang, A positive Barzilai-Borwein-Like stepsize and an extension for symmetric linear systems, Numer. Funct. Anal. Optim. 134 (2015), pp. 59-75.